PRACTICAL MATHEMATICS

NEW EDITION

REVISED UNDER THE SUPERVISION OF

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EXTRACT FROM ORIGINAL PREFACE

In preparing this treatise on Practical Mathematics considerable pains have been taken to explain, in the clearest manner, the method of solving the numerous problems. The rules have been expressed as simply and concisely as possible in common language, as well as symbolically by algebraic formulæ, which frequently possess, on account of their conciseness and precision, a great advantage over ordinary language; they have also in many instances been given logarithmically, because of the facility and expedition of logarithmic calculation. To understand the algebraic formulæ, nothing more is necessary than a knowledge of the simple notation of algebra; the method of computation by logarithms is explained in the Introduction to *Chambers's Mathematical Tables*.

PREFACE TO THIRD EDITION

In general plan the present edition of *Practical Mathematics* does not differ from its predecessors, which were prepared and edited by Dr Pryde. The aim has been to illustrate the use of mathematics in constructing diagrams; in measuring areas, volumes, strengths of materials; in calculating latitudes and longitudes on the earth's surface; and in solving similar problems. There is no attempt at a systematic development of any part of mathematics, except to

a certain extent in the sections on plane and spherical trigonometry. The plane trigonometry has been remodelled; but the spherical trigonometry, which is required for navigation and geodesy, has been left as it was. The greatest changes will, however, be found in the section upon strength of materials and associated problems in elasticity. New tables of constants have been added, and new types of problems have been worked out or indicated. Throughout this section graphical solutions are occasionally given; and a final section has been added to the book in which the elements of curve-tracing, a growingly important part of mathematics, both practical and theoretical, are discussed and illustrated by examples.

One great branch of Practical Mathematics, that dealing with electricity and magnetism, has not been included in the present work. It would have been impossible to give this peculiarly modern subject adequate space without increasing the volume to an inconvenient size.

When logarithms are required in solving any problem, the seven-place Logarithmic Tables are used; but since for many purposes a rough approximation is all that is needed, it has been thought advisable to make the book more complete in itself by the addition of six pages of four-place logarithms of the natural numbers and the trigonometrical ratios.

It should be stated that in the section dealing with strength of materials, as well as in other parts of the work, valuable aid has been rendered by Mr Forrest Sutherland, of Bloemfontein.

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PRACTICAL MATHEMATICS

DESCRIPTIVE GEOMETRY

DESCRIPTIVE GEOMETRY explains the methods of performing certain geometrical operations, such as the construction of mathematical figures, the drawing of lines in certain positions, and the application of geometrical principles to the accurate representation of plane surfaces and solids. Hence it is treated of under two heads—PLANE DESCRIPTIVE GEOMETRY, and SOLID DESCRIPTIVE GEOMETRY.

There are three kinds of geometrical magnitudes—lines, surfaces, and solids. **Lines** have one dimension—length; **surfaces** have two dimensions—length and breadth; and **solids** have three dimensions—length, breadth, and thickness.

I. PLANES

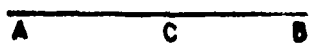
Plane Descriptive Geometry treats of the relations and dimensions of lines and figures formed by their combinations on planes or plane surfaces.

DEFINITIONS

1. A **point** has position only, but no magnitude.
2. A **line** has length without breadth.

Hence the extremities or ends of a line are points; and if two lines intersect or cross each other, the intersections are points.

A line is named by two letters placed one at each of its extremities. Thus, the line drawn here is named the line AB.



3. A **straight line** is that which lies evenly between its extreme points.

If a straight line, as AB, revolve like an axis, its two extremities, A and B, remaining in the same position, any other point of it, as C, will also remain in the same position.

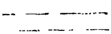
4. A **point of section** is any point in a line, and the two parts into which it divides the line are called **segments**. Thus the point C in the preceding line AB is a point of section, and AC, BC, are segments.

It is evident that two straight lines cannot enclose a space; and that two straight lines cannot have a common segment, or cannot coincide in part without coinciding altogether.

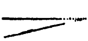
5. A **crooked line** is composed of two or more straight lines.

6. A **curve,** or a **curved line,** is one of which no part is straight.

7. **Parallel straight lines** are such as are in the same plane, and are at all points equidistant; hence, if they are produced indefinitely in either direction, they do not meet.



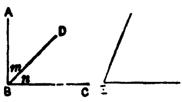
8. **Convergent lines** are those in the same plane, but not parallel, while they are supposed to be produced in the direction in which they would meet. Such lines are said to be *divergent,* when considered as receding from the same point.



9. A **surface** has only length and breadth.

The boundaries of a surface are lines; and the intersection of one surface with another is also a line.

10. A **plane surface,** or a **plane,** is a surface such that, if any two points are taken on it, the straight line joining them lies wholly on that surface.



11. A **plane rectilineal angle** is the inclination of two straight lines that meet, but are not in the same straight line.

12. The **angular point** is the point at which an angle is formed, as E or B.

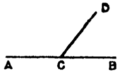
When there is only one angle at a point, it may be named by one letter, as angle E.

But when there are more angles than one at a point, they are named by three letters, the letter at the angular point being put in the middle. Thus the angle contained by the lines DB and BC is named the angle DBC or CBD. So the angle contained by the lines AB and DB is named the angle ABD or DBA.

An angle may also be named by means of a small letter placed in it. Thus angle ABD may be named angle m; angle DBC, n; and ABC, $m+n$.

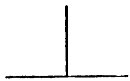
The two lines containing an angle are called its sides. Thus DB, BC, are the sides of the angle DBC, or n.

13. **Supplementary angles** are the two adjacent angles formed by one straight line standing upon another.

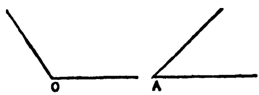


Thus the angles ACD, DCB, are said to be supplementary to one another; or the angle ACD is called the supplement of the angle DCB; and DCB is called the supplement of ACD.

14. A **right angle** is one of two supplementary angles which are equal; and the line which separates them is said to be a **perpendicular** to the line on which it stands.

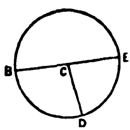


15. An **obtuse angle** is greater than a right angle, as O; and an **acute angle** is less than a right angle, as A.



16. A **figure** is that which is enclosed by one or more boundaries. The space contained within the boundary of the figure is called its **surface**; and the quantity of surface in reference to that of some other figure with which it is compared is called its **area.**

17. A **circle** is a figure formed on a plane by causing a line to revolve round one of its extremities which remains fixed.



18. The **circumference of a circle** is the line that bounds it.

19. The **centre of a circle** is the fixed point of the revolving line which describes it, as C.

An eccentric point in a circle is one which is not the centre of the circle, but spoken of in reference to it.

20. A **radius** is a straight line drawn from the centre to the circumference of a circle; CB, CD, and CE are radii.

It is evident that all radii of the same circle are equal in length.

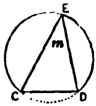
21. A **diameter** is a straight line passing through the centre of a circle, and terminated at each extremity by the circumference, as BE. The radius is sometimes called the **semi-diameter.**

22. An **arc** of a circle is any part of the circumference.

23. The **chord** of an arc is a straight line joining its extremities, as AB.

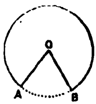
24. A **segment** of a circle is a figure contained by an arc and its chord.

25. A **semicircle** is a segment having a diameter for its chord, and is evidently half of the whole circle.



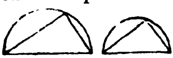
26. An **angle in a segment** of a circle is an angle contained by two straight lines, drawn from any point in the arc of the segment to the extremities of its chord, as m in the segment CED; the angle m is also said to *stand on* the arc CD.

27. A **sector of a circle** is a figure contained by two radii and the intercepted arc, as AOB.



28. A **quadrant** is a sector whose bounding radii are perpendicular to each other, and is evidently the fourth part of a circle.

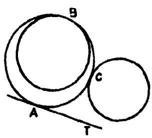
29. A **quadrantal arc,** or the arc of a quadrant, is the fourth part of the circumference, and is sometimes merely called a quadrant.



30. **Similar segments of circles** are those that contain equal angles.

31. **Similar arcs of circles** are those that subtend or are opposite to equal angles at the centre.

32. **Similar sectors** are those that are bounded by similar arcs.



33. **Equal circles** are those that have equal radii.

34. **Concentric circles** are those that have the same centre, and **eccentric circles** are those which have different centres.

35. A **tangent** is a straight line which meets a circle or curve in one point, and being produced, does not cut it, as AT.

36. **Tangent circles** are those of which the circumferences meet, but do not cut one another.

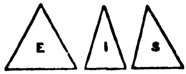
37. The **point of contact** is that point in which a tangent and a curve, or two tangent curves, meet; thus the points A, B, and C are points of contact.

38. **Rectilineal figures** are those contained by straight lines.

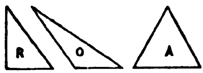
39. **Trilateral figures,** or triangles, are contained by three straight lines.

40. **Quadrilateral figures** are contained by four straight lines.

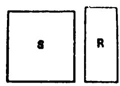
41. **Multilateral figures,** or **polygons,** are contained by more than four straight lines.



42. Of **three-sided figures,** an **equilateral triangle** has three equal sides, as E; an **isosceles triangle** has two equal sides, as I; and a **scalene triangle** has three unequal sides, as S.

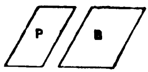


43. A **right-angled triangle** has one right angle, as R; an **obtuse-angled triangle** has one obtuse angle, as O; and an **acute-angled triangle** has all its angles acute, as A.



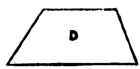
44. Of **quadrilateral figures,** a **square** has all its sides equal, and its angles right angles, as S.

45. A **rectangle** has all its angles right angles, but its sides are not all equal, as R.



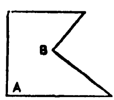
46. A **rhombus** has all its sides equal, but its angles are not right angles, as B.

47. A **parallelogram** has its opposite sides parallel, as P.



48. A **trapezium** has only two sides parallel, as D.

49. An angle of a rectilineal figure which is greater than two right angles is said to be **re-entrant,** as B.



50. Any side of a **rectilineal figure** may be called its **base.** In a **right-angled triangle,** the side opposite to the right angle is called the **hypotenuse;** either of the sides about the right angle, the **base;** and the other, the **perpendicular.** In an **isosceles triangle,** the unique side is called the **base;** the **angular point** opposite to the base of a triangle is called the **vertex;** and the **angle** at the vertex, the **vertical angle.**

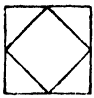
51. The **altitude of a triangle** is a perpendicular drawn from the vertex to the base. The **altitude of a parallelogram** is a perpendicular to the base from any point in the opposite side. The **altitude of a trapezium** is a perpendicular from any point in one of its parallel sides to the opposite side.

52. A **diagonal** of a quadrilateral is a straight line joining two of the opposite angular points.

A diagonal of any polygon is a straight line joining any two of its angular points which are not consecutive.

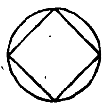
53. A **rectangle** is said to be **contained by two lines** when its two adjacent sides are these lines, or lines equal to them.

54. A line is said to be cut in **medial section,** or in **extreme and mean ratio,** when the rectangle contained by the whole line and one of its parts is equal to the square on the other part.

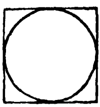


55. **A rectilineal figure is said to be inscribed in another rectilineal figure** when all the angular points of the inscribed figure are upon the sides of the figure in which it is inscribed.

56. **A rectilineal figure is said to be circumscribed about another** when its sides respectively pass through the angular points of the other figure about which it is circumscribed.

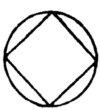


57. **A rectilineal figure is said to be inscribed in a circle** when all the angular points of the inscribed figure are upon the circumference of the circle.



58. **A rectilineal figure is said to be circumscribed about a circle** when each side of the rectilineal figure touches the circumference of the circle.

59. **A circle is said to be inscribed in a rectilineal figure** when the circumference of the circle touches each of the sides of the rectilineal figure.



60. **A circle is said to be circumscribed about a rectilineal figure** when the circumference of the circle passes through all the angular points of the figure.

61. A **regular polygon** has all its sides and angles equal; or it is both equilateral and equiangular.

62. A **polygon** of five sides is called a **pentagon**; of six, a **hexagon**; of seven, a **heptagon**; of eight, an **octagon**; of nine, a **nonagon**; of ten, a **decagon**; of twelve, a **dodecagon.**

63. The **centre of a regular polygon** is a point equidistant from its sides, or from its angular points.

64. The **apothem of a regular polygon** is a perpendicular from its centre upon any of its sides.

65. The **perimeter of any figure** is its circumference or whole boundary; it is also called the **periphery.**

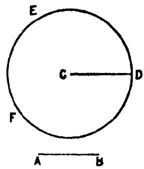
66. The **ratio of any two quantities to one another** is the number of times that the former contains the latter.

Thus, if a line A contain a line B three times, the ratio of A to B is 3, or the ratio of B to A is $\frac{1}{3}$. The ratio of A to B is denoted by A : B, or A ÷ B, or $\frac{A}{B}$.

67. A **proportion** consists of two equal ratios.

PROBLEMS

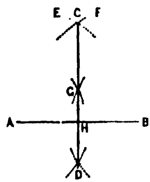
68. **Problem I.—To describe a circle with a given radius about a given point as a centre.**



Let AB be the given radius, and C the given point.

Place one point of the compasses on A, and extend the other point to B; then with that distance as a radius, and placing one point of the compasses on C, describe with the other point the circumference DEF; and the required circle will be formed.

69. **Problem II.—To bisect a given straight line; that is, to divide it into two equal parts.**



METHOD 1.—Let AB be the given straight line.

From A and B as centres, with a radius greater than the half of AB, describe arcs EC, FC, intersecting in C (68); describe arcs similarly intersecting in D; and join the points C, D, and CD will bisect AB in H.

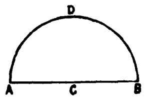
METHOD 2.—As before, describe arcs intersecting in C, and describe similarly two arcs intersecting in G; and if GC be then drawn and produced, it will bisect AB in H.

The first method can be proved by joining with straight lines

the points A and C, C and B, B and D, D and A. For then the two triangles thus formed—namely, ADC and BDC—would be equal in every respect (Eucl. I. 8); and hence the two angles thus formed at D would be equal. Then the two triangles ADH, BDH, would be equal (Eucl. I. 4), and hence AH = HB.

The second method can be similarly proved.

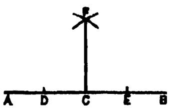
70. Problem III.—To describe a semicircle on a given finite straight line as a diameter.



Let AB be the given straight line.

Bisect it in C (69), and from C as a centre, with a radius equal to AC or CB, describe the semicircle ADB (68), and it will be the required semicircle.

71. Problem IV.—From a given point in a given straight line, to erect a perpendicular.

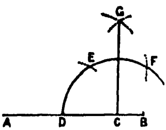


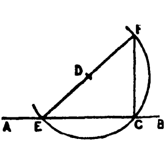
Let AB be the given straight line, and C the given point.

CASE 1.—When the point is near the middle of the line.

On each side of C lay off equal distances CD, CE; and from D and E as centres, with a radius greater than DC or CE, describe arcs intersecting in F; draw CF, which is the required perpendicular. (Eucl. I. 11.)

CASE 2.—When the point is near one of the extremities of the line.



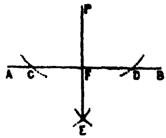


METHOD 1.—From C as a centre, with any radius, describe the arc DEF, and from D lay off the same radius to E, and from E to F; then from E and F as centres, with the same or any other radius not less than half the former, describe arcs intersecting in G; draw GC, and it will be perpendicular to AB.

This is evident from Eucl. IV. 15, Cor.

METHOD 2.—From any point D as a centre, and the distance DC as a radius, describe an arc ECF, cutting AB in E and C; draw ED, and produce it to cut the arc in F; then draw FC, which is the required perpendicular. (Eucl. III. 31.)

72. **Problem V.—From a given point without a given straight line, to draw a perpendicular to it.**

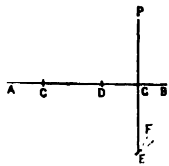


Let AB be the given line, and P the given point.

CASE 1.—When the point is nearly opposite to the middle of the line.

From P as a centre, with any convenient radius, describe arcs cutting AB in C and D; and from these two points as centres, with a radius greater than the half of DC, describe arcs cutting in the point E; draw PE, and PF will be the required perpendicular.

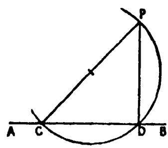
This may be proved as Prob. II.



CASE 2.—When the given point is nearly opposite to one end of the line.

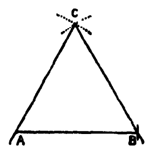
METHOD 1.—From any point C in AB as a centre, with the radius CP, describe an arc on the other side of AB; and from any other point D in AB, with the radius DP, describe an arc cutting the former in E; then draw PE, and PG is the perpendicular.

This is proved as the preceding case.



METHOD 2.—Take any point C in AB, and join PC, and on PC describe a semicircle (III.) PDC intersecting AB in D; draw PD, which is the perpendicular required. (Eucl. III. 31.)

73. **Problem VI.—On a given straight line, to describe an equilateral triangle.**



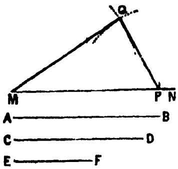
Let AB be the given line.

From A and B as centres, with a radius equal to AB, describe arcs intersecting in C, and draw AC, BC; then ABC is the required triangle. (Eucl. I. 1.)

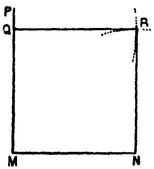
74. **Problem VII.—To describe a triangle whose three sides shall be respectively equal to three given lines, of which the length of any two together is greater than the third.**

Let AB, CD, and EF be the three given lines.

Draw any line MN, and from it cut off a part MP equal to AB; then from M as centre, with CD as radius, describe an arc at Q; and from P as centre, with EF as radius, describe another arc cutting the former in Q; and draw MQ and PQ; then MPQ is the required triangle. (Eucl. I. 22.)



75. Problem VIII.—On a given straight line, to describe a square.

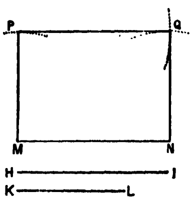


Let MN be the given straight line.

From M draw MP perpendicular to MN (71), and from MP cut off a part MQ equal to MN; then from Q and N as centres, with a radius equal to MN, describe arcs intersecting in R; draw QR and NR; MR* is the required square.

This is easily proved by Eucl. I. 8 and 32.

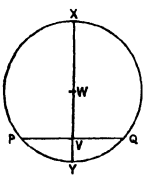
76. Problem IX.—To describe a rectangle whose length and breadth shall be respectively equal to two given straight lines.



Let HI and KL be the given straight lines.

Draw a line MN equal to HI; and draw MP perpendicular to MN (71), and equal to KL; from P as a centre, with a radius equal to MN, describe an arc at Q; and from N as centre, with a radius equal to MP, describe an arc cutting the former in Q; draw PQ, NQ; and MQ is the required rectangle.

This may be proved by Eucl. I. 8, 27 and 29.



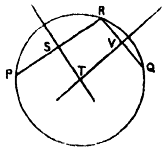
77. Problem X.—To find the centre of a given circle.

Let PQX be the given circle.

Draw any chord PQ in the circle; bisect the chord by the perpendicular XY, which is a diameter; then bisect XY in the point W, and the point W is the centre of the circle. (Eucl. III. 1.)

* Quadrilateral figures are thus concisely named by the letters at two opposite angular points.

78. **Problem XI.—To describe a circle through three given points, not in the same straight line.**



Let P, Q, and R be the three points.

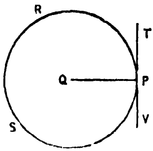
Join PR and QR; bisect PR by the perpendicular ST, and QR by the perpendicular VT; then from T as centre, with any of the distances TP, TR, TQ, describe a circle, and it will pass through the points P, Q, R, and be the required circle. (Eucl. IV. 5.)

79. **Problem XII.—Given a segment of a circle, to complete the circle of which it is a segment.**

Let P, Q, and R (fig. Prob. XI.) be any three points in the arc of the segment.

As in the preceding problem, find T the centre of the circle that passes through P, Q, and R, and it is the centre of the required circle, which can be described as in that problem.

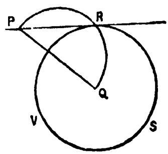
80. **Problem XIII.—To draw a tangent to a given circle from a given point in its circumference.**



Let PRS be the given circle, and P the given point.

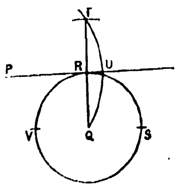
Find the centre of the circle, and from the point P draw the radius PQ; then draw a line TV through P perpendicular to PQ, and TV is the required tangent. (Eucl. III. 16.)

81. **Problem XIV.—To draw a tangent to a given circle from a given point without it.**



Let P be the given point, and RVS the given circle.

METHOD 1.—Find the centre Q, and join P and Q; on PQ describe a semicircle PRQ, cutting the given circle in R; draw PR, and it is the required tangent.



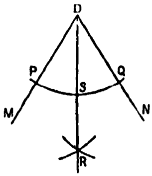
For if RQ is joined, then PRQ, being an angle in a semicircle, is a right angle. (Eucl. III. 31.)

METHOD 2.—Find Q the centre of the circle, and with the radius PQ describe the arc QUT; with the diameter of the circle VS as a radius, and Q as a centre, cut the arc QUT in T;

draw TQ, intersecting the given circle in R; draw PR, and it is the required tangent.

For PR bisects QT, and is therefore perpendicular to it. (Eucl. III. 3.)

82. Problem XV.—To bisect a given angle.



Let MDN be the given angle.

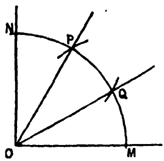
Lay off on the sides of the angle any equal distances DP, DQ; from P and Q as centres, describe arcs with equal radii intersecting in R; draw DR, and it bisects the given angle, or divides it into the two equal angles MDR and NDR. (Eucl. I. 9.)

83. Problem XVI.—To bisect an arc of a circle.

Let PSQ (fig. Art. 82) be the arc, of which D is the centre.

Find the point R, as in the preceding problem; and then the line DR divides the arc in S into the two equal arcs PS and SQ. (Eucl. III. 26.)

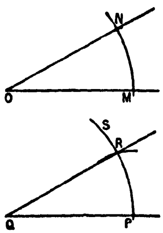
84. Problem XVII.—To trisect a right angle; that is, to divide it into three equal parts.



Let MON be the right angle.

From the point O, with any radius, describe an arc MPN, cutting the sides of the angle in M and N; with the same radius and the centres, M and N, cut the arc in P and Q; draw OP, OQ, which trisect the angle. This is evident from Eucl. IV. 15, Cor.

85. Cor.—The quadrantal arc NPQM is evidently trisected in the points P and Q.



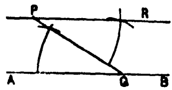
86. Problem XVIII.—At a given point in a given straight line, to make an angle equal to a given angle.

Let O be the given angle, QP the given straight line, and Q the given point.

From the centres O and Q, with the same radius, describe arcs MN and PS; with a radius equal to the chord of the arc MN, and P as centre, cut the arc PS in R; draw QR, and PQR is the required angle, being equal to angle MON.

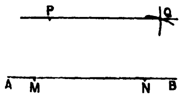
For if MN are joined, and also PR, the two triangles MON, PQR, will be equal in every respect (Eucl. I. 8); and hence angle Q=O.

87. Problem XIX.—Through a given point to draw a straight line parallel to a given straight line.



Let AB be the given line, and P the given point.

METHOD 1.—Take any point Q in AB, and draw PQ; make the angle RPQ equal to the angle PQA (86), and PR is parallel to AB. (Eucl. I. 27.)

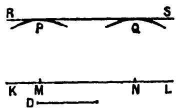


METHOD 2.—In AB take any two points M and N; from P as centre, with the radius MN, describe an arc at Q; from N as centre, with the radius MP, describe an arc cutting the former in Q; draw PQ, and it is parallel to AB.

For it is easily proved that if PM, NQ were joined, PN would be a parallelogram.

88. Problem XX.—To draw a straight line parallel to a given straight line, and at a given distance from it.

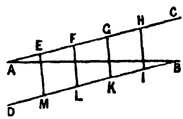
Let KL be the given line, and D the given distance.



From any two points M and N in KL as centres, with a radius equal to D, describe the arcs P and Q; draw a line RS to touch these arcs; that is, to be a common tangent to them; and RS is the line required parallel to KL.

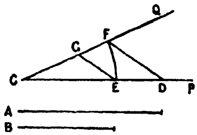
89. Problem XXI.—To divide a given straight line into any number of equal parts.

Let AB be the given straight line, and let the number of equal parts into which it is to be divided be five.



Draw a line AC through A at any inclination to AB, and through B draw another line BD parallel to AC; take any distance AE, and lay it off four times on AC, forming the equal parts AE, EF, FG, GH; lay off the same distance four times on BD in the same manner, from the point B; draw the lines HI, GK, FL, and EM, and they will divide AB into five equal parts. For AB, AH, and BM are cut proportionally. (Eucl. VI. 10.)

90. Problem XXII.—To find a third proportional to two given straight lines.

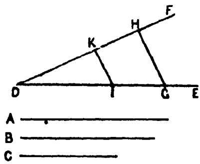


Let A and B be the given lines.

Draw a line CD equal to A, and through C draw a line CQ inclined at any angle to CD; make CE and CF each equal to B; join DF, and through E draw EG parallel to DF; and CG is the third proportional to A and B; that is,

$$A : B = B : CG. \quad \text{(Eucl. VI. 2.)}$$

91. Problem XXIII.—To find a fourth proportional to three given straight lines.

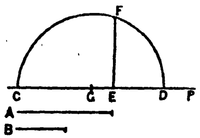


Let A, B, and C be the three given straight lines.

Draw two lines DE, DF, forming any angle; make DG equal to A; DH equal to B; DI equal to C; join G and H, and through I draw IK parallel to GH, cutting DF in K; then DK is the required fourth proportional; that is,

$$A : B = C : DK. \quad \text{(Eucl. VI. 2.)}$$

92. Problem XXIV.—To find a mean proportional between two given straight lines.

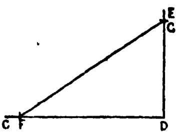


Let A and B be the given straight lines.

Draw any line CP, and lay off on it CE equal to A, and ED to B; on CD describe a semicircle CFD (70); from E draw EF perpendicular to CD, and EF is the mean proportional; that is,

$$A : EF = EF : B. \quad \text{(Eucl. II. 14, and VI. 17.)}$$

93. Cor.—If A and B be two adjacent sides of a rectangle, the line EF is the side of a square equal in area to it.



94. Problem XXV.—To find a square that shall be equal to the sum of two given squares.

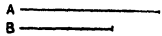
Let A and B (fig. in 95) be the sides of the two given squares.

Draw any line CD and DE perpendicular to it; make DF = A,

and DG = B; join F, G, and FG is the side of the required square; for

$$FG^2 = FD^2 + DG^2 = A^2 + B^2. \quad \text{(Eucl. I. 47.)}$$

95. **Problem XXVI.—To find a square that shall be equal to the difference between two given squares.**



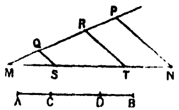
Let A and B be the sides of the two given squares.

Draw any line CP (fig. Prob. XXIV.); make CG and GD each = A, and GE = B; from centre G, with radius GD, describe the circle CFD, and draw FE perpendicular to CE, and FE is the side of the required square; for

$$EF^2 = GF^2 - GE^2 = A^2 - B^2. \quad \text{(Eucl. I. 47, Cor.)}$$

96. **Problem XXVII.—To divide a given straight line similarly to a given divided straight line.**

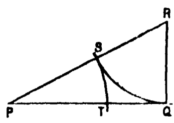
Let AB be the given divided line, C and D being its points of section; and MN the given line to be divided.



Draw through M a line MP at any inclination to MN, and equal to AB, and make its segments equal respectively to those of AB—namely, MQ to AC, QR to CD, and RP to DB. Join P and N, and draw through R and Q the lines RT, QS, each parallel to PN; and MN is divided in S and T similarly to AB; that is,

$$MS : ST = AC : CD, \text{ and } ST : TN = CD : DB. \quad \text{(Eucl. VI. 10.)}$$

97. **Problem XXVIII.—To cut a given straight line in medial section.**



Let PQ be the given line.

Erect a perpendicular QR equal to the half of PQ; join PR; from R as a centre, with the radius RQ, describe an arc cutting PR in S; from P as a centre, with the radius PS, describe an arc cutting PQ in T; then PQ is cut medially in T; that is,

$$PQ : PT = PT : TQ.$$

For $$PR^2 = PQ^2 + QR^2,$$

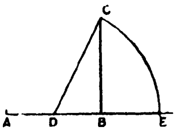
or $$PS^2 + SR^2 + 2RS \cdot PS = PQ \cdot QT + PQ \cdot PT + QR^2,$$

and $$RS^2 = QR^2, \text{ also } 2RS \cdot PS = PQ \cdot PT;$$

hence $$PQ \cdot QT = PS^2 = PT^2,$$

or $$PQ : PT = PT : QT.$$

98. **Problem XXIX.—To produce a line, so that the produced line may be cut medially at the extremity of the given line.**



Let AB be the given line.

Bisect AB in D; draw BC perpendicular to AB, and equal to it; from centre D, with radius DC, cut AB produced in E; and AE is cut medially in B; that is,

$$AE : AB = AB : BE.$$

For $DB^2 + BC^2 = DC^2 = DE^2 = DB^2 + BE^2 + 2DB \cdot BE$, and taking away DB^2 from both,

$$BC^2 \text{ or } AB^2 = BE^2 + AB \cdot BE = AE \cdot BE;$$

or

$$AE : AB = AB : BE.$$

99. **Problem XXX.—To describe an isosceles triangle having each of the angles at the base double of the third angle.**

CASE 1.—When one of the sides of the triangle is given.

Let AB be the given side.

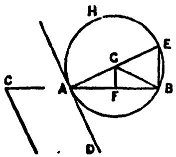
Cut AB medially in C (97), so that AC may be the greater segment; then construct an isosceles triangle on AC as a base, and having each of its sides equal to AB. (74.)

CASE 2.—When the base is given.

Let AB be the given base.

Produce AB to C, so that AC may be cut medially in B (98); then construct an isosceles triangle on AB as a base, and having each of its sides equal to AC. (Eucl. IV. 10.)

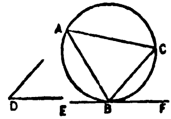
100. **Problem XXXI.—On a given straight line, to describe a segment of a circle containing an angle equal to a given angle.**



Let AB be the given line, and C the given angle.

Draw AD, making angle BAD equal to C; draw AE perpendicular to AD, and GF bisecting AB perpendicularly; from centre G, with radius GA, describe the circular segment AHB, and it is the segment required; for any angle in it, as AEB, is equal to C. (Eucl. III. 33.)

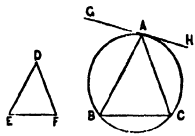
101. **Problem XXXII.—From a given circle, to cut off a segment that shall contain an angle equal to a given angle.**



Let ABC be the given circle, and D the given angle.

At any point B in the circumference, draw a tangent EBF; draw a chord BC, making angle CBF equal to D; and BAC is the required segment; for any angle in it, as BAC, is equal to D. (Eucl. III. 34.)

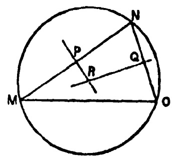
102. **Problem XXXIII.—In a given circle, to inscribe a triangle equiangular to a given triangle.**



Let ABC be the given circle, and DEF the given triangle.

Draw a tangent GAH to the circle at the point A; draw the chord AC, making the angle HAC equal to E, and the chord AB, making angle GAB equal F; join BC, and ABC is the required triangle similar to DEF, having angle B equal to E, angle C to F, and BAC to D. (Eucl. IV. 2.)

103. **Problem XXXIV.—To describe a circle about a given triangle.**

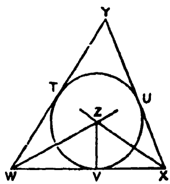


Let MON be the given triangle.

Bisect the side MN by the perpendicular PR; bisect NO similarly by the perpendicular QR; from R, the point of intersection, as a centre, with any of the distances RM, RN, or RO, describe the circle MNO, which is the required circumscribing circle. (Eucl. IV. 5.)

Compare Problem XI.

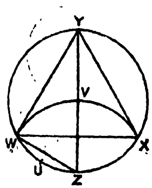
104. **Problem XXXV.—To inscribe a circle in a given triangle.**



Let WXY be the given triangle.

Bisect the angle XWY by the line WZ, and the angle WXY by the line XZ (82); from the intersection Z draw ZV perpendicular to WX, with VZ as a radius and Z as a centre, describe the circle VTU, and it is the required inscribed circle. (Eucl. IV. 4.)

105. Problem XXXVI.—To inscribe an equilateral triangle in a given circle.

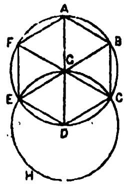


Let WXY be the given circle, and V its centre.

Draw a diameter ZY, and from Z as a centre, and the radius ZV, cut the circumference in W and X; draw WX, WY, and XY, and WXY is the equilateral triangle.

For the arcs WZ, ZX are each one-sixth of the circumference. (Eucl. IV. 15, Cor.)

106. Problem XXXVII.—In a given circle, to inscribe a regular hexagon.



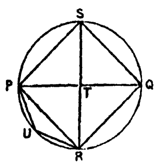
Let WXY be the given circle (fig. Prob. XXXVI.) With the radius of the given circle, and any point Z in the circumference as a centre, cut the circumference in W; draw WZ, and it is a side of the regular hexagon, which may be laid off six times on the circumference of the circle, and every two successive points of section being joined, the resulting figure will be the regular hexagon.

In this manner, the regular hexagon in the adjoining figure is described. (Eucl. IV. 15.)

107. Problem XXXVIII.—In a given circle, to inscribe a regular dodecagon.

Let WXY be the given circle (fig. Art. 105).

Let WZ be a side of the inscribed regular hexagon; bisect the arc WZ in U, and the distance WU being laid off twelve times on the circumference, and every two successive points of section being joined, the resulting figure is the regular dodecagon.



108. Problem XXXIX.—To inscribe a square in a given circle.

Let PRQS be the given circle.

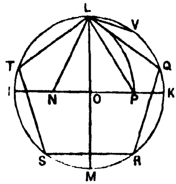
Draw two diameters PQ, SR perpendicular to each other; and join their extremities by PS, SQ, QR, and RP; and PSQR is the required square. (Eucl. IV. 6.)

109. Problem XL.—To inscribe a regular octagon in a given circle.

Let the given circle be PSQ (fig. Art. 108).

Find a side PR of the inscribed square, and bisect the arc PUR in U (83); join R and U, and RU may be laid off eight times on the circumference, and the adjacent points of section being joined, the regular octagon will be formed.

110. Problem XLI.—To inscribe a regular pentagon in a given circle.



Let SLR be the given circle.

Draw two perpendicular diameters IK, LM; bisect the radius OI in N; from N as a centre, with NL as a radius, cut OK in P; with radius LP, and centre L, cut the circumference in Q; join LQ, and other four chords equal to it being drawn in succession in the circle, the required polygon will be formed.

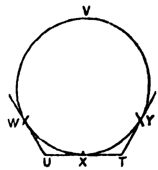
This construction depends on this theorem:—The square of a side of a regular pentagon inscribed in a circle is equal to the sum of the squares of the sides of the inscribed regular hexagon and decagon.

111. Problem XLII.—To inscribe a regular decagon in a given circle.

Let SLR be the given circle (fig. Art. 110).

Find a side LQ of the inscribed regular pentagon; bisect the arc LQ in V, and the chord LV being drawn, it is a side of the regular decagon; and ten chords equal to it being successively placed in the circle, will form the polygon.

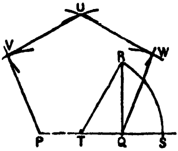
112. Problem XLIII.—To describe a regular polygon about a given circle.



Let WVY be the given circle.

Find the angular points of the corresponding inscribed polygon of the same number of sides by the preceding problems; let W, X, Y be three of these angular points; through these points draw the tangents WU, UT, TY; and UT is a side of the required polygon; in the same manner, the other sides are found, and the circumscribing polygon is thus described.

113. **Problem XLIV.—On a given straight line, to construct a regular pentagon.**

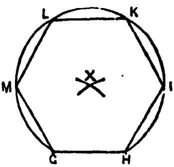


Let PQ be the given line.

Produce PQ to S, so that PS may be cut medially in Q (98); then with a radius equal to PS, from P and Q as centres, describe arcs cutting in U; from P and U as centres, with the radius PQ, describe arcs cutting in V; from Q and U as centres, with the same radius, describe arcs cutting in W; join in order the points Q, W, U, V, and P, and the required pentagon will be formed.

Since PS is cut in medial section in Q, an isosceles triangle, of which PQ is the base, and PS the sides, has each of its angles at the base double of the vertical angle (Eucl. IV. 10); hence if UP and UQ were joined, UPQ would be such a triangle, and hence the figure PQWUV is the required pentagon. (Eucl. IV. 11.)

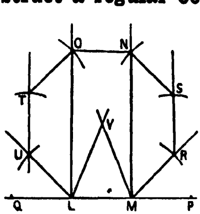
114. **Problem XLV.—On a given straight line, to construct a regular hexagon.**



Let GH be the given line.

From G and H as centres, with the radius GH, describe arcs intersecting in X, and X is the centre of the circumscribing circle; hence from the centre X, with the radius XG, describe a circle, and apply GH six times along the circumference, and GHKL is the required hexagon. This is evident. (Eucl. IV. 15, Cor.)

115. **Problem XLVI.—On a given straight line, to construct a regular octagon.**



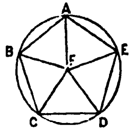
Let LM be the side of the octagon.

METHOD 1.—Draw from L and M two perpendiculars of indefinite length, LO and MN; produce LM in both directions, and bisect the angles QLO, PMN by the lines LU, MR, which are to be made equal to LM; from U and R draw UT, RS parallel to LO and MN, and equal to LM; from T and S as centres, with the radius LM, describe arcs cutting LO and MN in O and N; draw TO, SN, and ON, and the octagon is then constructed.

METHOD 2.—After drawing the lines LU and MR, as in the first method, bisect the angles ULM, LMR, by LV and MV, and V is the centre of the circumscribing circle, and LV its radius. Hence if this circle be described, and the line LM be applied eight times along the circumference, the required octagon will be constructed.

Since the angles at L and M in triangle VLM are together $=135^\circ$, therefore $V=45^\circ=\frac{1}{8}$ of 360°, or four right angles. Hence LM is the side of an octagon inscribed in a circle, of which VL is the radius.

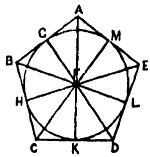
116. **Problem XLVII.—To describe a circle about any given regular polygon.**



Let ABCDE be the regular polygon.

Bisect the angles BCD and CDE by the straight lines CF, DF intersecting in F; from the centre F, with the radius FC, or FD, describe a circle, and it will pass through all the angular points of the polygon, and be described about it. In the same manner, a circle may be described about any other polygon.

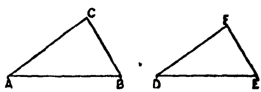
117. **Problem XLVIII.—To inscribe a circle in any given regular polygon.**



Let ABCDE be any regular polygon.

Bisect the angles BCD and CDE by the straight lines CF and DF, intersecting in F; and from F draw FK perpendicular to CD. From the centre F, with radius FK, describe a circle, and it will touch all the sides of the polygon, and therefore be inscribed in the polygon (59).

118. **Problem XLIX.—On a given straight line, to construct a triangle similar to a given triangle.**

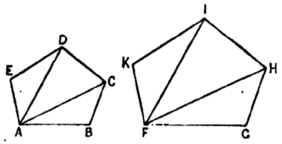


Let DE be the given line, and ABC the given triangle.

At the point D draw DF, making angle D equal to angle A (86); and at E draw EF, making angle E equal to B; then DEF is the triangle required.

For the third angle F is then $=$C. (Eucl. I. 32.)

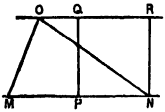
119. **Problem L.—On a given straight line, to construct a figure similar to a given rectilineal figure.**



Let FG be the given line, and ABCDE the given figure.

Divide the given figure into triangles by drawing diagonals AC, AD; on FG describe a triangle FGH similar to ABC (118); on FH describe the triangle FIH similar to ADC; on FI describe a triangle FKI similar to AED: then the whole figure FGHIK is similar to ABCDE. (Eucl. VI. 18.)

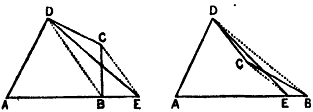
120. **Problem LI.—To construct a rectangle equivalent to a given triangle.**



Let MNO be the given triangle.

Through the vertex O draw OR parallel to the base MN; through P, the middle point of the base, draw PQ perpendicular to it; through N draw NR parallel to PQ, and PQRN is the required rectangle equal to the triangle MNO. (Eucl. I. 42.)

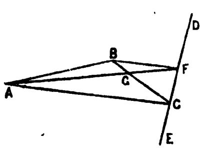
121. **Problem LII.—To construct a triangle equivalent to a given quadrilateral.**



Let ABCD be the given quadrilateral.

Join DB, and through C draw CE parallel to DB; then join DE, and ADE is the required triangle equivalent to ABCD.

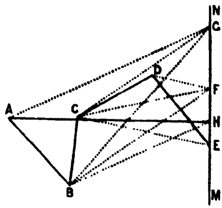
122. **Problem LIII.—To rectify a crooked boundary; that is, to find a straight line that will cut off the same surface on each side of it that a given crooked boundary does.**



CASE 1.—Let ABC be the crooked boundary, and DE a fixed line.

Join A and C, and through B draw BF parallel to AC; join AF, and AF is the required boundary, the triangle ABG, taken from the original space on one side of AF, being equivalent to FGC added on the other; or the triangle ABC is = AFC. (Eucl. I. 37.)

CASE 2.—Let ABCDE be the crooked boundary, and MN the fixed line.



Join EC, and through D draw DF parallel to CE; then join C and F, and the line CF may now be taken instead of the lines CD, DE (*Case* 1). Again, join BF, and through C draw CG parallel to BF; join BG; then BG may now be taken instead of BC, CF. Again, join AG, and through B draw BH parallel to AG; join AH; and AH may now be taken instead of AB and BG. Hence AH is the required line.

The method may easily be extended to a crooked boundary consisting of any number of lines.

Instead of drawing the lines DF, CG, &c., only the points F, G, &c. may be marked.

CONSTRUCTION OF SCALES, AND PROBLEMS TO BE SOLVED BY THEM

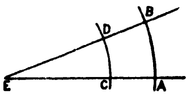
Scales are lines with divisions of various kinds marked upon them, according as they are to be used for measuring lines or angles. In Geometry, they are employed for the construction and measurement of mathematical figures.

The **values of the magnitudes of lines or angles** are numbers representing the number of times that some unit of the same kind is contained in them.

The **unit of measure for lines** is some line of given length—as a foot, a yard, a mile, and so on.

The **unit of measure of an angle** is the 90th part of a right angle. Hence a quadrant of a circle which measures a right angle is supposed to be divided into 90 equal parts called **degrees.** The whole circumference of a circle, therefore, is supposed to be divided into 360 degrees; each degree into 60 equal parts called **minutes**; and each minute into 60 equal parts called **seconds**; and so on. Degrees, minutes, and seconds are respectively denoted by the marks °, ′, ″; thus 23° 27′ 54″ denotes 23 degrees, 27 minutes, and 54 seconds.

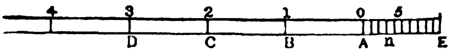
An **angle is measured** by the number of degrees, minutes, &c., in the circular arc intercepted by the sides of the angle, the angular point being the centre of the arc. The numerical measure of the angle in degrees, &c., will evidently be the same, whatever be the length of the radius of the arc.



Thus, if E is the centre of the arcs AB, CD, and if AB contain 30°, CD will also contain 30°, for these arcs are the same parts of the circles to which they belong.

123. Problem LIV.—To construct a scale of equal parts.

Lay off a number of equal divisions, AB, BC, CD, &c., and AE, and divide AE into 10 equal parts (89). When a large division, as



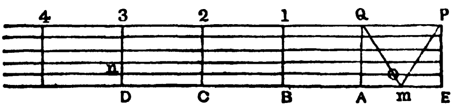
AB, represents 10, each of the small divisions in AE will represent 1. When each of the large divisions represents 100, each of the small divisions in AE represents 10. Hence, on the latter supposition, the distance from C to n is $=230$; and on the former supposition, it is $=23$.

If the large divisions represent units, the small ones on AE represent tenths; that is, each of them is $\frac{1}{10}$ or $\cdot 1$. On this supposition, the distance Cn is $=2\cdot 3$.

124. Problem LV.—To construct a plane diagonal scale.

1. *A diagonal scale for two figures.*

Draw five lines parallel to DE and equidistant, and lay off the equal divisions AE, AB, BC, CD, &c., and make EP, AQ, B1,

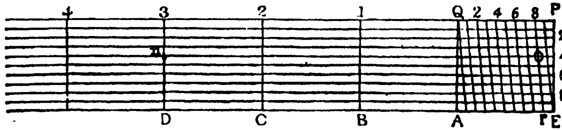


C2, &c., perpendicular to DE. Find m the middle of AE, and draw the lines Qm, mP.

The mode of using this scale is evident from the last. If the large divisions denote tens, then from n to o is evidently $=34$.

2. *A diagonal scale for three figures.*

Draw ten lines parallel to DE, and equidistant. Lay off the equal parts AB, BC, CD, &c., and AE, and draw EP, AQ, B1, C2,...&c., perpendicular to DE. Divide QP and AE into 10 equal parts. Join the 1st, 2nd, 3rd,...divisions on QP with the 2nd, 3rd, 4th,...divisions on AE respectively.



If the divisions on AD each represent 100, each of those on QP will represent 10. Thus from 3 on AD to 8 on QP is = 380; but by moving the points of the compasses down to the fourth line, and extending them from *n* to *o*, the number will be = 384. For the distance of 8 on QP from Q is = 80, and of *r* from A is = 90; and hence that of *o* from the line AQ is = 84.

When the divisions on AD denote tens, those on QP denote units; from *n* to *o* would then = 38·4.

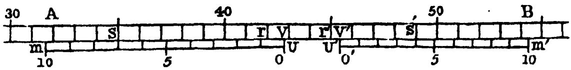
NOTE.—When the numbers representing the lengths of the sides of any figure would give lines of an inconvenient size taken from the scale, they may be all multiplied or all divided by such a number as will adapt the lengths of the lines to the required dimensions of the figure.

125. **Problem LVI.—To construct a vernier scale adapted to a scale of equal parts.**

Let AB be a part of the scale, and *mu* the vernier.

1. *When the divisions of the vernier lie in a direction opposite to those of the principal scale.*

The zero or 0 of the vernier, in the accompanying diagram, lies between 42 and 43 on the scale, and the use of the vernier is to



determine what part *rv* is of a division of the scale. If it be required to estimate *rv* in tenth-parts of a division of the scale, then let 10 divisions on the vernier *mu* be = 11 on the principal scale; then 1 division of the vernier will exceed 1 on the principal scale

by $\frac{1}{10}$ of a division of the latter. Therefore the 7 divisions of the vernier from s to v exceed the 7 on the principal scale from s to r by $\frac{7}{10}$; that is, rv is $\frac{7}{10}$. Hence this rule:—

RULE.—The number of the division on the vernier that coincides with a division on the principal scale shows the numerical value of the part to be measured; that is, the part between the zero of the vernier and the preceding division on the principal scale.

2. *When the divisions on the vernier lie in the same direction as those on the principal scale.*

Let the vernier $m'u'$ have its zero at v', and let it be required to estimate the part $r'v'$, as in the former case, in tenths of a division of the principal scale. Make 10 divisions of the vernier equal to 9 on the principal scale; then each division on the latter will exceed each on the vernier by $\frac{1}{10}$. Therefore the 4 divisions on the principal scale from s' to r' will exceed the 4 on the vernier from s' to v' by $\frac{4}{10}$; that is, $r'v'$ is $\frac{4}{10}$; and the rule thus obtained is the same as in the former case.

The zero of the vernier stands, therefore, opposite to 42·7 in the former case, and to 45·4 in the latter.

Let d, d', denote the divisions on the principal scale and vernier respectively; then in the first case, $10d' = 11d$, or $d' = d + \frac{1}{10}d$; and in the second case, $9d = 10d'$, or $d = d' + \frac{1}{9}d' = d' + \frac{1}{10}d$.

If the divisions 40, 50, &c., are reckoned as 400, 500, &c., the small divisions on the principal scale will denote 10, and the part rv will then be estimated in units. The zeros of the verniers are thus opposite to 427 and 454 on the principal scale.

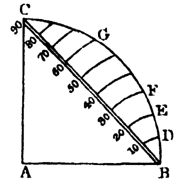
In the same manner, a vernier for a circular arc is constructed. If the arc be divided into half-degrees, and the distance of 29 of these divisions on the vernier be divided into 30 equal parts, the angle can then be read to minutes, when the vernier reads in the same direction as the readings on the arc; and generally if r represent the value of one division on the arc, and n the divisions on the vernier for the length of $(n-1)$ divisions on the arc, then $\frac{r}{n}$ is the value of one division, as read by the vernier.

126. Problem LVII.—To construct a scale of chords.

Let AB be the radius to which the scale is to be adapted.

With centre A and radius AB describe a quadrant BEC. Divide the quadrantal arc BEC into 9 equal parts, BD, DE, &c., which is easily done by first dividing it into 3 equal parts, BF,

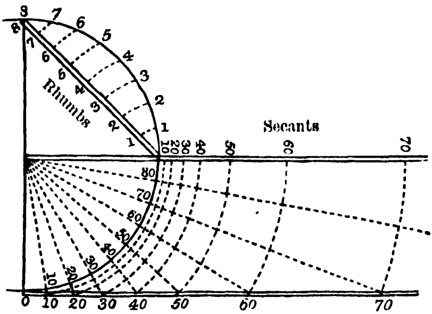
FG, GC (XVII.), and then trisecting each of these parts by trial, as no direct method is known. Draw the chord of the quadrant BC; from B as a centre, and the chord of BD as a radius, describe an arc cutting BC at 10; with the chord of BE as a radius, describe an arc cutting BC in 20; with the chord of BF, describe an arc cutting BC in 30; and in a similar manner, find the divisions 40, 50, 60, 70, 80. Then the arcs BD, BE, BF, &c., being arcs of 10°, 20°, 30°, &c., respectively, the distances from B to 10, 20, 30, &c., are the chords of arcs of 10°, 20°, 30°, &c.; so that BC is a scale of chords for every 10°, from 0° to 90°.



127. **Problem LVIII.—To construct scales of tangents, secants, and rhumbs.**

For definitions of sines, tangents, secants, see TRIGONOMETRY.

Rhumbs are lines drawn to the points of the compass; there are eight points in each quadrant, so that one point contains



11° 15′. The scale of rhumbs consists of the chords of one, two, three, &c., points, or of 11° 15′, 22° 30′, 33° 45′, 45°, &c. (See NAVIGATION, page 535).

The line of tangents is constructed by dividing the quadrant into 9 equal parts, each = 10°, and drawing through the points of division, from the centre, straight lines; then these lines produced will cut a tangent drawn at the extremity of the quadrant in the required points 0, 10, 20, &c.

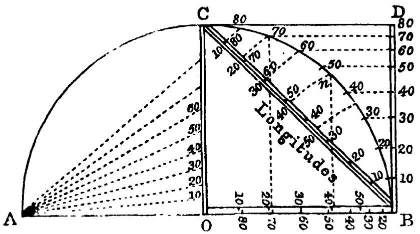
The distances from the centre to the divisions on the scale of tangents, being laid off on a straight line, give the divisions 0, 10, 20, 30, &c., of the scale of secants.

128. Problem LIX.—To construct a scale of sines, and of semitangents, and a line of longitudes.

Divide the quadrant BC into 9 equal parts; through them draw parallels to OB, meeting the perpendicular BD, and it will be a scale of sines.

The **semitangent** of an arc is the tangent of half that arc. Thus the semitangent of 48° is = tangent of 24°.

A line of longitudes is a scale of the lengths of a degree of longitude at different latitudes. For example, the length of a degree of longitude at latitude 60° is = 30 geographical miles, being exactly = the half of a degree of longitude at the equator.



Join A with the divisions of the quadrant BC, and the connecting lines will cut OC in the divisions required for a line of semitangents, which are just the tangents of half their respective arcs; which the divisions on OC evidently are, for the angle at the circumference is half of the angle at the centre. (Eucl. III. 20.)

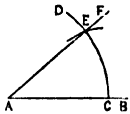
To construct a line of longitudes—first make OB a line of sines, beginning at O, and then it will also be a line of cosines; thus from O to 80 is the cosine of 80°; from O to 70, the cosine of 70°; and so on. Now if OB be divided into 60 equal parts, represented by the numbers marked above it, they will show the number of miles in a degree of longitude corresponding to the latitude denoted by the numbers marked below OB. Thus 30 above OB is opposite to 60° below it, denoting that in latitude 60° the length of a degree of longitude is = 30 miles. If now, instead of the divisions on OB, there be taken on BC the chords

of the corresponding arcs determined by drawing perpendiculars to OB from these divisions, as from 70 to 70, then the higher divisions on BC being taken for latitudes, the divisions below BC show the length of a degree of longitude corresponding to the latitude on the upper side. Thus 40 below BC corresponds to about 48° above; that is, in latitude 48°, the length of a degree of longitude is = 40 miles; and this appears also from the divisions on OB.

The lengths of a degree of longitude in two different latitudes are evidently proportional to the circumference of the parallels of latitude at these places, or proportional to their radii, which are just the cosines of the latitudes, the radius of the earth being radius, and the earth being supposed a sphere.

Note.—All the above scales are laid down on the common Gunter's scale.

129. Problem LX.—At a given point in a given straight line, to make an angle of a given number of degrees and minutes.



Let AB be the given line, and A the given point, and the number of degrees = 38° 30′.

With a radius equal to the chord of 60°, taken from the scale of chords, describe an arc CD from A as centre; with a radius equal to the chord of 38° 30′, taken from the same scale, and from C as centre, cut the arc CD in E; draw AE, and A is the required angle.

When the angle exceeds a right angle (= 90°), lay off on the arc, first the chord of 60°, and then the chord of the remaining number of degrees; or lay off the chords of any two numbers of degrees whose sum is = the given number of degrees.

130. Problem LXI.—To measure a given angle; that is, to find the number of degrees, &c., it contains.

Let BAF (fig. Art. 129) be the given angle.

With the chord of 60° as radius, and A as centre, describe an arc CE; take the chord CE of the arc in the compasses, and apply it to the scale of chords; one point of the compasses being placed at 0°, the other point will extend to the number of degrees and minutes which the angle contains.

When the chord of the intercepted arc exceeds the chord of 90°, lay off the chord of 90° on the arc; take the chord of the remaining arc, and find on the scale of chords the number of degrees belonging

to it, and this number added to 90°, will give the whole number of degrees in the angle.

131. Problem LXII.—To find a third proportional to two given straight lines.

Measure the two given lines by any scale of equal parts; then find a number that is a third proportional to these two numbers; this number, taken on the scale, will be the length of the third proportional. 8,032 .

Generally, let a and b represent in numbers the measures of the two given lines on the scale, and let x be the unknown number which gives on the scale the length of the third proportional; then

$$a : b = b : x ; \therefore x = \frac{b^2}{a}.$$

Hence, divide the square of the number denoting the length of the second line by the number denoting the length of the first, and the quotient is the number that denotes the length of the required line.

Let $a = 128$, and $b = 160$; then

$$x = \frac{b^2}{a} = \frac{160^2}{128} = \frac{25600}{128} = 200, \text{ the third proportional.}$$

132. Problem LXIII.—To find a fourth proportional to three given straight lines.

Measure the three given lines on the scale; then find a number that is a fourth proportional to these three; and this number on the scale will give the line that is the fourth proportional.

Let the measures of the three given lines be denoted by a, b, and c, and the required line by x, then

$$a : b = c : x ; \therefore x = \frac{bc}{a}.$$

Let $a = 225$, $b = 270$, $c = 235$; then $x = \frac{bc}{a} = \frac{270 \times 235}{225} = 282$, the fourth proportional.

133. Problem LXIV.—To find a mean proportional between two given straight lines.

Find on a scale of equal parts the numbers that represent the lines; find their product, and its square root will be the number that expresses the length of the required line.

Generally, let a and b be the numbers representing the two given lines, and x the mean proportional; then

$$a : x = x : b ; \therefore x^2 = ab, \text{ or } x = \sqrt{ab}.$$

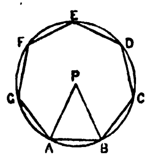
Hence, find the product of the numbers representing the given lines, and the square root of this product will be the number denoting the length of the mean proportional.

Let $a=240$, and $b=364$; then

$$x=\sqrt{ab}=\sqrt{240\times 364}=\sqrt{87360}=295\cdot 5.$$

134. **Problem LXV.—To inscribe any regular polygon in a given circle.**

Let ABE be the given circle, and let the polygon to be inscribed be a regular heptagon.



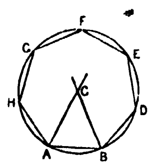
Divide 360° by 7, and the quotient 51° 26′ nearly is the angle at the centre of the circle subtended by the side of the polygon.

Hence, make an angle APB at the centre $=51° 26'$; then the chord AB, laid off 7 times along the circumference, will form the heptagon.

If the number of sides of the polygon be denoted by n, then the angle at the centre is denoted by $\frac{360°}{n}$; hence,

To find the central angle of any regular polygon—Divide 360° by the number of its sides.

135. **Problem LXVI.—On a given straight line, to describe any regular polygon.**



Let AB be the given straight line, and let the polygon to be described upon it be a regular heptagon.

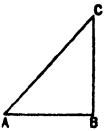
Multiply 90° by 5, and divide by 7; then the half of one of the interior angles BAH is $=\frac{90°\times 5}{7}=\frac{450°}{7}=64° 17'$.

Make the angles BAC, ABC each $=64° 17'$, and C is the centre of the circumscribing circle; and AB being applied 7 times along the circumference, will form the heptagon ABDEFGH.

For any regular polygon, let $n=$the number of its sides, and $a=$the half of one of its interior angles, then $a=\frac{(n-2)90°}{n}$; hence,

To find the half of one of the interior angles of any regular polygon—Multiply 90° by the number of sides diminished by 2, and divide the product by the number of sides, and the quotient is the required angle in degrees.

136. **Problem LXVII.—Given the hypotenuse and a side of a right-angled triangle, to construct it, and to measure the other parts of the triangle.**

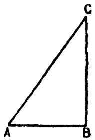


Given the hypotenuse AC = 326, and the side AB = 200.

Draw a line AB = 200; draw BC perpendicular to AB; and from centre A, with radius = 326, cut BC in C; draw AC, and ABC is the required triangle.

By measurement, it will be found that BC = 257, angle A = $52^\circ\ 9'$, and C = $37^\circ\ 51'$.

137. **Problem LXVIII.—Given the two sides about the right angle of a right-angled triangle, to construct it, and to measure the other parts.**

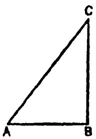


Given the side AB = 162, and BC = 216.

Make AB = 162; draw BC perpendicular to it, and = 216; and draw AC; ABC is the required triangle.

By measurement, it is found that AC = 270, angle A = $53^\circ\ 8'$, and C = $36^\circ\ 52'$.

138. **Problem LXIX.—Given the hypotenuse and one of the acute angles of a right-angled triangle, to construct it, and measure the other parts.**

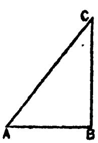


Let the hypotenuse AC = 324, and angle A = $48^\circ\ 17'$.

Draw any line AB; then draw AC, making angle A = $48^\circ\ 17'$; make AC = 324; from C draw CB perpendicular to AB, and ABC is the required triangle.

By measurement, AB = 215, BC = 242, angle C = $41^\circ\ 43'$.

139. **Problem LXX.—Given a side and an acute angle of a right-angled triangle, to construct it, and measure the other parts.**



Given the base AB = 125, and the adjacent angle A = $51^\circ\ 19'$.

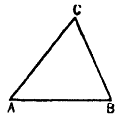
Make AB = 125; draw AC, making angle A = $51^\circ\ 19'$; from B draw BC perpendicular to AB, and ABC is the required triangle.

By measurement, AC = 200, BC = 156, and angle C = $38^\circ\ 41'$.

Note.—When a side, as AB, and the opposite acute angle C

are given, the triangle may be constructed in the same manner. For the three angles of every triangle are equal to two right angles, or 180° (Eucl. I. 32); and as B is one right angle, A and C together are equal to one right angle, or $=90°$; hence when C is given, A is found by subtracting C from 90°. Thus, let there be given $AB=125$, and angle $C=38° 41'$; then angle A is found thus, $A=90°-C=90°-38° 41'=51° 19'$.

140. **Problem LXXI.—Given a side and two angles of a triangle, to construct it.**



Given angle $A=49° 25'$, $B=66° 47'$, and $AB=275$.

As the three angles of every triangle are equal to 180°, therefore

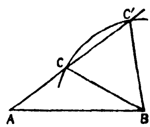
Angle $C=180°-(A+B)=180°-(49° 25'+66° 47')=180°-116° 12'=63° 48'$.

Make $AB=275$, angle $A=49° 25'$, and angle $B=66° 47'$.

By measurement, $AC=282$, $BC=233$, and it was found above that angle $C=63° 48'$.

Note.—When any other two angles of a triangle are given, the third angle may be found in the same manner; that is, by subtracting the sum of the two given angles from 180°; and the triangle can then be constructed as shown above.

141. **Problem LXXII.—Given two sides of a triangle, and an angle opposite to one of them, to construct the triangle.**

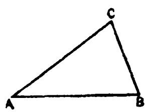


Given $AB=345$, $BC=232$, and angle $A=37° 20'$.

Make $AB=345$, angle $A=37° 20'$, and from B as a centre, with a radius $BC=232$, describe an arc cutting AC in C, and C′; then either of the two triangles ABC, ABC′ is the required triangle. (See Art. 187.)

By measurement, it is found that in triangle ABC, $AC=174$, angle $ABC=27° 4'$, angle $ACB=115° 36'$; also in triangle ABC′, $AC'=375$, angle $ABC'=78° 16'$, angle $AC'B=64° 24'$.

142. **Problem LXXIII.—Given two sides of a triangle, and the contained angle, to construct the triangle.**

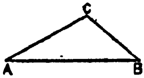


Let $AB=176$, $BC=133$, and angle $B=73°$.

Make side $AB=176$, and then angle $B=73°$, and side $BC=133$.

By measurement, angle $C=64° 9'$, angle $A=42° 49'$, and $AC=187$.

143. **Problem LXXIV.—Given the three sides of a triangle, to construct it, and measure its angles.**



Let AB=345, AC=232, and BC=174.

Make AB=345; from A and B as centres, with the respective radii 232, 174, describe arcs cutting in C; then draw AC, BC, and ABC is the required triangle.

By measuring the three angles of the triangle, it is found that A=27° 2′, B=37° 20′, and C=115° 38′.

COMPUTATION BY LOGARITHMS

144. The **logarithm** of a number is the exponent of the power to which another given number must be raised in order to produce the former number.

Thus $1000=10\times10\times10$; that is, $10^3=1000$, and the exponent 3 is the logarithm of 1000.

So $100=10\times10$; that is, $10^2=100$, and 2 is the L 100, where L denotes logarithm.

Or $10^1=10$, $10^2=100$, $10^3=1000$, $10^4=10,000$,......and 1=L 10, 2=L 100, 3=L 1000, 4=L 10,000,......

145. Since the logarithms of 100 and 1000 are respectively 2 and 3, the logarithm of some intermediate number, as of 856, will be between 2 and 3, or=2+a fraction. So the logarithm of a number between 1000 and 10,000 is between 3 and 4, or=3+a fraction.

146. The integral part of a logarithm is called the **characteristic**, and is one less than the number of integral figures in the number; thus, if the number contains 5 integral figures, the characteristic of its logarithm is 4; if it contains 4 integral figures, the characteristic is 3; if 7, the characteristic is 6; and so on.

147. Since $10^{-2}=1/10^2=1/100=\cdot01$, the log. of ·01 is −2; so $10^{-3}=\cdot001$, $10^{-4}=\cdot0001$, $10^{-5}=\cdot00001$,......and −3=L ·001, −4=L ·0001, −5=L ·00001,......

Since the logarithms of ·01 and ·001 are −2 and −3, the logarithm of an intermediate number, as ·00754, will be between −2 and −3, or= −3+a fraction. So the logarithm of a number between ·001

and ·0001, as ·000754, is between -3 and -4, or $= -4 +$ a fraction; hence the characteristic of the logarithm of a decimal fraction is a negative number, a unit greater than the number of prefixed ciphers.

148. The decimal parts of the logarithms of numbers that consist of the same figures are the same wherever the decimal place is marked, for they differ only in their characteristics.

Thus the logarithms of 43625, 4362·5, 436·25, 43·625, 4·3625, ·43625, ·043625, ·0043625,... ..are the same in the decimal part, but the characteristics are respectively 4, 3, 2, 1, 0, $\bar{1}$, $\bar{2}$, $\bar{3}$; the negative sign being written over the characteristic.

Note.—For additional information in reference to the nature and construction of logarithms, see the Introduction to *Chambers's Mathematical Tables* and *Chambers's Algebra for Schools*, by W. Thomson.

LOGARITHMIC SCALES

149. These scales are constructed by making the distances of the divisions from one extremity equal to the logarithms of the numbers marked on the divisions; and by means of them, several processes of arithmetical and trigonometrical calculation can be easily performed approximately, and the results may be used as a check against errors in the ordinary methods of calculation. A scale of this kind is usually called Gunter's scale. The logarithmic lines of **numbers**, **sines**, and **tangents** are laid down on sectors, and are marked respectively N, S, and T.

150. **Problem I.—To construct a line of logarithmic numbers.**

The line of logarithmic numbers is constructed by making the distances from the extremity of the scale marked 1 equal to the logarithms of the series of natural numbers 1, 2, 3, 4, &c.; that is, to 0, ·301, ·477, ·602, &c.; and if a scale of equal parts be constructed of the same length as the line of logarithmic numbers, and

divided into 1000 equal parts, then the division marked 1 on the logarithmic line will be 0 distant from its extremity; that is, 1 will be at its extremity; the division 2 will be at the distance 301 from the extremity, or from 1; 3 will be at the distance 477; 4 at the distance 602; and so on for the other divisions of this line, that marked 10 being at the distance 1000.

Let $a : b = c : d$, and consequently $\frac{a}{b} = \frac{c}{d}$.

Then $La - Lb = Lc - Ld$; hence

151. **The difference between the logarithms of the first and second terms of a proportion is equal to the difference between the logarithms of the third and fourth.**

Since $1 : 2 = 10 : 20$, and $2 : 3 = 20 : 30$; $\therefore$ $L\,1 \sim L\,2 = L\,10 \sim L\,20$, and $L\,2 \sim L\,3 = L\,20 \sim L\,30$.

Hence the line may be easily extended beyond the division 10, for the extent from 10 to 20 is equal to that from 1 to 2; the extent from 20 to 30 is equal to that from 2 to 3; and so on.

The divisions reckoned above, as 1, 2, 3, 4,... may also be considered as 10, 20, 30, 40,... or as 100, 200, 300, 400;... or, in fact, any numbers proportional to these.

152. **Problem II.—To perform proportion by the line of numbers.**

RULE.—Extend the points of the compasses from the first to the second term, and this extent will reach in the same direction from the third term to the fourth.

EXAMPLE.—Find a fourth proportional to 124, 144, and 186.

The distance from 124 to 144 on the line of numbers will extend from 186 to 216, the term required.

153. **Problem III.—To construct the line of logarithmic sines, cosines, secants, and cosecants.**

On Gunter's scale the logarithm of 100, and the sine of 90°, which is equal radius, are the same length, and therefore the sines are laid down on the scale to radius 100, of which the logarithm is 2; but in the tables of logarithmic sines, the logarithm of radius is 10; hence, if 8 be subtracted from the logarithmic sines, the remainders will be the length of the logarithmic sines on the scale, taken from a scale of the same length divided into 200 equal parts.

Or, the natural sines may be multiplied by 100, and the logarithms of the products taken from the scale will give the same results.

This scale may also be used as a scale of logarithmic cosines, for the cosine of an angle is the sine of its complement. It may also be used as a scale of logarithmic cosecants and secants, for sin $A \times$ cosec $A = R^2$, therefore since on this scale $R = 100$, log. sin $A +$ log. cosec $A = 4$, hence log. cosec $A = 4 -$ log. sin $A =$ the whole length of the scale + the excess of the whole scale above the logarithmic sine A. In the same manner, it can be shown that the logarithmic secant $A =$ the whole scale + its excess above the logarithmic cos A.

Generally, for any radius, since sin $A \times$ cosec $A = R^2$, and cos $A \times$ sec $A = R^2$; log. cosec $A = 2$ log. $R -$ log. sin A, and log. sec $A = 2$ log. $R -$ log. cos A. Also tan $A \times$ cot $A = R^2$; hence log. cot $A = 2$ log. $R -$ log. tan A.

154. **Problem IV.—Given two numbers and an angle, to find another angle such that the two numbers and the sines of the angle shall be proportional.**

The distance between the numbers on the line of numbers will extend in the same direction on the line of sines from the sine of the given angle to the sine of the required angle.

If $a : b =$ sine c : sine d, $\frac{a}{b} = \frac{\text{sine } c}{\text{sine } d}$, and $La - Lb = L \text{ sine } c - L \text{ sine } d$.

EXAMPLES.—1. Given the numbers 121 and 100, and an angle of 90°, to find another angle a such that

$$121 : 100 = \text{sine } 90° : \text{sine } a.$$

The extent from 121 to 100 on the line of numbers reaches from 90° on the line of sines to 55° 44′.

2. Find an angle such that 121 is to 68·5 as the sine of 90° to the cosine of the required angle.

Let a be the required angle, then its complement $b = 90 - a$, and cos $a =$ sine $(90 - a)$ or sine b. Hence

$$121 : 68{\cdot}5 = \text{sine } 90° : \text{sine } b,$$

b is found, as in the preceding example, to be $= 34° \ 29'$; hence

$$a = 90° - b = 90° - 34° \ 29' = 55° \ 31'.$$

3. Find an angle a such that

$$135 : 111 = \text{sine } 79° \ 23' : \text{sine } a.$$

The distance from 135 to 111 extends from 79° 23′ to 53° 55′, which is therefore the value of a.

155. **Problem V.—To construct a scale of logarithmic tangents and cotangents.**

The arcs $(45° - A)$ and $(45° + A)$ are evidently complements of

each other, for their sum is 90°; and (Art. 153) tan (45° - A) × tan (45° + A) = R^2, and therefore log. tan (45° + A) = 2 log. R - log. tan (45° - A); hence, since tan 45° = R, and on the scale log. R = 2, and in the logarithmic tables log. R = 10; if 8 be subtracted from the tabular log. tangents, the remainders will be the lengths of the log. tangents on the scale. These being laid down on the scale, from a scale of the same length divided into 200 equal parts, will give the scale of log. tangents up to 45°. It is also evident from the above, that the log. tangents of angles greater than 45° may be found from the same scale by taking the tangent of 45° + the distance from the tangent of 45° to the tangent of the complement of the angle.

This will also be a scale of logarithmic cotangents, for the cotangent of an angle is the tangent of its complement.

156. **Problem VI.—Given two numbers and an angle, to find another angle such that the lines shall be proportional to the tangents of the angles.**

The distance between the numbers on the Line of Numbers will extend in the proper direction from the given number of degrees on the line of tangents to the required number of degrees.

Note.—When the distance extends beyond 45°, take the excess in the compasses, and apply it back upon the scale from 45°, and it will reach to the complement of the angle sought.

EXAMPLES.—1. Given two numbers 420 and 650, and an angle 45°, to find another angle, such that the numbers shall be proportional to the tangents of the angles.

420 : 650 = tan 45° : tan a, where a = the required angle.

The extent from 420 to 650 on the line of numbers reaches from 45° to 32° 52′ on the line of tangents; but as the second term exceeds the first, the fourth will exceed the third; therefore 32° 52′ is the complement of the angle sought; hence it is 90° - 32° 52′ = 57° 8′.

2. Given two numbers 142 and 42, and an angle of 71° 34′ to find another angle a, such that

142 : 42 = tan 71° 34′ : tan a.

In this example, either 71° 34′ or its complement is taken—namely, 18° 26′, and the distance from 142 to 42 on the line of numbers extends from 18° 26′ beyond 45° on the line of tangents, and the distance *beyond* it being taken with the compasses, will extend from 45° back to 41° 35′, the angle required.

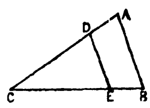
THE LINES OF THE SECTOR

157. Besides the **logarithmic lines** already explained, there are also on each of the legs of the sector a **line of equal parts,** as well as one of **chords,** of **sines,** of **tangents,** and of **secants**; there is also a line of **polygons.**

These lines proceed from the centre of the sector, which is the centre of the joint about which its legs are movable. The **line of lines** is a line of equal parts, the number of large divisions being 10, beginning at the centre, and marked 10 at the extremity; the **line of chords** is a scale of chords, as far as 60°; the **line of sines** is a scale of sines, as far as 90°; and the **line of tangents** is a scale of tangents, as far as 45°. The lengths of these four **lines of lines, chords, sines,** and **tangents** are the same; the **chord** of 60°, the **sine** of 90°, and the **tangent** of 45° being all equal to 10 on the **line of lines,** which is the **radius** of the sector. Another **scale** for **tangents** above 45° begins at one-fourth of the radius 10; that is, at 2·5, and is extended beyond 75°. The **line of secants** also begins at the distance 2·5, which is the radius of the circle to which these secants and the tangents above 45° belong; the beginning of the **line of secants** being marked 0°, for the secant of 0° is = the radius = 2·5. The **line of polygons** is of the same length as that of lines or chords, being marked 4 at the extremity, and 5, 6, 7, &c., towards the centre.

The general principle on which the use of the **sector** is founded is this:—

Let ACB, DCE be two similar isosceles triangles, so that CA = CB, and CD = CE; then CA : AB = CD : DE.



Now, CA and CB being two lines of lines, of chords, of sines, or tangents, it is evident from the above proportion, the distance CA being the radius of the sector, and AB the radius of any other circle, the extremity of the sector being opened to this distance, that—

158. The radius of the sector is to the radius of the circle, as the length of any line (CD) belonging to the circle whose radius is that of the sector, to the length of a corresponding line (DE) of the other circle, whether that line is a chord, a sine, a tangent, or the side of an inscribed regular polygon.

159. The two lines of lines, of chords, sines, and tangents, have the same inclination, so that when the sector is opened till the distance between 10 and 10 on the lines of lines is any given distance, the distances between 60 and 60 on the lines of chords, 90 and 90 on the lines of sines, and 45 and 45 on the lines of tangents will all be the same.

The distance from the centre of the sector on any of the lines proceeding from its centre is called the **lateral distance.**

The distance from any point in one of the lines of the sector, to the corresponding point in the similar line on the other leg, is called the **parallel distance.**

Any two lateral distances are evidently proportional to their corresponding parallel distances, as appears from the preceding proportion.

THE LINE OF LINES

The **line of lines** is one of equal parts. The two following are the most useful problems to be performed by this line.

160. Problem I.—To find a fourth proportional to three given numbers or lines.

RULE.—Make the parallel distance of the first term = the lateral distance of the second, then the parallel distance of the third term will be = the fourth term.

EXAMPLE.—Find a fourth proportional to 72, 48, and 60.

Considering the large divisions as each = 10, take the lateral distance of 48, and make the parallel distance of 72 equal to it; then the parallel distance of 60 applied to the line of lines will give 40, the fourth term required.

If the lengths of the lines represented by the given numbers are too large, any parts of them may be taken, and then the fourth term will be the same part of the number sought.

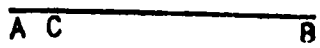
161. Problem II.—To divide a line into any number of equal parts.

RULE.—Find some number on the line of lines which is a multiple of the number of parts into which the line is to be divided, and make the parallel distance of this number = the given line; then

the parallel distance of the corresponding aliquot part of the latter number will be the required aliquot part of the given line.

EXAMPLE.—Let it be required to find the 8th part of the line AB.

Since 80 is divisible by 8, make the parallel distance of 80 = the line AB; then as 10 is the 8th part of 80, the parallel distance of 10 will be = AC, the 8th part of the given line.



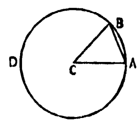
Instead of 80, 48 may be taken, and its parallel distance being made = AB, the parallel distance of 6, the 8th part of 48, will be = AC.

THE LINE OF CHORDS

The **chord of an arc is double the sine of half that arc**; hence the chord of 30° is twice the sine of 15°. If, therefore, the natural sine of 15°, to a radius = 10, which in a table of natural sines is 2·5882, be doubled, the product 5·17 will be the length of the chord of 30° to the same radius. Hence, take 5·17 from the line of lines, and lay it off on the line of chords, and this will determine the division for 30°. The divisions for the other degrees are found in the same manner.

162. **Problem III.—To cut off an arc of any number of degrees from the circumference of a given circle.**

RULE.—Open the sector till the parallel distance of 60° on the line of chords is = the radius of the given circle; then the parallel distance of the required number of degrees will be the chord of the required arc, which can therefore be cut off.



EXAMPLE.—Cut off from the circumference of the circle ABD an arc = 48°.

Make the parallel distance of 60° on the scale of chords = the radius AC, then the parallel distance of 48° will be AB, the chord of the required arc.

THE LINE OF SINES

163. The **line of sines is constructed** by taking the numerical values of the sines from a table of natural sines, supposing the radius = 10, and then taking in the compasses the lateral distances from the line of lines corresponding to these values, and laying them off on the line of sines.

Thus, the natural sine of 40° to a radius = 10 is, by the table, = 6·43; hence, if 6·43 be taken from the line of lines, and laid off on

that of sines, it will determine the division for 40°. In the same manner the other divisions are found.

The sine of any arc of a circle, whose radius is given, is found exactly in the same manner as the chord of an arc was found in the preceding problem; observing that the distance between 90° and 90° is to be made = the radius. The same remark applies to the line of tangents, the distance between 45 and 45 being made = the radius of the given circle.

THE LINE OF TANGENTS

164. The **line of tangents is constructed** in the same manner as that of sines, taking the tangents of the degrees from a table of natural tangents; or since tangent $a = R \sin a/\cos a$, the tangents can be found by dividing the sine by the cosine, and multiplying the quotient by the given radius.

Thus, tangent 30° to a radius R of 10 is = 5·77, and this distance taken on the line of lines, and laid on that of tangents, gives the division of 30°.

For tangents above 45°, the radius is taken = one-fourth of 10 or = 2·5, and the numbers for the natural tangents to R = 10 are divided by 4. Thus, tangent 60° = 17·32, the 4th of which is 4·33; and this distance taken from the line of lines, and laid off on that of tangents above 45, gives the division for 60°.

The tangent of any arc of any circle above 45° is found from the lines of tangents in the same manner as those below 45°; observing that it is the distance between 45° on the two lines that is made = the radius of the given circle.

THE LINE OF SECANTS

The **line of secants is constructed** exactly as that of tangents above 45°; only in this case secant 0° = radius = one-fourth of 10 = 2·5. So for the division of 60° on this line, the secant of 60° by the table is = 20 to radius 10, and one-fourth of this is 5; and the distance 5 taken from the line of lines, and applied to that of secants, gives the division of 60°; and in a similar manner the other divisions are found.

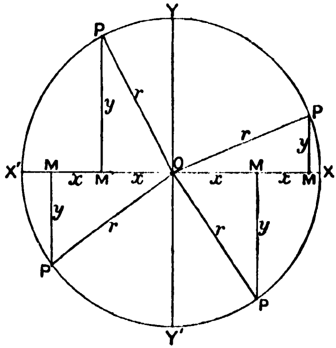
This line is used for finding the secant of any arc of a given circle, in the same manner as those for chords, sines, and tangents; observing that the distance from 0° to 0° on the lines of secants is to be made = the radius of the given circle.

PLANE TRIGONOMETRY

DEFINITIONS

165. The **object of Plane Trigonometry** was originally the calculation of the sides and angles of plane triangles. It now investigates the general relations that subsist between any angles and their trigonometrical functions.

166. In trigonometry, for the purposes of calculation, the circumference of a circle, and also all the angles round a point, are each divided into 360 equal parts called **degrees**; each degree is subdivided into 60 equal parts called **minutes**; and each minute into 60 equal parts called **seconds**: degrees, minutes, and seconds are thus indicated—5° 17′ 28″; which is read—five degrees, seventeen minutes, and twenty-eight seconds.



167. The **complement of an angle** is its difference from a right angle.

168. The **supplement of an angle** is its difference from two right angles.

169. Let XX′ and YY′, which are perpendicular to each other, be two diameters of the circle whose centre is O; the quadrants XOY, YOX′, X′OY′, Y′OX are called respectively the 1st, 2nd, 3rd, 4th quadrants.

170. If OP be supposed to start at OX and to revolve counter-clockwise round O, it will generate with OX angles of all sizes, the angles increasing the more OP revolves. Suppose OP to have generated the four angles XOP, one in each quadrant. From P draw PM perpendicular to XX′; then the lines MP, OM, OP are called respectively the **ordinate**, the **abscissa**, the **radius vector** of the point P.

171. The trigonometrical functions, or trigonometrical ratios, of the angle XOP are called **sine, cosine, tangent, cotangent, secant, cosecant** (abbreviated into sin, cos, tan, cot, sec, cosec), and are defined as follows:

$$\sin XOP = \frac{\text{ordinate}}{\text{radius}} = \frac{MP}{OP} = \frac{y}{r} \qquad \cos XOP = \frac{\text{abscissa}}{\text{radius}} = \frac{OM}{OP} = \frac{x}{r}$$

$$\tan XOP = \frac{\text{ordinate}}{\text{abscissa}} = \frac{MP}{OM} = \frac{y}{x} \qquad \cot XOP = \frac{\text{abscissa}}{\text{ordinate}} = \frac{OM}{MP} = \frac{x}{y}$$

$$\sec XOP = \frac{\text{radius}}{\text{abscissa}} = \frac{OP}{OM} = \frac{r}{x} \qquad \operatorname{cosec} XOP = \frac{\text{radius}}{\text{ordinate}} = \frac{OP}{MP} = \frac{r}{y}$$

172. From these definitions the following formulæ are derived. For shortness, let $\angle XOP$ be denoted by A.

$$\sin A = \frac{1}{\operatorname{cosec} A} \qquad \operatorname{cosec} A = \frac{1}{\sin A} \qquad \sin A \operatorname{cosec} A = 1$$

$$\cos A = \frac{1}{\sec A} \qquad \sec A = \frac{1}{\cos A} \qquad \cos A \sec A = 1$$

$$\tan A = \frac{1}{\cot A} \qquad \cot A = \frac{1}{\tan A} \qquad \tan A \cot A = 1$$

173. Other useful formulæ are also obtained immediately from the definitions.

Thus $\dfrac{\sin A}{\cos A} = \dfrac{y}{r} : \dfrac{x}{r} = \dfrac{y}{x} = \tan A,$ $\qquad \dfrac{\sec A}{\operatorname{cosec} A} = \dfrac{r}{x} : \dfrac{r}{y} = \dfrac{y}{x} = \tan A;$

and consequently $\dfrac{\cos A}{\sin A} = \cot A,$ $\qquad \dfrac{\operatorname{cosec} A}{\sec A} = \cot A.$

174. Again, from any one of the four right-angled triangles OMP we obtain by Pythagoras's theorem (Eucl. I. 47),

$$y^2 + x^2 = r^2.$$

Divide both sides of this equality successively by r^2, x^2, y^2.

There are obtained:

(1) $\dfrac{y^2}{r^2} + \dfrac{x^2}{r^2} = 1;$ that is, $\sin^2 A + \cos^2 A = 1.$

(2) $\dfrac{y^2}{x^2} + 1 = \dfrac{r^2}{x^2};$ that is, $\tan^2 A + 1 = \sec^2 A.$

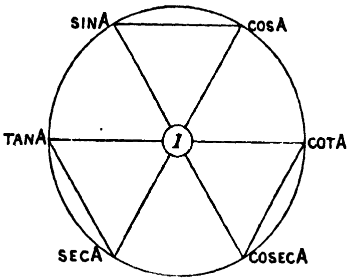
(3) $1 + \dfrac{x^2}{y^2} = \dfrac{r^2}{y^2};$ that is, $1 + \cot^2 A = \operatorname{cosec}^2 A.$

175. It will be observed that $\sin^2 A$, $\cos^2 A$, &c., are written for $(\sin A)^2$, $(\cos A)^2$, &c.

176. The relations just found may be put in other forms. For example,

$\sin A = \sqrt{1-\cos^2 A}$, $\cos A = \sqrt{1-\sin^2 A}$, $\tan A = \sqrt{\sec^2 A - 1}$,
$\cot A = \sqrt{\operatorname{cosec}^2 A - 1}$, $\sec A = \sqrt{1+\tan^2 A}$, $\operatorname{cosec} A = \sqrt{1+\cot^2 A}$.

177. All the formulæ which occur in Articles 172, 173, 174, and many other forms of them, may be obtained from the inspection of a figure which is not difficult to construct, and which may be called Alison's Diagram.



Take any three consecutive values, either diametrically or circumferentially; then the middle one is equal to the product of the other two. Thus, of the three values diametrically, sin A, 1, cosec A,

$$\sin A \operatorname{cosec} A = 1.$$

Of the three values circumferentially, sin A, tan A, sec A,

$$\sin A \sec A = \tan A.$$

In each triangle the square of the value written at the vertex turned downwards is equal to the sum of the squares of the values written at the other two vertices. Thus, $1^2 = \sin^2 A + \cos^2 A$.

178. The following convention regarding the signs to be attributed to the four sets of three lines MP, OM, OP is observed by all mathematicians.

Of the four MP lines, those situated above XX′ are considered positive, those situated below XX′ negative; of the four OM lines, those situated to the right of YY′ are considered positive, those to the left of YY′ negative; OP in any of its positions is always considered positive.

179. To find the signs of the trigonometrical functions in the four quadrants.

Take first sin XOP or sin A and find the sequence of signs for it, according as the angle is in the 1st, 2nd, 3rd, 4th quadrant.

$$\text{Sin } A = \frac{MP}{OP} \text{ in all cases.}$$

Now, $\frac{MP}{OP} = \frac{\text{positive}}{\text{positive}}, \frac{\text{positive}}{\text{positive}}, \frac{\text{negative}}{\text{positive}}, \frac{\text{negative}}{\text{positive}}$ according as $\angle A$
is in quadrant 1 2 3 4

Hence the sequence of signs for sin A is, + + − −.

Again, $\cos A = \dfrac{OM}{OP}$ in all cases.

Now, $\dfrac{OM}{OP} = \dfrac{\text{positive}}{\text{positive}}, \dfrac{\text{negative}}{\text{positive}}, \dfrac{\text{negative}}{\text{positive}}, \dfrac{\text{positive}}{\text{positive}}$ according as $\angle A$ is in quadrant 1 2 3 4

Hence the sequence of signs for cos A is, $+ \ - \ - \ +$.

Since $\tan A = \sin A/\cos A$, the sequence of signs for tan A may be obtained from the sequences of signs for sin A and cos A.

It is $\dfrac{+}{+} \ \dfrac{+}{-} \ \dfrac{-}{-} \ \dfrac{-}{+}$; that is, $+ \ - \ + \ -$.

The sequences of signs for cosec A, sec A, cot A are the same as those for sin A, cos A, tan A.

180. To trace the variation in magnitude of the various functions.

$\sin A = MP/OP$ in all cases, and as OP does not vary in length, the variation of sin A will depend on MP.

Now, when $\angle A$ is very small, MP is very small; therefore sin A is very small. Also, we can make sin A as small as we please (that is, less than any assignable magnitude) by diminishing $\angle A$. This statement is usually expressed by saying $\sin 0° = 0$.

If $\angle A$ from being very small increases up to 90°, MP increases till it equals OP; hence sin A increases till $\sin 90° = 1$.

If $\angle A$ increases beyond 90°, MP begins to diminish, till when $\angle A$ is 180° MP has vanished. Hence $\sin 180° = 0$.

In the 3rd quadrant, as $\angle A$ increases MP increases in magnitude, till when $\angle A$ is 270° MP equals OP; hence $\sin 270° = -1$.

In the 4th quadrant, as $\angle A$ increases MP diminishes in magnitude, till when $\angle A$ is 360° MP disappears; hence $\sin 360° = 0$.

A similar discussion may be made of the variations of cos A which are dependent on the variations of OM.

Because $\tan A = MP/OM$, the variations of tan A will not be so easy to trace, since they depend on the simultaneous variations of MP and OM. If $\angle A$ is very small, MP is very small, and OM is nearly equal to OP; therefore tan A is very small. Also, we can make tan A as small as we please by diminishing $\angle A$. This statement is usually expressed by saying $\tan 0° = 0$.

If $\angle A$ from being very small increases up to 90°, MP increases till it equals OP, and OM shrinks down to 0. Hence, as the

numerator of tan A increases and the denominator diminishes, tan A itself increases rapidly. Also, we can make tan A as great as we please (that is, greater than any assignable magnitude) by making $\angle A$ as near as we please to 90°. This statement is usually expressed by saying tan $90° = \infty$ (infinity).

As $\angle A$ increases from 90° to 180°, MP diminishes and OM increases; that is to say, tan A diminishes in magnitude. When $\angle A$ is 180° MP has vanished, and OM has become equal to OP; hence tan $180° = 0$.

In the 3rd quadrant, as $\angle A$ increases MP increases, and OM diminishes; hence tan A increases and tan $270° = \infty$.

In the 4th quadrant, as $\angle A$ increases MP diminishes, and OM increases; hence tan A diminishes and tan $360° = 0$.

The variations in magnitude of the functions cosec A, sec A, cot A may be investigated directly from the figure, or indirectly from the variations of their reciprocals sin A, cos A, tan A.

181. The two following Tables give the variations of all the functions, first in sign and second in magnitude.

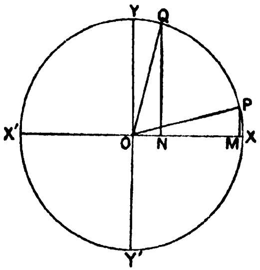
SIGN

A being between	0° and 90°	90° and 180°	180° and 270°	270° and 360°
	are	are	are	are
Sin A, cosec A	+	+	−	−
Cos A, sec A	+	−	−	+
Tan A, cot A	+	−	+	−

MAGNITUDE

A being between	0° and 90°	90° and 180°	180° and 270°	270° and 360°
	Varies from	Varies from	Varies from	Varies from
Sin A	0 to 1	1 to 0	0 to −1	−1 to 0
Cos A	1 " 0	0 " −1	−1 " 0	0 " 1
Tan A	0 " ∞	∞ " 0	0 " ∞	∞ " 0
Cot A	∞ " 0	0 " ∞	∞ " 0	0 " ∞
Sec A	1 " ∞	∞ " −1	−1 " ∞	∞ " 1
Cosec A	∞ " 1	1 " ∞	∞ " −1	−1 " ∞

182. To find the relations between the functions of an angle and the functions of its complement.



Let $\angle XOP$ be denoted by A; then if $\angle YOQ$ be equal in magnitude to $\angle XOP$, the complement of A (namely, $90° - A$) will be $\angle XOQ$.

Draw the ordinates PM, QN.

Then the triangles ONQ, OMP are equal in all respects (Eucl. I. 26);

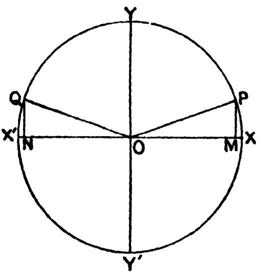
therefore $NQ = OM$, $ON = MP$.

Hence
$$\sin(90° - A) = \frac{NQ}{OQ} = \frac{OM}{OP} = \cos A,$$
$$\cos(90° - A) = \frac{ON}{OQ} = \frac{MP}{OP} = \sin A,$$
as far as magnitude is concerned.

But since (in the figure) A and $90° - A$ are both in the 1st quadrant, therefore their sines and cosines are all positive, and consequently
$$\sin(90° - A) = \cos A, \qquad \cos(90° - A) = \sin A.$$

The values of the other functions of $90° - A$ may be deduced from those just found. For example,
$$\tan(90° - A) = \frac{\sin(90° - A)}{\cos(90° - A)} = \frac{\cos A}{\sin A} = \cot A;$$
and so on.

183. To find the relations between the functions of an angle and the functions of its supplement.



Let $\angle XOP$ be denoted by A; then if $\angle X'OQ$ be equal in magnitude to $\angle XOP$, the supplement of A (namely, $180° - A$) will be $\angle XOQ$.

Draw the ordinates PM, QN.

Then the triangles ONQ, OMP are equal in all respects (Eucl. I. 26);

therefore $NQ = MP$, $ON = OM$.

Hence
$$\sin(180° - A) = \frac{NQ}{OQ} = \frac{MP}{OP} = \sin A,$$

$$\cos(180° - A) = \frac{ON}{OQ} = \frac{OM}{OP} = \cos A,$$

as far as magnitude is concerned.

But since (in the figure) A and 180° - A are respectively in the 1st and 2nd quadrants, sin A and sin (180° - A) will both be positive, while cos A will be positive and cos (180° - A) negative.

Consequently, $\sin(180° - A) = \sin A$; $\cos(180° - A) = -\cos A$.

The values of the other functions of 180° - A may be deduced from those just found. For example,

$$\sec(180° - A) = \frac{1}{\cos(180° - A)} = \frac{1}{-\cos A} = -\sec A;$$

and so on.

184. When A is associated, either by addition or subtraction, with an *odd* number of right angles (90° - A, 90° + A, A - 90°, 270° - A, 270° + A, A - 270°, &c.), the sin, tan, sec of such angle is equal to the cos, cot, cosec of A, and the cos, cot, cosec of such angle is equal to the sin, tan, sec of A.

Thus, as far as magnitude is concerned,

$$\sin(90° - A) = \cos A, \ \sin(90° + A) = \cos A, \ \sin(A - 90°) = \cos A;$$

and so on.

The proper signs to be affixed may be determined from the table of sequence for sines. Thus, if A is an angle in the 1st quadrant, say 20°, 90° - A is in the 1st quadrant, 90° + A in the 2nd, A - 90° in the 4th; consequently, the signs of the sines of these angles are + + -, and

$$\sin(90° - A) = \cos A, \ \sin(90° + A) = \cos A, \ \sin(A - 90°) = -\cos A.$$

185. When A is associated with an *even* number of right angles (180° - A, 180° + A, 360° - A, 360° + A, &c.), the sin, tan, sec of such angle is equal to the sin, tan, sec of A; and similarly for the cos, cot, cosec of such angle.

Thus, as far as magnitude is concerned,

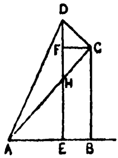
$$\cos(180° - A) = \cos A, \ \cos(180° + A) = \cos A, \ \cos(360° - A) = \cos A.$$

The proper signs to be affixed may be determined from the table of sequence for cosines.

Thus, if A is an angle in the 1st quadrant, say 20°, 180° - A is in the 2nd quadrant, 180° + A in the 3rd, 360° - A in the 4th; consequently, the signs of the cosines of these angles are - - +, and

$$\cos(180° - A) = -\cos A, \ \cos(180° + A) = -\cos A,$$
$$\cos(360° - A) = \cos A.$$

186. To find the sine of the sum of two angles, having given the sine and cosine of each of the angles.



Let $\angle$ BAC be denoted by A, and $\angle$ CAD by B; then $\angle$ BAD will be denoted by (A + B).

From any point D in AD draw DE perpendicular to AB, and DC perpendicular to AC; also through C draw CB and CF perpendicular to AB and DE; then FB is a rectangle; hence FE = CB, and FC = EB. And since the triangles AHE, DHC, are right-angled at E and C, and have the angles at H vertically opposite, the third angle EAH = CDH or CDF; hence angle CDF = A.

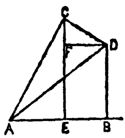
$$\text{Now, } \sin(A+B)=\frac{DE}{AD}=\frac{CB+DF}{AD}=\frac{CB}{AD}+\frac{DF}{AD}$$
$$=\frac{CB}{AC}\cdot\frac{AC}{AD}+\frac{DF}{DC}\cdot\frac{DC}{AD}$$
$$=\sin A\cos B+\cos A\sin B.$$

187. To find the cosine of the sum of two angles, having given the sine and cosine of each of the angles.

From the above diagram,

$$\cos(A+B)=\frac{AE}{AD}=\frac{AB-FC}{AD}=\frac{AB}{AD}-\frac{FC}{AD}$$
$$=\frac{AB}{AC}\cdot\frac{AC}{AD}-\frac{FC}{DC}\cdot\frac{DC}{AD}$$
$$=\cos A\cos B-\sin A\sin B.$$

188. To find the sine of the difference of two angles, having given the sine and cosine of each of the angles.



Let $\angle$ BAC = A, and $\angle$ CAD = B, then $\angle$ BAD = (A − B).

From D, any point in AD, draw DB and DC perpendicular to AB and AC, and from D draw DF perpendicular to CE.

BF is a rectangle, therefore DB = FE, and FD = EB; and since the angles EAC and ACE are together equal to the right angle ACD, from each take the angle ACE, and there remains the angle DCF = CAE = A.

$$\text{Now, } \sin(A-B)=\frac{DB}{AD}=\frac{CE-CF}{AD}=\frac{CE}{AD}-\frac{CF}{AD}$$
$$=\frac{CE}{AC}\cdot\frac{AC}{AD}-\frac{CF}{DC}\cdot\frac{DC}{AD}$$
$$=\sin A\cos B-\cos A\sin B.$$

189. To find the cosine of the difference of two angles, having given the sine and cosine of each of the angles.

From the diagram to Art. 188,

$$\cos(A-B)=\frac{AB}{AD}=\frac{AE+FD}{AD}=\frac{AE}{AD}+\frac{FD}{AD}$$
$$=\frac{AE}{AC}\cdot\frac{AC}{AD}+\frac{FD}{CD}\cdot\frac{CD}{AD}$$
$$=\cos A\cos B+\sin A\sin B.$$

190. Adding and subtracting the values in Arts. 186 and 188, and also those in Arts. 189 and 187, gives

$$\text{Sin }(A+B)+\sin(A-B)=2\sin A\,.\cos B \quad \ldots \quad \ldots \quad \ldots \quad [a].$$
$$\text{Sin }(A+B)-\sin(A-B)=2\cos A\,.\sin B \quad \ldots \quad \ldots \quad \ldots \quad [b].$$
$$\text{Cos }(A-B)+\cos(A+B)=2\cos A\,.\cos B \quad \ldots \quad \ldots \quad \ldots \quad [c].$$
$$\text{Cos }(A-B)-\cos(A+B)=2\sin A\,.\sin B \quad \ldots \quad \ldots \quad \ldots \quad [d].$$

191. If $A+B=S$, and $A-B=D$, then $A=\frac{1}{2}(S+D)$, and $B=\frac{1}{2}(S-D)$; and substituting these values in the last four expressions, they become

$$\text{Sin }S+\sin D=2\sin\tfrac{1}{2}(S+D)\cos\tfrac{1}{2}(S-D) \quad \ldots \quad \ldots \quad \ldots \quad [a].$$
$$\text{Sin }S-\sin D=2\cos\tfrac{1}{2}(S+D)\sin\tfrac{1}{2}(S-D) \quad \ldots \quad \ldots \quad \ldots \quad [b].$$
$$\text{Cos }D+\cos S=2\cos\tfrac{1}{2}(S+D)\cos\tfrac{1}{2}(S-D) \quad \ldots \quad \ldots \quad \ldots \quad [c].$$
$$\text{Cos }D-\cos S=2\sin\tfrac{1}{2}(S+D)\sin\tfrac{1}{2}(S-D) \quad \ldots \quad \ldots \quad \ldots \quad [d].$$

192. These four expressions prove the four following propositions:—

(*a*) The sum of two sines is equal to the product of twice the sine of half the sum of the angles, into the cosine of half their difference.

(*b*) The difference of two sines is equal to the product of twice the cosine of half the sum of the angles, into the sine of half their difference.

(*c*) The sum of two cosines is equal to the product of twice the cosine of half the sum of the angles, into the cosine of half their difference.

(*d*) The difference of two cosines is equal to the product of twice the sine of half the sum of the angles, into the sine of half their difference.

193. But S and D are any two arcs, and may therefore be represented by any two letters; hence, putting A for S, and B for D,

$$\text{Sin }A+\sin B=2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) \quad \ldots \quad \ldots \quad \ldots \quad [a].$$
$$\text{Sin }A-\sin B=2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B) \quad \ldots \quad \ldots \quad \ldots \quad [b].$$
$$\text{Cos }B+\cos A=2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) \quad \ldots \quad \ldots \quad .. \quad [c].$$
$$\text{Cos }B-\cos A=2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B) \quad \ldots \quad .. \quad \ldots \quad [d].$$

194. In Arts. 186–189 let $B=A$, then Arts. 186, 187, 188, and 189 become

$\sin(A+A)=\sin A\cos A+\cos A\sin A$;

$\therefore \sin 2A=2\sin A\cos A$ ... ... [a].

$\cos(A+A)=\cos^2 A-\sin^2 A$;

$\therefore \cos 2A=\cos^2 A-\sin^2 A$ ... ... [b].

$\sin(A-A)=\sin A\cos A-\cos A\sin A$;

$\therefore \sin 0°=0$ ... ... [c].

$\cos(A-A)=\cos^2 A+\sin^2 A=1$ (Art. 174, (1));

$\therefore \cos 0°=1$ ... ... [d].

Also the expressions (Arts. 190, c, d) become

$\cos 0+\cos 2A=2\cos^2 A$; $\therefore 1+\cos 2A=2\cos^2 A$ ... ... [e].

$\cos 0-\cos 2A=2\sin^2 A$; $\therefore 1-\cos 2A=2\sin^2 A$ ... ... [f].

From b, c, and f of this Art. transposed, we obtain

$\cos 2A=\cos^2 A-\sin^2 A=2\cos^2 A-1=1-2\sin^2 A$ ... ... [g].

195. **To find the sine and cosine of 3A; in Arts. 186 and 187, for B put 2A, and reduce by Art. 194 (a and g).**

$$\begin{aligned}\sin 3A=\sin(A+2A)&=\sin A\cos 2A+\cos A\sin 2A\\ &=\sin A(\cos^2 A-\sin^2 A)+\cos A\times 2\sin A\cos A\\ &=\sin A\cos^2 A-\sin^3 A+2\cos^2 A\sin A\\ &=3\sin A\cos^2 A-\sin^3 A=3\sin A(1-\sin^2 A)-\sin^3 A;\end{aligned}$$

$\therefore \sin 3A=3\sin A-4\sin^3 A$ ... ... [a].

$$\begin{aligned}\cos 3A=\cos(A+2A)&=\cos A\cdot\cos 2A-\sin A\cdot\sin 2A\\ &=\cos A(\cos^2 A-\sin^2 A)-2\sin^2 A\cos A\\ &=\cos^3 A-3\cos A\sin^2 A=\cos^3 A-3\cos A(1-\cos^2 A);\end{aligned}$$

$\therefore \cos 3A=4\cos^3 A-3\cos A$ ... ... [b].

196. **To find the tangent of the sum and difference of two angles.**

$\tan(A+B)=\dfrac{\sin(A+B)}{\cos(A+B)}=\dfrac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$, by 186 and 187.

Dividing both numerator and denominator of the last value by $\cos A\cos B$, and remembering that $\dfrac{\sin}{\cos}=\tan$, gives

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B} \quad \text{... ... [}a\text{].}$$

$\tan(A-B)=\dfrac{\sin(A-B)}{\cos(A-B)}=\dfrac{\sin A\cos B-\cos A\sin B}{\cos A\cos B+\sin A\sin B}$; and dividing as in the last, it becomes

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B} \quad \text{... ... [}b\text{].}$$

If in (*a*) B be made equal to A, we obtain

$$\text{Tan } 2\text{A}=\frac{2\tan \text{A}}{1-\tan^2\text{A}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad [c].$$

197. From Art. 193 (*a*, *b*, *c*, *d*) the following six expressions may be easily derived :—

$$\frac{\text{Sin A}+\sin \text{B}}{\text{Sin A}-\sin \text{B}}=\frac{2\sin \frac{1}{2}(\text{A}+\text{B})\cos \frac{1}{2}(\text{A}-\text{B})}{2\cos \frac{1}{2}(\text{A}+\text{B})\sin \frac{1}{2}(\text{A}-\text{B})}=\frac{\tan \frac{1}{2}(\text{A}+\text{B})}{\tan \frac{1}{2}(\text{A}-\text{B})} \quad \ldots \quad [a].$$

$$\frac{\text{Sin A}+\sin \text{B}}{\text{Cos A}+\cos \text{B}}=\frac{2\sin \frac{1}{2}(\text{A}+\text{B})\cos \frac{1}{2}(\text{A}-\text{B})}{2\cos \frac{1}{2}(\text{A}+\text{B})\cos \frac{1}{2}(\text{A}-\text{B})}=\tan \tfrac{1}{2}(\text{A}+\text{B}) \quad \ldots \quad [b].$$

$$\frac{\text{Sin A}+\sin \text{B}}{\text{Cos B}-\cos \text{A}}=\frac{2\sin \frac{1}{2}(\text{A}+\text{B})\cos \frac{1}{2}(\text{A}-\text{B})}{2\sin \frac{1}{2}(\text{A}+\text{B})\sin \frac{1}{2}(\text{A}-\text{B})}=\cot \tfrac{1}{2}(\text{A}-\text{B}) \quad \ldots \quad [c].$$

$$\frac{\text{Sin A}-\sin \text{B}}{\text{Cos A}+\cos \text{B}}=\frac{2\cos \frac{1}{2}(\text{A}+\text{B})\sin \frac{1}{2}(\text{A}-\text{B})}{2\cos \frac{1}{2}(\text{A}+\text{B})\cos \frac{1}{2}(\text{A}-\text{B})}=\tan \tfrac{1}{2}(\text{A}-\text{B}) \quad \ldots \quad [d].$$

$$\frac{\text{Sin A}-\sin \text{B}}{\text{Cos B}-\cos \text{A}}=\frac{2\cos \frac{1}{2}(\text{A}+\text{B})\sin \frac{1}{2}(\text{A}-\text{B})}{2\sin \frac{1}{2}(\text{A}+\text{B})\sin \frac{1}{2}(\text{A}-\text{B})}=\cot \tfrac{1}{2}(\text{A}+\text{B}) \quad \ldots \quad [e].$$

$$\frac{\text{Cos B}+\cos \text{A}}{\text{Cos B}-\cos \text{A}}=\frac{2\cos \frac{1}{2}(\text{A}+\text{B})\cos \frac{1}{2}(\text{A}-\text{B})}{2\sin \frac{1}{2}(\text{A}+\text{B})\sin \frac{1}{2}(\text{A}-\text{B})}=\frac{\cot \frac{1}{2}(\text{A}+\text{B})}{\tan \frac{1}{2}(\text{A}-\text{B})}$$

$$=\frac{\cot \frac{1}{2}(\text{A}-\text{B})}{\tan \frac{1}{2}(\text{A}+\text{B})} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad [f].$$

If in the last three expressions B=0, they become

$$\frac{\sin \text{A}}{1+\cos \text{A}}=\tan \tfrac{1}{2}\text{A}, \text{ for } \sin 0=0, \text{ and } \cos 0=1 \quad \ldots \quad \ldots \quad \ldots \quad [g].$$

$$\frac{\sin \text{A}}{1-\cos \text{A}}=\cot \tfrac{1}{2}\text{A} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad [h].$$

$$\frac{1+\cos \text{A}}{1-\cos \text{A}}=\frac{\cot \frac{1}{2}\text{A}}{\tan \frac{1}{2}\text{A}}=\cot^2 \tfrac{1}{2}\text{A} \quad \ldots \quad \ldots \quad [i].$$

Also by inverting, we have

$$\frac{1-\cos \text{A}}{1+\cos \text{A}}=\frac{\tan \frac{1}{2}\text{A}}{\cot \frac{1}{2}\text{A}}=\tan^2 \tfrac{1}{2}\text{A} \quad \ldots \quad \ldots \quad [k].$$

198. Again, in the expressions of Art. 193 (*a*, *b*, *c*), let A=90°, and we shall have

$$1+\sin \text{B}=2\sin\ (45^\circ+\tfrac{1}{2}\text{B})\cos (45^\circ-\tfrac{1}{2}\text{B})$$
$$=2\sin^2 (45^\circ+\tfrac{1}{2}\text{B}) \quad \ldots \quad \ldots \quad [a].$$

$$1-\sin \text{B}=2\cos\ (45^\circ+\tfrac{1}{2}\text{B})\sin (45^\circ-\tfrac{1}{2}\text{B})$$
$$=2\cos^2 (45^\circ+\tfrac{1}{2}\text{B}) \quad \ldots \quad \ldots \quad [b].$$

$$\cos \text{B}=2\cos\ (45^\circ+\tfrac{1}{2}\text{B})\cos (45^\circ-\tfrac{1}{2}\text{B})$$
$$=2\cos^2 \tfrac{1}{2}\text{B}-1 \quad \ldots \quad \ldots \quad [c].$$

199. If in Art. 194 (a) we put $\frac{1}{2}(A+B)$ for A, we shall have

Sin $(A+B)=2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A+B)$; and dividing this by each of the expressions in Art. 193 (a, b, c, d) successively, there results

$$\frac{\operatorname{Sin}(A+B)}{\operatorname{Sin}A+\sin B}=\frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)} \quad \ldots \quad \ldots \quad [a].$$

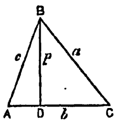
$$\frac{\operatorname{Sin}(A+B)}{\operatorname{Sin}A-\sin B}=\frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)} \quad \ldots \quad \ldots \quad [b].$$

$$\frac{\operatorname{Sin}(A+B)}{\operatorname{Cos}B+\cos A}=\frac{\sin\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)} \quad \ldots \quad \ldots \quad [c].$$

$$\frac{\operatorname{Sin}(A+B)}{\operatorname{Cos}B-\cos A}=\frac{\cos\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)} \quad \ldots \quad \ldots \quad [d].$$

RELATION BETWEEN THE SIDES AND ANGLES OF TRIANGLES

200. To investigate the relation that subsists between the sides and the trigonometrical functions of the angles of a plane triangle,



Let ABC be any plane triangle, having the three angles A, B, and C; calling the sides opposite to these angles respectively a, b, and c. Draw $BD=p$ perpendicular to AC.

From the right-angled triangles ABD and CBD, we have $\sin A=p/c$, and $\sin C=p/a$; and dividing the former by the latter,

$$\left\{\begin{array}{l}\dfrac{\operatorname{Sin}A}{\operatorname{Sin}C}=\dfrac{p}{c}\times\dfrac{a}{p}=\dfrac{a}{c}. \text{ Similarly,}\\[2ex] \dfrac{\operatorname{Sin}A}{\operatorname{Sin}B}=\dfrac{a}{b}, \text{ and}\\[2ex] \dfrac{\operatorname{Sin}B}{\operatorname{Sin}C}=\dfrac{b}{c}; \text{ that is,}\end{array}\right.$$

the sides are proportional to the sines of the opposite angles.

201. Again, since $\dfrac{a}{b}=\dfrac{\sin A}{\sin B}$,

$$\frac{a+b}{a-b}=\frac{\sin A+\sin B}{\sin A-\sin B}=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}.$$

From the above diagram, we have also

$$AD=c\cos A, \text{ and } CD=a\cos C;$$

therefore $$AD+CD=b=c\cos A+a\cos C.$$

202. In the same manner, by drawing perpendiculars on each of the other sides, we obtain,

$$\left.\begin{cases} a=b\cos C+c\cos B, \\ b=a\cos C+c\cos A, \\ c=a\cos B+b\cos A, \end{cases}\right\} \quad \ldots \quad \ldots \quad [a].$$

Multiplying the first of the above by a, the second by b, and the third by c, gives

$$\left.\begin{cases} a^2=ab\cos C+ac\cos B, \\ b^2=ab\cos C+bc\cos A, \\ c^2=ac\cos B+bc\cos A, \end{cases}\right\} \quad \ldots \quad \ldots \quad [b].$$

Adding the second and third, and subtracting the first, gives

$$b^2+c^2-a^2=2bc\cos A\text{ ; hence,}$$

$$\begin{cases} \text{Cos } A=\dfrac{b^2+c^2-a^2}{2bc}\text{; and similarly} \quad \ldots \quad [c], \\ \text{Cos } B=\dfrac{a^2+c^2-b^2}{2ac} \quad \ldots \quad \ldots \quad \ldots \quad [d], \\ \text{Cos } C=\dfrac{a^2+b^2-c^2}{2ab} \quad \ldots \quad \ldots \quad \ldots \quad [e]. \end{cases}$$

203. Since $2bc\cos A=b^2+c^2-a^2$ by transposition,

$$a^2=b^2+c^2-2bc\cos A.$$

Similarly, $b^2=a^2+c^2-2ac\cos B$. ... ... [f],

and $c^2=a^2+b^2-2ab\cos C$.

Whence $1+\cos A=1+\dfrac{b^2+c^2-a^2}{2bc}=\dfrac{2bc+b^2+c^2-a^2}{2bc}$

$$=\frac{(b+c)^2-a^2}{2bc}=\frac{(a+b+c)(b+c-a)}{2bc}.$$

But since $\cos 2A=2\cos^2A-1$, $1+\cos 2A=2\cos^2A$, and hence $1+\cos A=2\cos^2\frac{1}{2}A$. Therefore

$$2\cos^2\tfrac{1}{2}A=\frac{(a+b+c)(b+c-a)}{2bc},$$

or

$$\cos^2\tfrac{1}{2}A=\frac{\frac{1}{2}(a+b+c)\times\frac{1}{2}(b+c-a)}{bc} \quad \ldots \quad [g].$$

204. Put $\frac{1}{2}(a+b+c)=s$, then $\frac{1}{2}(b+c-a)=s-a$, $\frac{1}{2}(a+c-b)=s-b$; and $\frac{1}{2}(a+b-c)=s-c$; and inserting these values in the above, it becomes

$$\cos^2\tfrac{1}{2}A=\frac{s(s-a)}{bc}\text{; similarly, }\cos^2\tfrac{1}{2}B=\frac{s(s-b)}{ac},\ \cos^2\tfrac{1}{2}C=\frac{s(s-c)}{ab}.$$

Extracting the square root of each of the above, we have

$$\left.\begin{aligned} \text{Cos}\ \tfrac{1}{2}A&=\sqrt{\frac{s(s-a)}{bc}},\quad \cos\tfrac{1}{2}B=\sqrt{\frac{s(s-b)}{ac}},\\ \text{and}\qquad \cos\tfrac{1}{2}C&=\sqrt{\frac{s(s-c)}{ab}},\end{aligned}\right\}\quad \dots\ [\alpha].$$

205. Again, since $\cos A=\frac{b^2+c^2-a^2}{2bc}$, subtracting both sides from 1,

$$1-\cos A=1-\frac{b^2+c^2-a^2}{2bc}=\frac{2bc-b^2-c^2+a^2}{2bc}=\frac{a^2-(b-c)^2}{2bc}$$

$$=\frac{(a+b-c)(a+c-b)}{2bc}.$$

But $1-\cos A=2\sin^2\tfrac{1}{2}A$; hence,

$$2\sin^2\tfrac{1}{2}A=\frac{(a+b-c)(a+c-b)}{2bc};$$

therefore $$\sin^2\tfrac{1}{2}A=\frac{\tfrac{1}{2}(a+b-c)\times\tfrac{1}{2}(a+c-b)}{bc}=\frac{(s-b)(s-c)}{bc}.$$

$$\text{Sin}\ \tfrac{1}{2}A=\sqrt{\frac{(s-b)(s-c)}{bc}};\quad \text{similarly,}\ \sin\tfrac{1}{2}B=\sqrt{\frac{(s-a)(s-c)}{ac}},$$

and $$\sin\tfrac{1}{2}C=\sqrt{\frac{(s-a)(s-b)}{ab}}.$$

206. Now, since $\tan\tfrac{1}{2}A=\frac{\sin\tfrac{1}{2}A}{\cos\tfrac{1}{2}A}$, by dividing the value of $\sin\tfrac{1}{2}A$ by that of $\cos\tfrac{1}{2}A$, we obtain

$$\left.\begin{aligned} \tan\tfrac{1}{2}A&=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\sqrt{\frac{1}{s}\,\frac{(s-a)(s-b)(s-c)}{(s-a)^2}}.\\ \text{Similarly,}&\\ \tan\tfrac{1}{2}B&=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}=\sqrt{\frac{1}{s}\,\frac{(s-a)(s-b)(s-c)}{(s-b)^2}},\\ \tan\tfrac{1}{2}C&=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}=\sqrt{\frac{1}{s}\,\frac{(s-a)(s-b)(s-c)}{(s-c)^2}},\end{aligned}\right\}\quad \dots\ [\alpha].$$

Extracting the root of the square factor in the denominator of each of the last values of the tangent given above, we obtain

$$\left.\begin{cases} \operatorname{Tan} \frac{1}{2}A = \dfrac{1}{s-a}\sqrt{\dfrac{1}{s}(s-a)(s-b)(s-c)}, \\ \operatorname{Tan} \frac{1}{2}B = \dfrac{1}{s-b}\sqrt{\dfrac{1}{s}(s-a)(s-b)(s-c)}, \\ \operatorname{Tan} \frac{1}{2}C = \dfrac{1}{s-c}\sqrt{\dfrac{1}{s}(s-a)(s-b)(s-c)}, \end{cases}\right\} \quad \dots \quad \dots \quad [b].$$

From Art. 194 (a) we have $\sin A = 2 \sin \frac{1}{2}A \cos \frac{1}{2}A$, and by 204 ($a$), $\cos \frac{1}{2}A = \sqrt{\dfrac{s(s-a)}{bc}}$; and by 205, $\sin \frac{1}{2}A = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$.

Substituting these values in Art. 194 (a),

$$\sin A = 2\sqrt{\frac{s(s-a)}{bc}} \times \sqrt{\frac{(s-b)(s-c)}{bc}} = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

And from the symmetry of the expression, the sines of the other angles may be written; hence,

$$\left.\begin{cases} \operatorname{Sin} A = \dfrac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}, \\ \operatorname{Sin} B = \dfrac{2}{ac}\sqrt{s(s-a)(s-b)(s-c)}, \\ \operatorname{Sin} C = \dfrac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}, \end{cases}\right\} \quad \dots \quad \dots \quad [c].$$

207. **To find the numerical values of sine, cosine, and tangent of 30°.**

Let $x = \sin 30°$; then $\cos 30° = \sqrt{1-x^2}$, but $\cos 30° = \sin 60° = 2 \sin 30 \,.\, \cos 30°$;

$$\therefore\ 2x\sqrt{1-x^2} = \sqrt{1-x^2}. \quad \text{Divide by } 2\sqrt{1-x^2},$$

$$x = \tfrac{1}{2}\,;\ \therefore\ \sin 30° = \tfrac{1}{2},$$

and $\qquad \cos 30° = \sqrt{1-\sin^2 30°} = \sqrt{1-\frac{1}{4}} = \dfrac{\sqrt{3}}{2}$;

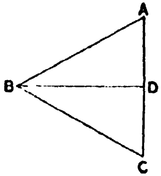
also $\qquad \tan 30° = \dfrac{\sin 30°}{\cos 30°} = \frac{1}{2} \times \dfrac{2}{\sqrt{3}} = \dfrac{1}{\sqrt{3}}$.

COR.—Since $\sin 30° = \cos 60°$, $\cos 60° = \frac{1}{2}$; and similarly, we find

$$\sin 60° = \frac{\sqrt{3}}{2}, \text{ and } \tan 60° = \sqrt{3}.$$

208. Otherwise thus:

Let ABC be an equilateral triangle, and let BD be drawn perpendicular to CA; then BD bisects CA and also angle ABC.



Hence angle $ABD = 30°$.

Denote AD by 1, then AB will be 2 and (Eucl. I. 47) $BD = \sqrt{3}$.

$$\text{Sin } 30° = \frac{AD}{AB} = \frac{1}{2}, \cos 30° = \frac{BD}{AB} = \frac{\sqrt{3}}{2},$$

$$\tan 30° = \frac{AD}{BD} = \frac{1}{\sqrt{3}}, \text{ \&c.}$$

209. To find the numerical values of the sine, cosine, and tangent of 45°.

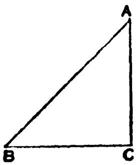
By Art. 174 (1), $\sin^2 45° + \cos^2 45° = 1$, but $\sin 45° = \cos 45°$;

$$\therefore 2 \sin^2 45° = 1, \text{ and hence } \sin 45° = \frac{1}{\sqrt{2}} = \cos 45°.$$

Whence, $$\tan 45° = \frac{\sin 45°}{\cos 45°} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{1} = 1.$$

Otherwise thus:

Let ABC be a right-angled triangle having $AC = BC$; then $\angle B = 45°$.



Denote AC and BC by 1; then (Eucl. I. 47) $AB = \sqrt{2}$.

$$\text{Sin } 45° = \frac{AC}{AB} = \frac{1}{\sqrt{2}}, \cos 45° = \frac{BC}{AB} = \frac{1}{\sqrt{2}},$$

$$\tan 45° = \frac{AC}{BC} = 1.$$

Exercises

1. Prove that $\tan A + \cot A = 2 \operatorname{cosec} 2A$.

2. Prove that $\sec A = 1 + \tan A \cdot \tan \frac{1}{2}A$; and $\operatorname{cosec} 2A = \dfrac{1 + \cot^2 A}{2 \cot A}$.

3. Prove that $\cot^2 A \cdot \cos^2 A = \cot^2 A - \cos^2 A$; and $\operatorname{cosec}^2 A \cdot \sec^2 A = \sec^2 A + \operatorname{cosec}^2 A$.

4. Prove that $\dfrac{\tan A + \tan B}{\cot A + \cot B} = \tan A \cdot \tan B$; and $\dfrac{\cos^2 A - \sin^2 B}{\sin^2 A \cdot \sin^2 B} = \cot^2 A \cdot \cot^2 B - 1$.

5. Prove that $\cos 2A + \cos 2B = 2 \cos (A + B) \cdot \cos (A - B)$.

6. Prove that $\cos (A + B) \cdot \cos (A - B) = \cos^2 A - \sin^2 B$; and $\sin (A + B) \cdot \sin (A - B) = \sin^2 A - \sin^2 B$.

7. If $A+B+C=180^\circ$, prove (1) that $\sin A+\sin B+\sin C=4\cos\frac{1}{2}A\,.\cos\frac{1}{2}B\,.\cos\frac{1}{2}C$; (2) that $\cos A+\cos B+\cos C=4\sin\frac{1}{2}A\,.\sin\frac{1}{2}B\,.\sin\frac{1}{2}C+1$; (3) that $\tan A+\tan B+\tan C=\tan A\,.\tan B\,.\tan C$; (4) that $\cot A+\cot B+\cot C=\cot A\,.\cot B\,.\cot C+\operatorname{cosec} A\,.\operatorname{cosec} B\,.\operatorname{cosec} C$.

8. Determine the value of A in degrees from the following equations: (1) $\sin A=\sin 2A$; (2) $\tan 2A=3\tan A$; (3) $\tan A+3\cot A=4$; (4) $2\sin^2 3A+\sin^2 6A=2$.

9. In any right-angled triangle ABC, in which C is the right angle, c the hypotenuse, a the side opposite the angle A, and b the side opposite the angle B, prove (1) that $\sin(A-B)=\frac{a^2-b^2}{c^2}$; (2) $\cos(A-B)=\frac{2ab}{c^2}$; (3) $\tan(A-B)=\frac{a^2-b^2}{2ab}$; (4) $\sin^2\frac{1}{2}A=\frac{c-b}{2c}$; (5) $\cos^2\frac{1}{2}A=\frac{c+b}{2c}$; and (6) $\tan^2\frac{1}{2}A=\frac{c-b}{c+b}$.

10. If a line CD bisect the angle C of any triangle, and meet the base in D; $\tan ADC=\frac{a+b}{a-b}\tan\frac{1}{2}C$, and $CD=\frac{2ab}{a+b}\cos\frac{1}{2}C$.

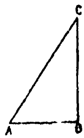
11. Prove (1) that $\tan^2(45^\circ+\frac{1}{2}A)=\frac{1+\sin A}{1-\sin A}$; (2) $\sec(45^\circ+A)\,.\sec(45^\circ-A)=2\sec 2A$; (3) $\tan(30^\circ+A)\,.\tan(30^\circ-A)=\frac{2\cos 2A-1}{2\cos 2A+1}$; and (4) $\sin(60^\circ+A)-\sin(60^\circ-A)=\sin A$.

12. If the sides a, b, of a triangle include an angle of 120°, show that $c^2=a^2+ab+b^2$; and if they include an angle of 60°, $c^2=a^2-ab+b^2$.

SOLUTION OF TRIANGLES

210. First, let ABC be a right-angled triangle.

When the two sides are given, the hypotenuse may be found by taking the sum of the squares of the sides and extracting the square root.



When the hypotenuse and a side are given, the other side may be found by taking the difference of the squares of the hypotenuse and the given side and extracting the square root; or, what comes to the same thing, find the product of the sum and difference of the hypotenuse and the given side, and extract the square root.

211. CASE 1.—Given the hypotenuse and a side of a right-angled triangle, to find the other parts.

EXAMPLE.—Given AC=415, and AB=249, to find the other parts of triangle ABC.

1. To find angle A

$$\text{Cos A}=\frac{\text{AB}}{\text{AC}}=\frac{249}{415}=\cdot 6;$$

therefore, from a table of natural cosines, the value of A= 53° 7′ 48·4″.

Or thus:

$$\sec \text{A}=\frac{\text{AC}}{\text{AB}}=\frac{415}{249};$$

therefore
$$\begin{aligned}\text{L} \sec \text{A} &= \text{L } 415-\text{L } 249+10\\ &= 2\cdot 6180481-2\cdot 3961993+10\\ &= 10\cdot 2218488\\ &= \text{L} \sec 53° \, 7' \, 48\cdot 4''.\end{aligned}$$

2. To find BC

$$\frac{\text{BC}}{\text{AB}}=\tan \text{A}; \text{ therefore BC}=\text{AB} \tan \text{A};$$

therefore
$$\begin{aligned}\text{L BC} &= \text{L AB}+\text{L} \tan \text{A}-10\\ &= \text{L } 249+\text{L} \tan 53° \, 7' \, 48\cdot 4''-10\\ &= 2\cdot 3961993+10\cdot 1249388-10\\ &= 2\cdot 5211381\\ &= \text{L } 232.\end{aligned}$$

3. To find angle C

$$\text{C}=90°-\text{A}=90°-53° \, 7' \, 48\cdot 4''=36° \, 52' \, 11\cdot 6''.$$

But BC may be found, independently of any of the angles, thus,

$$\text{BC}^2=\text{AC}^2-\text{AB}^2=415^2-249^2=172225-62001=110224,$$

and
$$\text{BC}=\sqrt{110224}=332.$$

Or
$$\text{BC}^2=(\text{AC}+\text{AB})(\text{AC}-\text{AB})=(415+249)(415-249)=664\times 166=110224,$$

and
$$\text{BC}=\sqrt{110224}=332.$$

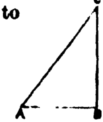
This latter method is well adapted to logarithmic calculation; thus,

L (AC+AB) 664, ...	=	2·8221681
L (AC−AB) 166, ...	=	2·2201081
L BC^2, ...	=	5·0422762
hence (I. T.*), L BC 332, ...	=	2·5211381

* I. T. refers to the Introduction to the *Mathematical Tables*.

212. CASE 2.—Given a side and one of the oblique angles.

EXAMPLE.—In the right-angled triangle ABC, given the hypotenuse AC 324 feet, and angle A 48° 17′, to find the other parts.



1. To find angle C

$$C = 90° - A = 90° - 48° 17'$$
$$= 41° 43'.$$

2. To find BC

$$\frac{BC}{AC} = \sin A \text{ ; therefore } BC = AC \sin A$$
$$= 324 \sin 48° 17' ;$$

therefore
$$L\ BC = L\ 324 + L \sin 48° 17' - 10$$
$$= 2{\cdot}5105450 + 9{\cdot}8729976 - 10$$
$$= 2{\cdot}3835426$$
$$= L\ 241{\cdot}848.$$

3. To find AB

$$\frac{AB}{AC} = \cos A \text{ ; therefore } AB = AC \cos A$$
$$= 324 \cos 48° 17' ;$$

therefore
$$L\ AB = L\ 324 + L \cos 48° 17' - 10$$
$$= 2{\cdot}5105450 + 9{\cdot}8231138 - 10$$
$$= 2{\cdot}3336588$$
$$= L\ 215{\cdot}605.$$

These sides may also be found by natural sines, independently of logarithms; thus,

1. To find BC

$$\frac{BC}{AC} = \sin A \text{; therefore } BC = AC \sin A$$
$$= 324 \sin 48° 17'$$
$$= 324 \times {\cdot}746446$$
$$= 241{\cdot}848.$$

2. To find AB

$$\frac{AB}{AC} = \cos A \text{; therefore } AB = AC \cos A$$
$$= 324 \cos 48° 17'$$
$$= 324 \times {\cdot}6654475$$
$$= 215{\cdot}605.$$

Exercises

1. In a right-angled triangle, the hypotenuse is 1246, and one of the oblique angles 25° 30′; find the other angle and the two sides. . . . . . . =64° 30′, 536·4168, and 1124·621.

2. The hypotenuse is 645, and an oblique angle 39° 10′; find the other sides. . . . . =500·076, and 407·368.

3. In a triangle right-angled at B, given the side AB 125, and angle A 51° 19′, to find the other parts.
C=38° 41′, BC=156·1186, AC=199·9949.

4. In a right-angled triangle ABC, having a right angle at B, the side AB is 180, and angle A 62° 40′, find other parts.
C=27° 20′, AC=392·0147, BC=348·2464.

5. In a right-angled triangle ABC, given the hypotenuse AC 645, and the base AB 500; required the other parts.
BC=407·459, angle A=39° 10′ 38″, and angle C=50° 49′ 22″.

6. Given the base and hypotenuse 288 and 480, to find the other parts.
The perpendicular=384, and the oblique angles=53° 7′ 48″, and 36° 52′ 12″.

7. The two sides about the right angle of a right-angled triangle are 360 and 270; required the hypotenuse and the oblique angles.
=450, 36° 52′ 12″, and 53° 7′ 48″.

8. What are the hypotenuse and oblique angles in a right-angled triangle, of which the two sides are 389 and 467?
=607·79, 30° 47′ 37″, and 50° 12′ 23″.

9. Given the base=530, the perpendicular=670, to find the hypotenuse and the acute angles. =854·284, 51° 39′ 16″, 38° 20′ 44″.

COMPUTATION OF THE SIDES AND ANGLES OF OBLIQUE-ANGLED TRIANGLES

213. Case I.—When two angles and a side opposite to one of them are given.

Rule.—The sides are proportional to the sines of the opposite angles. Hence,

To find a side, begin with an angle—namely, the angle opposite to the given side; thus, the sine of the angle opposite to the given side is to the sine of the angle opposite to the required side as the given side to the required side.

When two angles of a triangle are known, the third is found by subtracting their sum from two right angles.

Let the three angles of any triangle be represented by the letters A, B, C, and the sides opposite to these angles respectively by the letters a, b, c; and let A, B, and a be given, to find the other parts.

1. To find angle C

$$C = 180° - (A + B).$$

2. To find the side b or AC

$$\text{Sin A} : \sin B = a : b.$$

By natural sines, $$b = \frac{a \sin B}{\sin A}.$$

By logarithms, $$Lb = La + L \sin B - L \sin A.$$

By the same means, the side c can be found. The two preceding formulæ can be adopted for finding c, by merely changing b into c, and B into C; thus,

$$\sin A : \sin C = a : c, \text{ and } c = \frac{a \sin C}{\sin A},$$

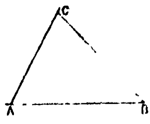
or $$Lc = La + L \sin C - L \sin A;$$

also $$Lc = L \operatorname{cosec} A + L \sin C + La - 20,$$

which is the most convenient formula for calculating this case.

EXAMPLE.—In the triangle ABC, there are given angle A = 63° 48′, angle B = 49° 25′, and BC = 275.

1. To find angle C



$$C = 180° - (A + B) =$$
$$180° - (63° \ 48' + 49° \ 25') =$$
$$180° - 113° \ 13' = 66° \ 47'.$$

2. To find the side AC

$$\text{Sin A} : \sin B = BC : AC = a : b.$$

L cosec A 63° 48′, ...	=	10·0470825
L sin B 49° 25′, ...	=	9·8805052
L BC 275, ...	=	2·4393327
L AC 232·7665, ...	=	2·3669204

3. To find AB

$$\text{Sin A} : \sin C = BC : AB = a : c.$$

L cosec A 63° 48′,	=	10·0470825
L sin C 66° 47′,	=	9·9633253
L BC 275,	=	2·4393327
L AB 281·67,	=	2·4497405

214. CASE 2.—When two sides and an angle opposite to one of them are given.

RULE.—The sides are proportional to the sines of the opposite angles. Hence,

To find an angle, begin with a side—namely, the side opposite to the given angle; thus, the side opposite to the given angle is to the side opposite to the required angle as the sine of the given angle to the sine of the required angle.

When two of the angles are known, the third is found by subtracting their sum from two right angles.

Let a, b, and A be given, to find the other parts.

To find angle B

$$a : b = \sin A : \sin B,$$

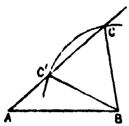
and by natural sines, $\sin B = \dfrac{b \sin A}{a}$.

By logarithms, $L \sin B = L \sin A + Lb - La$,

or $L \sin B = \text{ar. co. } La + Lb + L \sin A - 10$,

which is the most convenient formula for calculating this case.

EXAMPLE.—In the triangle ABC are given the sides AB and BC 345 and 232 feet, and angle A 37° 20′.



In this case, when the side opposite to the given angle is greater than the other given side, only one triangle can be formed; but when less, there can be two constructed.

1. In the triangle ABC.

1. To find angle C

$$BC : AB = \sin A : \sin C,$$

or $$a : c = \sin A : \sin C.$$

Ar. co. L BC 232,	=	7·6345120
L AB 345,	=	2·5378191
L sin A 37° 20′,	=	9·7827958
L sin C 64° 24′ 1″,	=	9·9551269

2. To find angle B

$$B = 180° - (A + C) = 180° - 101° \, 44' \, 1'' = 78° \, 15' \, 59''.$$

3. To find the side AC

Sin A : sin B = BC : AC,

or sin A : sin B = a : b.

L cosec A 37° 20′,	=	10·2172042
L sin B 78° 15′ 59″,	=	9·9908287
L BC 232,	=	2·3654880
L AC 374·559,	=	2·5735209

2. In the triangle ABC′.

The first proportion above gave angle C, but it gives also angle C′ in triangle ABC′, observing that, instead of the angle 64° 24′ 1″, its supplement must be taken; for angle AC′B is the supplement of BC′C, which is equal to C. Hence angle C′ = 180° − 64° 24′ 1″ = 115° 35′ 59″.

Then angle ABC′ = 180 − (A + C′) = 180° − 152° 55′ 59″ = 27° 4′ 1″.

The last proportion will then give AC′, if for angle ABC 78° 15′ 59″ the angle ABC′ 27° 4′ 1″ is substituted. The student will find, by making this substitution, that AC′ = 174·0738.

EXERCISES.

1. In a triangle ABC are given the angles A and C 59° and 52° 15′, and also the side AB 276·5, to find its other parts.

AC = 325·9183, BC = 299·7469, and angle B = 68° 45′.

2. In a triangle ABC, the angles A and B are respectively 54° 20′ and 62° 36′, and the side AB is 245; required the other parts of the triangle. AC = 243·978, BC = 223·26, and angle C = 63° 4′.

3. In a triangle ABC, the angles A and B are = 56° 6′ 13″ and 59° 50′ 27″, and the side AB is = 130; required the remaining parts of the triangle. Angle C = 64° 3′ 20″, AC = 125, and BC = 120.

4. In a triangle ABC are given the side AB = 142·02, AC = 104, and angle B = 44° 12′.

BC = 133·639 or 69·992, and angle C = 72° 10′ 55″, or 107° 49′ 5″.

5. In a triangle ABC are given the side AB = 456, AC = 780, and angle B = 125° 40′; required the other parts.

Angle C = 28° 21′ 23″, A = 25° 58′ 37″, and BC = 420·520.

6. In a triangle ABC are given AB=520, BC=394, and the angle C=64° 20′; required the other parts.

A=43° 4′ 23″, B=72° 35′ 37″, and AC=550·507.

215. CASE 3.—Given two sides and the contained angle.

Let the given sides be a and b, and C the given contained angle.

1. To find the sum of the angles opposite to the given sides, or A+B.

RULE.—The sum of the angles opposite to the given sides is found by subtracting the given angle from two right angles.

Or, $A+B=180°-C$, and $\frac{1}{2}(A+B)=90°-\frac{1}{2}C$.

2. To find the angles opposite to the given sides, or A and B.

RULE.—The sum of the two sides is to their difference as the tangent of half the sum of the angles at the base to the tangent of half their difference.

Or $$a+b : a-b = \tan\tfrac{1}{2}(A+B) : \tan\tfrac{1}{2}(A-B).$$

By natural sines, $$\tan\tfrac{1}{2}(A-B)=\frac{(a-b)\,.\,\tan\frac{1}{2}(A+B)}{a+b};$$

and by logarithms,

$$L\tan\tfrac{1}{2}(A-B)=L\tan\tfrac{1}{2}(A+B)+L(a-b)-L(a+b),$$

or $$L\tan\tfrac{1}{2}(A-B)=\text{ar. co. }L(a+b)+L(a-b)+L\tan\tfrac{1}{2}(A+B)-10.$$

When A−B is thus found, then

Half the difference of the two angles, added to half their sum, gives the greater; and taken from half the sum, gives the less.

Or $$A=\tfrac{1}{2}(A+B)+\tfrac{1}{2}(A-B),$$
and $$B=\tfrac{1}{2}(A+B)-\tfrac{1}{2}(A-B).$$

216. When only the third side C is wanted, it can be found by the formula

$$c^2=a^2+b^2\pm 2ab\,.\cos C.$$

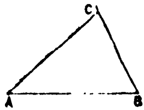
When C is obtuse, the upper sign + is to be used; and when C is acute, the lower sign − is to be taken.

The natural cosine of C is of course to be used. The third term of the value of c^2 may be found by logarithms; thus, let it=M,... then $L\,\frac{1}{2}M=La+Lb+L\cos C-10$.

EXAMPLE.—Of a triangle ABC, given the sides AC, BC respectively=176 and 133, and the contained angle C=73°.

1. To find the angles A and B

The side AC being greater than BC, angle B opposite to the former exceeds angle A opposite to the latter; also $A+B=180°-C=180°-73°=107°$, and $\frac{1}{2}(A+B)=53°\ 30'$.



And $AC+BC:AC-BC=\tan\frac{1}{2}(A+B):\tan\frac{1}{2}(B-A)$,
or $b+a:b-a=\tan\frac{1}{2}(B+A):\tan\frac{1}{2}(B-A)$.

Ar. co. L (AC + BC) 309,	=	7·5100415
L (AC − BC) 43,	=	1·6334685
L tan $\frac{1}{2}$(A + B) 53° 30′,	=	10·1307911
L tan $\frac{1}{2}$(B − A) 10° 39′ 3″	=	9·2743011

Hence, angle B = 64° 9′ 3″
" A = 42 50 57

2. To find AB

Sin B : sin C = AC : AB.

L cosec B 64° 9′ 3″,	=	10·0457840
L sin C 73°,	=	9·9805963
L AC 176,	=	2·2455127
L AB 187·022,	=	2·2718930

When the third side AB only is wanted, it may be found thus
$$AB^2=BC^2+AC^2-2BC\,.\,AC\,.\,\cos C,$$
or $c^2=a^2+b^2-2ab\,.\cos C=133^2+176^2-2\times133\times176\times\cos 73°=17689+30976-46816\times{\cdot}2923717=48665-13687{\cdot}67=34977{\cdot}33$, and $c=\sqrt{34977{\cdot}33}=187{\cdot}022$.

The sign − is used above because C is acute.

EXERCISES

1. In a triangle ABC, given AB and BC respectively = 180 and 200, and angle B = 69°, to find the other parts.

Angle A = 59° 52′ 45″, C = 51° 7′ 15″, and AC = 215·864.

2. Two sides of a triangle are respectively = 240 and 180, and the contained angle is = 25° 40′; required the other angles and the third side.

The angles are = 109° 15′ 30″ and 45° 4′ 30″, and the third side = 110·114.

3. Two sides of a triangle are respectively = 3754 and 3226·4, and the contained angle = 58° 53′; required the other angles, and the third side.

The angles = 68° 11′ 8″ and 52° 55′ 52″; third side = 3461·75.

4. Two sides of a triangle are respectively $=375{\cdot}4$ and $327{\cdot}763$, and the contained angle $=57^\circ\ 53'$; required the other parts.

The third side $=342{\cdot}818$, and the angles $=68^\circ\ 2'\ 35''$ and $54^\circ 4'\ 25''$.

217. CASE 4.—When the three sides of a triangle are given.

This case may be solved by any of the following five rules :—

RULE I.—Draw a perpendicular from one of the angles upon the opposite side, or this side produced; then—calling this side the base—twice the base is to the sum of the two sides as the difference of these sides to the distance of the perpendicular from the middle of the base; then the sum of half the base and this distance is = the greater segment, and their difference is = the less.

The given triangle is thus divided by the perpendicular into two right-angled triangles, in each of which two sides are known; and hence the angles at the base can be found, and consequently the third angle.

RULE II.—From half the sum of the three sides subtract each of the sides containing the required angle; then add together the logarithms of the two remainders, and the arithmetical complements of the logarithms of these two sides, and half the sum is the logarithmic sine of half the required angle.

Let C be the required angle, and $s=\frac{1}{2}(a+b+c)$;

then, by natural sines, $\sin^2\frac{1}{2}\,C=\dfrac{(s-a)(s-b)}{ab}$;

and by logarithms,

$$2\mathrm{L}\sin\tfrac{1}{2}\mathrm{C}=\mathrm{L}(s-a)+\mathrm{L}(s-b)+(10-\mathrm{L}a)+(10-\mathrm{L}b).$$

RULE III.—From half the sum of the three sides subtract the side opposite to the required angle; then add together the logarithms of the half sum and of this difference, and the arithmetical complements of the logarithms of the other two sides, and half the sum is the logarithmic cosine of half the required angle.

By natural sines, $\cos^2\frac{1}{2}\,C=\dfrac{s(s-c)}{ab}$; and by logarithms,

$$2\mathrm{L}\cos\tfrac{1}{2}\mathrm{C}=\mathrm{L}s+\mathrm{L}(s-c)+(10-\mathrm{L}a)+(10-\mathrm{L}b).$$

RULE IV.—From half the sum of the three sides subtract each side separately; then subtract the logarithm of the half sum from 20, and under the result write the logarithms of the three remainders; **half** the sum of these will be a **constant**, from which, if the logarithms of the three remainders be successively subtracted, the new remainders will be the logarithmic tangents of half the angles of the triangle.

By natural tangents, $\tan^2 \frac{1}{2} C = \frac{(s-a)(s-b)(s-c)}{s(s-c)^2}$.

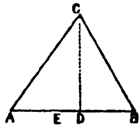
Rule V.—The angle may also be found by the formula,

$$\cos C = \frac{a^2+b^2 \sim c^2}{2ab}.$$

This method is simple when the sides are small numbers; when c^2 is less than a^2+b^2, angle C is acute; but when greater, this angle is obtuse. When $c^2 = a^2+b^2$, angle C is a right angle.

The proof of these rules is given in the articles 201–205.

The second method ought not to be used when the angle is a large obtuse angle, for then $\frac{1}{2}$ C will be nearly a quadrant, and the sines of angles near 90° vary slowly, and the seconds will not be accurately obtained. For a similar reason, the third method ought not to be used when C is very small. When all the angles are required, the fourth method is much more expeditious than any of the others.



Example.—In the triangle ABC there are given the three sides AB, BC, and AC respectively = 150, 130, and 140, to find the angles.

By Rule I

1. To find the difference of the segments AD, DB

$$2\ AB : AC + CB = AC - CB : DE.$$

$$300 : 270 = 10 : 9 = DE; \text{ and}$$

$$AD = \tfrac{1}{2}\ AB + DE = 75 + 9 = 84$$

$$BD = \tfrac{1}{2}\ AB - DE = 75 - 9 = 66.$$

2. To find angle A

$$\text{Cos } A = \frac{AD}{AC} = \frac{84}{140};$$

therefore

$$\begin{aligned} L \cos A &= 10 + L\ 84 - L\ 140 \\ &= 11\cdot9242793 - 2\cdot1461280 \\ &= 9\cdot7781513 \\ &= L \cos 53°\ 7'\ 48''. \end{aligned}$$

3. To find angle B

$$\text{Cos } B = \frac{BD}{BC} = \frac{66}{130};$$

therefore

$$\begin{aligned} L \cos B &= 10 + L\ 66 - L\ 130 \\ &= 11\cdot8195439 - 2\cdot1139434 \\ &= 9\cdot7056005 \\ &= L \cos 59°\ 29'\ 23''. \end{aligned}$$

4. To find angle ACB

$$ACB = 180 - (A + B) = 180 - 112^\circ\ 37'\ 11'' = 67^\circ\ 22'\ 49''.$$

By Rule II

This rule may be used exactly as the following, taking $(s-a)$ and $(s-b)$ instead of s and $(s-c)$, and $\sin \frac{1}{2} C$ for $\cos \frac{1}{2} C$.

By Rule III

To find angle C

$$s = \tfrac{1}{2}(a+b+c) = \tfrac{1}{2}(130+140+150) = \tfrac{1}{2} \times 420 = 210,$$
$$s - c = 210 - 150 = 60.$$

L s 210, ...	=	2·3222193
L $(s-c)$ 60, ...	=	1·7781513
10 − L a 130, ...	=	7·8860566
10 − L b 140, ...	=	7·8538720
		2)19·8402992
L cos $\frac{1}{2}$ C 33° 41′ 24″·2 . .	=	9·9201496
2		
C = 67° 22′ 48″·4		

By Rule IV. all the angles may be found, and being added together, when the work is correct, their sum will be = 180°.

By Rule IV

$a = 130$
$b = 140$
$c = 150$

$2s = 420$

$s = 210$,	and $20 - Ls$	=	17·6777807
$s - a =$ 80	$L(s-a)$	=	1·9030900
$s - b =$ 70	$L(s-b)$	=	1·8450980
$s - c =$ 60	$L(s-c)$	=	1·7781513
			2)23·2041200
	Constant	=	11·6020600

Tan $\frac{1}{2}$ A = 26° 33′ 54·2″ = 9·6989700 {cont. − $L(s-a)$}.
Tan $\frac{1}{2}$ B = 29 44 41·6 = 9·7569620 {cont. − $L(s-b)$}.
Tan $\frac{1}{2}$ C = 33 41 24·2 = 9·8239087 {cont. − $L(s-c)$}.

Hence
A = 53° 7′ 48·4″
B = 59° 29′ 23·2″
C = 67° 22′ 48·4″; and adding

A + B + C = 180°

By Rule V.,...Nat. $\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{130^2+140^2-150^2}{2 \times 130 \times 140}$

$= \frac{36500-22500}{2 \times 130 \times 140} = \frac{14000}{2 \times 130 \times 140} = \frac{5}{13} = \cdot 3846154 =$ nat. cos 67° 22′ 48″·5.

When the sides are small numbers, as in the present example, this method is very expeditious. The angle C being found, A and B may be similarly calculated by these formulæ,

$$\cos A = \frac{b^2+c^2-a^2}{2bc}, \text{ and } \cos B = \frac{a^2+c^2-b^2}{2ac}.$$

EXERCISES

1. The three sides of a triangle AB, BC, AC are = 100, 80, and 60; find the angles. A = 53° 7′ 48″, B = 36° 52′ 12″, and C = 90°.

2. The three sides of a triangle AB, AC, BC are = 457, 368, and 325; find the angles.
A = 44° 48′ 15″, B = 52° 55′ 56″, and C = 82° 15′ 49″.

3. The three sides of a triangle are AB = 562, BC = 320, and AC = 800; required the angles.
A = 18° 21′ 24″, B = 128° 3′ 49″, C = 33° 34′ 47″.

PROMISCUOUS EXERCISES IN TRIGONOMETRY

1. Given the hypotenuse of a right-angled triangle = 774, and one of the oblique angles = 57° 8′; to find the other parts.
The other acute angle is = 32° 52′, and the other two sides are = 420·039 and 650·11.

2. Given one of the sides about the right angle of a triangle = 2456, and the opposite angle = 44° 26′; to find the other parts.
The other acute angle is = 45° 34′, and the other sides are = 2505·068 and 3508·176.

3. Given the hypotenuse and another side = 3604·5 and 2935·2; to find the other parts.
The angles are = 35° 28′ 48″·8 and 54° 31′ 11·2″, and the other side is = 2092·13.

4. Given the two sides about the right angle = 1260 and 1950; to find the other parts.
The angles are = 57° 7′ 53″ and 32° 52′ 7″, and the hypotenuse is = 2321·66.

5. Given two angles, A and C, of a triangle = 32° 42′ and 28° 58′, and the side AC = 6364; to find the other parts.
The angle B is = 118° 20′, the side AB = 3501·57, and BC is = 3906·02.

6. The sides AB, BC of a triangle are $=1000$ and 1200, and angle A is $=36° \ 50'$; required the other parts.

Angle B is $=113° \ 11' \ 41''$, angle $C=29° \ 58' \ 19''$; and the side AC is $=1839 \cdot 909$.

7. The two sides AC, BC of a triangle are $=281 \cdot 67$ and 275, and angle C is $=49° \ 25'$; required the other parts.

Angles A and B are $=63° \ 48'$ and $66° \ 47'$, and the side BC is $=232 \cdot 7665$.

8. The three sides of a triangle are $=133$, 176, and $187 \cdot 022$; required the angles.

The angles are $=73°$, $64° \ 9' \ 3''$, and $42° \ 50' \ 57''$.

MENSURATION OF HEIGHTS AND DISTANCES

218. For the measurement of lines, some line of a determinate length is assumed—as an inch, a foot, a yard, &c. The assumed line is called the **lineal unit.** The number of lineal units contained in a line is its **measure** or **numerical value.**

The heights and distances of objects are represented by lines, and are therefore expressed in terms of some lineal unit.

The measure of any height or distance might be ascertained by applying the lineal unit to its length, were it possible to reach it; but many heights and distances are of such a nature that their measures can be obtained only by the application of the principles of trigonometry.

219. **Heights** and **distances** are said to be **accessible** or **inaccessible** according as it is possible or not to reach the base of the perpendiculars that measure the heights, or according as the distance between two objects can be directly measured or not.

220. A **vertical line** is the direction of the plumb-line.

221. A **vertical plane** is a plane passing through a vertical line.

222. A **horizontal plane** is perpendicular to a vertical line.

223. A **horizontal line** is one in a horizontal plane.

224. An **oblique plane** is one that is neither vertical nor horizontal.

225. A **vertical angle** is an angle in a vertical plane.

226. A **horizontal angle** is an angle in a horizontal plane.

227. An **inclined angle** is an angle in an oblique plane.

228. An **angle of elevation of one point above another** is the vertical angle formed by a line joining the two points and a horizontal line passing through the latter point.

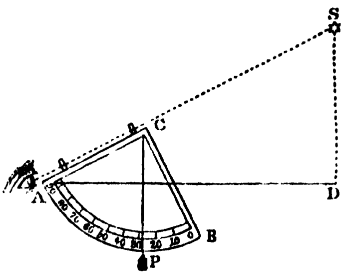
An **angle of elevation** is also called an **angle of altitude.**

229. An **angle of depression of one point below another** is the vertical angle contained by a line joining the two points and a horizontal line passing through the latter point.

230. The **angular** distance between two objects at any point is the angle formed at that point by two lines drawn from it to the objects; this angle is therefore the angle of a triangle opposite to the line joining the objects.

Horizontal and vertical angles can be measured most conveniently by means of the **Theodolite**; for an account and engraving of which, see LAND-SURVEYING.

When much accuracy is not required, vertical angles can be measured by means of a **quadrant** of simple construction, represented in the adjoining figure. The arc AB is a quadrant, graduated into degrees from B to A; C, the point from which the plummet P is suspended, being the centre of the quadrant.

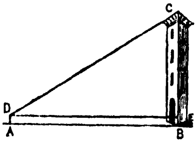


When the sights A, C are directed towards any object, S, the degrees in the arc BP are the measure of the angle of elevation SAD of the object. For AD being a horizontal line, and SD (supposing S and D joined) a vertical line, and therefore CP parallel to SD, the angle ACP = ASD: now BCP is the complement of ACP, and SAD of ASD; therefore angle SAD = BCP, which is measured by the arc BP.

231. Problem I.—To compute the height of an accessible object.

Let the object whose height is required be a tower BC.

Measure a horizontal line AB from the base of the object to any convenient distance A, and then measure the angle of elevation of the top of the object at A.



Then if AD denote the height of the eye, the angle CDE is the given angle, DE being parallel to AB. Hence, in the triangle DEC, the side DE=AB, and angle D are given; therefore CE can be found by 180.

EXAMPLE.—Required the height of the tower BC, having given the horizontal line DE=120 feet, the angle of elevation CDE =39° 49′, and the height of the eye=5 feet 2 inches.

To find CE in triangle CDE

$$\frac{CE}{DE}=\tan D,$$

therefore $CE=DE \tan D$.

Tan D 39° 49′,	. . .	=	9·9209898
DE 120,	. . . . .	=	2·0791812
			12·0001710
			10·
CE 100·039,	. . . .	=	2·0001710

Height of eye = 5·166

" tower = 105·205

EXERCISES

1. The breadth of a ditch in front of a tower is=48 feet; and from the outer edge of the ditch the angle of elevation of the top of the tower is=53° 13′; what is the height of the tower?
=64·20184 feet.

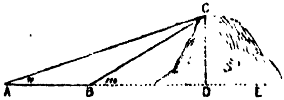
2. Required the height of an accessible building, the angle of elevation of its top being=41° 4′ 34″ at a point=101·76 feet distant from it, the height of the eye being=5 feet . . =93·696 feet.

3. At the top of a ship's mast=120 feet high, the angle of depression of another ship's hull was=15° 45′; what is the distance between the ships? . . . . =425·49 feet.

4. Required the height of a tower, a horizontal base of 245 feet being measured, and the angle of elevation being=35° 24′.
=174·112 feet.

232. Problem II.—To compute an inaccessible height, when a horizontal line in the same level with its base, and in the same vertical plane with its top, can be found.

Let the object whose height is wanted be a hill, CDE, such that the foot D of the vertical line CD that measures its altitude is inaccessible.



Measure a horizontal base AB, and the angles of elevation of the top C at A and B—namely, n and m.

In the triangle ABC angle $C=m-n$ (Eucl. I. 32), and is hence known. Then to find BC in the same triangle, sin C : sin n = AB : BC; thus BC is found to be $\frac{\text{AB}\sin n}{\sin C}$. Again, to find CD in the triangle BCD:

$$\frac{\text{CD}}{\text{BC}}=\sin m\text{; therefore CD}=\text{BC}\sin m.$$

Hence, using the value of BC,

$$\text{CD}=\frac{\text{AB}\sin n\sin m}{\sin C}=\text{AB}\sin n\sin m\operatorname{cosec}C,$$

$$\text{CD}=\frac{\text{AB}\sin n\,.\,\sin m}{\text{R}\sin C},\ \text{CD}=\text{AB}\sin n\,.\,\sin m\,.\operatorname{cosec}C;$$

$$\therefore \text{L}\,.\,\text{CD}=\text{L}\,.\,\text{AB}+\text{L}\sin n+\text{L}\,.\sin m+\text{L}\operatorname{cosec}(m-n)-30.$$

Since $C=(m-n)$; and 30 has to be subtracted, because each of the logarithmic trigonometrical functions is 10 greater than the logarithm of the natural function.

EXAMPLE.—From the base of a hill a horizontal line of 384 feet was measured in a direction from the hill, and such that the line and the top of the hill were in one vertical plane, the angles of elevation of the top of the hill, taken at two stations at the extremities of this base-line, were $=40°\ 12'$ and $50°\ 42'$; required the height of the hill.

Here $C=m-n=50°\ 42'-40°\ 12'=10°\ 30'$.

L AB 384, ...	=	2·5843312
L sin n 40° 12′, ...	=	9·8098678
L sin m 50° 42′, ...	=	9·8886513
L cosec $(m-n)$ 10° 30′, ...	=	10·7393670
L . CD, ...	=	3·0222173

$\therefore$ CD = 1052·488.

The two proportions may also be wrought separately.

In the following exercises the base-line is measured as in the above example :—

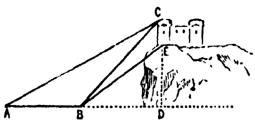
EXERCISES

1. In order to find the height of a hill, a base-line was measured $=130$ feet, and the angles of elevation of the top of the hill, measured at the extremities of the base, were $=31°$ and $46°$; required its height. . . . . $=186·089$ feet.

2. Required the height of an inaccessible tower on the opposite side of a river, the length of the base being $=170$ feet, and the angles of elevation at its extremities $=32°$ and $58°$; the height of the eye being $=5$ feet. . . . . $=179·276$ feet.

3. Required the height of a hill from these measurements : AB $=1356$, angle $m=36°\ 50'$, and $n=25°\ 36'$. . . $=1803·06$ feet.

233. Problem III.—To measure the height of an object situated on an inaccessible height, when a horizontal base can be measured in the same vertical plane with the top of the object.



Let EC be the object situated on the hill ED, AB the horizontal base; measure the angles of elevation CBD, EBD of the top and bottom of the tower at B; then measure at A, the angle of elevation of the top of the tower C.

Find BC in the triangle ABC thus :—Angle $C=CBD-A$, and $\sin C : \sin A = AB : BC$. Then find EC in triangle BCE thus :— Angle $BEC=D+EBD=90°+EBD$, and angle $B=CBD-EBD$, then $\sin BEC : \sin CBE = BC : CE$.

EXAMPLE.—Required the height of a fort CE, situated on the top of a hill, the angles of elevation of the top of the hill and the top of the fort at B being $=48°\ 20'$ and $61°\ 25'$; at A, the elevation of the top of the fort, being $=38°\ 19'$; and the base $AB=360$ feet.

1. To find BC in triangle ABC

$C=CBD-A=61°\ 25'-38°\ 19'=23°\ 6'$.

L cosec C 23° 6', ...	=	10·4063406
L sin A 38° 19', ...	=	9·7923968
L AB 360, ...	=	2·5563025
∴ L BC, ...	=	2·7550399

2. To find CE in triangle BCE

$$E = 90° + EBD = 90° + 48° 20' = 138° 20'.$$
$$B = CBD - DBE = 61° 25' - 48° 20' = 13° 5'.$$

L cosec E 138° 20′ (41° 40′), . .	=	10·1773117
L sin B 13° 5′, ...	=	9·3548150
L BC, ...	=	2·7550399
L CE, ...	=	2·2871666

$\therefore$ CE = 193·7165.

These two calculations may also be easily combined into one by compounding the two proportions from which they arise.

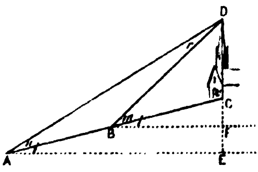
EXERCISES

1. Find the height of a tower on the top of a hill from these measurements:—The angles of elevation of the top of the hill, and the top of the tower at the nearer station, are = 40° and 51°; at the farther station, the angle of elevation of the top of the tower is = 33° 45′, and the horizontal base = 240 feet. . . = 111·9978.

2. In order to determine the height of a lighthouse, situated on the top of an inaccessible eminence, the following data were obtained:—A base-line = 368 feet, the angles of elevation at the nearer station of the top and bottom of the lighthouse = 36° 24′, and 24° 36′, and the angle of elevation of the top of the lighthouse at the farther station = 16° 40′; what was its height?

= 70·304 feet.

234. **Problem IV.—To calculate the height of an object standing on an inclined plane.**



Let CD be the object, and AC the inclined plane.

Measure a base AB on the plane, and the angles of elevation DBF, DAE of the top of the steeple, taken at the extremities of the base, and also the angle of inclination i of the plane with the horizon.

To find the angles m, n, r, and v

$$m = DBF - i,\ n = DAE - i, \text{ and } r = m - n;$$
also
$$v = F + i = 90° + i.$$

To find BD in triangle ABD, $\sin r : \sin n = AB : BD$; and to find CD in triangle BCD, $\sin v : \sin m = BD : CD$.

EXAMPLE.—Required the height of the steeple CD, situated on the inclined plane AC, from these measurements:—AB = 112, the angles of elevation at A and B = 44° 25′ and 63° 40′, and the inclination of the plane = 15° 20′.

To find angles m, n, r, and v

$$m = \text{DBF} - i = 63° 40' - 15° 20' = 48° 20'.$$
$$n = \text{DAE} - i = 44° 25' - 15° 20' = 29° 5'.$$
$$r = m - n = 48° 20' - 29° 5' = 19° 15'.$$
$$v = 90° + i = 90° + 15° 20' = 105° 20'.$$

To find BD		To find CD	
L cosec r 19° 15′,	= 10·4818934	L cosec v 105° 20′,	= 10·0157411
L sin n 29° 5′, .	= 9·6867088	L sin m 48° 20′,	= 9·8733352
L AB 112, .	= 2·0492180	L BD 165·12, .	= 2·2178202
L BD . 165·12, .	= 2·2178202	L CD, . .	= 2·1068965

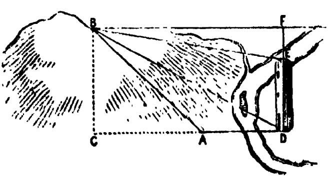
∴ CD = 127·9077.

EXERCISE

Required the height of an object standing on an inclined plane from these data:—AB = 124 feet, angle DBF = 58° 20′, DAE = 40° 30′, and the inclination of the plane = 14° 10′. . . = 129·068 feet.

235. **Problem V.—To find the height of an inaccessible object, when only one station can be taken on the same horizontal plane with its base, and a base-line on an inclined plane, and in the same vertical plane with its top; and also to find the distances of the stations from the object.**

Let DE be the inaccessible object; A, the station in a horizontal plane with D; AB, the acclivity.



Measure a base AB in the same vertical plane with DE; and BF being a horizontal line, measure the angles of depression FBA, FBD, and FBE.

Then in triangle ABD, angle D = FBD (Eucl. I. 29), and angle B = FBA − FBD, and A = 180° − ABF; for ABF = BAC. Again, in triangle BDE, B = FBD − FBE, and E = F + FBE = 90° + FBE.

Hence, in triangle ABD, find BD from sin D : sin A = AB : BD; then in triangle BDE, find DE from sin E : sin B = BD : DE.

If the distance of A from D were required, it could also be found from sin D : sin B = AB : AD.

EXAMPLE.—In order to find the height of a tower on the other side of a river, a base of 204 feet was measured up an acclivity from a station on the same horizontal plane with the bottom of the object, and at the upper station the angles of depression of the first station, and of the bottom and top of the object, were = 47° 42′, 17° 52′, and 11° 40′; required the height of the tower, and the distance of the two stations from its bottom.

In triangle ADB, angle D = FBD = 17° 52′, and B = FBA − FBD = 47° 42′ − 17° 52′ = 29° 50′, and A = 180° − ABF = 180° − 47° 42′ = 132° 18′. And in triangle BDE, angle B = FBD − FBE = 17° 52′ − 11° 40′ = 6° 12′, and E = F + FBE = 90° + 11° 40′ = 101° 40′.

1. To find BD in triangle ABD

L cosec D 17° 52′, ...	=	10·5131405
L sin A 132° 18′ (47° 42′), . .	=	9·8690152
L AB 204, ...	=	2·3096302
BD 491·797, ...	=	2·6917859

2. To find DE in triangle DBE

L cosec E 101° 40′ (78° 20′), . .	=	10·0090662
L sin B 6° 12′, ...	=	9·0334212
L BD 491·797, ...	=	2·6917859
DE 54·2342, ...	=	1·7342733

3. To find AD in triangle ADB

L cosec D 17° 52′, ...	=	10·5131405
L sin B 29° 50′, ...	=	9·6967745
L AB 204, ...	=	2·3096302
AD 330·7846, ...	=	2·5195452

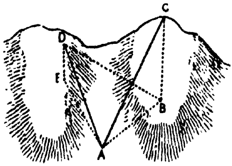
EXERCISE

Required the height of an inaccessible object from measurements similar to those in the above example, and also the distance of the lower station from the object, the angles of depression of the first station, and of the bottom and top of the object, taken from the upper station, being = 42°, 27°, and 19°, and the distance between the stations = 165 yards.

The height = 35·796, and distance = 94·066 yards.

236. **Problem VI.—To measure an inaccessible height when a horizontal base can be obtained, but not in the same vertical plane with the top of the object.**

Let BC, the altitude of a hill, be the inaccessible height, and AD the horizontal base.



Measure the base AD and the angle of elevation of the top C of the hill at the station A; let AB, DB be horizontal to the vertical line CB; and measure the horizontal angles ADB and DAB with the theodolite, as well as the vertical angle CAB.

In the triangle ADB the angles at A and D are known, and the side AD; hence angle $B = 180° - (A + D)$, and AB is found by the proportion $\sin B : \sin D = AD : AB$. Then in the triangle ABC, having the right angle B, BC is found from $\frac{BC}{AB} = \tan A$.

Using the value found for AB, we obtain

$$BC = \frac{AD \sin D \tan A}{\sin B};$$

hence $L . BC = L . AB + L \sin D + L \tan A + L \operatorname{cosec} B - 30$.

Example.—Required the height of a hill from these measurements:—$AD = 1284$, angle $ADB = 74° \ 15'$, $DAB = 85° \ 40'$, and angle $BAC = 25° \ 56'$.

In triangle ADB

$$\text{Angle } B = 180° - (A + D) = 180° - 159° \ 55' = 20° \ 5'.$$

To find BC

L AB 1284, ...	=	3·1085650
L cosec B 20° 5′, ...	=	10·4642168
L sin D 74° 15′, ...	=	9·9833805
L tan A 25° 56′, ...	=	9·6868981
L BC, ...	=	3·2430604

$\therefore BC = 1750·09$.

Exercises

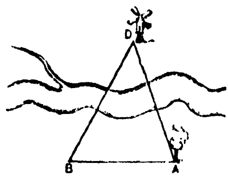
1. In order to find the height of a mountain, a base of 1648 feet was measured in a valley, and at the station at one of its extremities, the angle of elevation of the top of the hill was found to be $= 32° \ 25'$; and the horizontal angle at it, formed by the base, and a horizontal line drawn from this station to the vertical line

from the top of the hill, was $=78° 16'$; also the horizontal angle, similarly formed at the other station, was $=80° 12'$; what is the height of the hill? . . . . The altitude is $=2809{\cdot}63$ feet.

2. Find the height of a mountain BC from these measurements:—In the horizontal triangle ABD, the base $AD=1245$ feet, angle $A=74° 12'$, $D=84° 20'$, and the angle of elevation $CAB=25° 45'$. . . . . Height $=1632{\cdot}92$ feet.

If the base AD (last figure) were not horizontal, and if AE is a horizontal line, and DE a vertical one, the angle of acclivity DAE could be measured, and the base AD; and then the horizontal base AE could be found thus: $AE=AD \cos DAE$; then the base AE is known, and the horizontal angle measured at D, as formerly described, is = to the horizontal angle contained by AE, and a horizontal line from E to a point in CB. Considering, therefore, AE as the base, the height of the hill could be found exactly as before.

237. **Problem VII.—To measure a distance inaccessible at one extremity.**



Let AD be the inaccessible distance between two objects at A and D, on opposite sides of a river.

Measure a base AB, and the angles A and B at its extremities.

Angle $D=180°-(A+B)$; and AD is found by the proportion

$$\text{Sin } D : \sin B = AB : AD.$$

EXAMPLE.—Required the distance between a tree and a windmill on the other side of a river, a base of 1140 feet being measured from the tree to another station; the angular distance of the tree and windmill, measured at the latter station, being $=43°$; and the angular distance of the windmill and second station, measured at the tree, being $=60°$.

$$\text{Angle } D=180°-(A+B)=180°-103°=77°.$$

To find AD

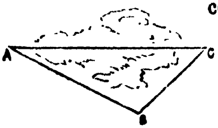
L cosec D 77°,	. . . . .	=	10·0112761
L sin B 43°, .	. . . . .	=	9·8337833
L AB 1140, .	. . . . .	=	3·0569049
AD 797·929, .	. . . . .	=	2·9019643

Exercises

1. Having taken two stations on the side of a river, and measured a base between them of 440 yards, and also the angles at the stations formed by the base and lines drawn from the stations to a house on the other side of a river, which were $=73° 15'$ and $68° 2'$; what are the distances of the stations from the house? . . . . $=673·624$ and $652·4$ yards.

2. A line was measured on the side of a lake of 500 yards, and the angles at its extremities contained by it and lines drawn to a castle on the other side of the lake were $=79° 23'$ and $54° 22'$; what is the distance of the castle from the extremities of the base? . . . . $=680·323$ and $562·57$ yards.

238. **Problem VIII.—To find the distance between two objects that are either invisible from each other or inaccessible in a straight line.**



Let A and C be the two objects, inaccessible in a straight line from each other on account of a marsh.

Measure two lines AB, BC to the objects and the contained angle B.

In the triangle ABC, two sides AB, BC, and the contained angle B, are known; hence AC may be found.

Example.—Given the two lines AB, BC, 562 and 320, and the contained angle B 128° 4′, to find AC.

1. To find the angles at A and C

$$A + C = 180° - 128° 4' = 51° 56'.$$

Ar. co. L (AB + BC) 882, ...	=	7·0545314
L (AB − BC) 242, ...	=	2·3838154
Tan ½(A + C) 25° 58′, ...	=	9·6875402
Tan ½(A ~ C) 7° 36′ 40″, . . .	=	9·1258870

∴ Angles C = 33° 34′ 40″
" A = 18 21 20

2. To find AC

L cosec C 33° 34′ 40″, ...	=	10·2572213
L sin B 128° 4′, ...	=	9·8961369
L AB 562, ...	=	2·7497363
L AC 800·008, ...	=	2·9030945

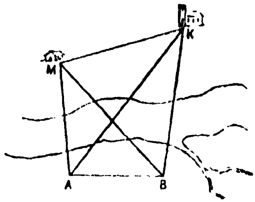
Exercises

1. Find the distance between two objects that are invisible from each other, having given their distances from a station at which they are visible = 882 and 1008 yards, and the angle at this station, subtended by the distance of the objects = 55° 40'. = 889·45 yards.

2. The distance of a given station from two objects situated at opposite sides of a hill are = 564 and 468 fathoms, and the angle at the station, subtended by their distance, is = 64° 28'; what is their distance? . . . . . Distance = 556·394 fathoms.

239. Problem IX.—To find the distance between two inaccessible objects.

Let M and K be the two objects on the opposite side of a river from the observer.



Measure a base AB; and at each station measure the angular distances between the other station and the two objects—namely, the angles BAK, BAM, ABM, and ABK.

Then in the triangle ABK, angle K = 180° - (A + B), and the side AB is known; hence find AK thus:—sin K : sin B = AB : AK.

Again, in triangle AMB, angle M = 180° - (A + B), and AM is found by the proportion sin M : sin B = AB : AM.

Hence, in triangle AMK, the two sides AM, AK are known, and the contained angle A = BAM - BAK; therefore AK + AM : AK ~ AM = tan $\frac{1}{2}$(M + K) : tan $\frac{1}{2}$(M ~ K); and M ~ K being thus found, each of the angles M and K can then be found. Hence MK will now be found in triangle AMK by the proportion sin K : sin A = AM : MK.

Example.—Required the distance between the two objects M and K in the preceding figure from the following data:—

BAK = 64° 25', ABM = 56° 15', BAM = 104° 25', ABK = 106° 23', and the base AB = 520 yards.

In triangle ABK, angle K = 180° - (A + B) = 180° - 170° 48' = 9° 12'.
In triangle ABM, angle M = 180° - (A + B) = 180° - 160° 40' = 19° 20'.
And in triangle AMK, angle A = BAM - BAK = 40°.

1. To find AK in triangle ABK

L cosec K 9° 12', ...	=	10·7962026
L sin B 106° 23' (73° 37'), . . .	=	9·9819979
L AB 520, ...	=	2·7160033
L AK 3120·35, ...	=	3·4942038

2. To find AM in triangle AMB

L cosec M 19° 20′,	=	10·4800888
L sin B 56° 15′,	=	9·9198464
L AB 520,	=	2·7160033
L AM 1305·98,	=	3·1159385

3. To find the angles M and K in triangle AMK

$$M + K = 180° - A = 180° - 40° = 140°.$$

Ar. co. L(AK + AM) 4426·33,	=	6·3539563
L(AK − AM) 1814·37,	=	3·2587259
L tan ½(M + K) 70°,	=	10·4389341
Tan ½(M − K) 48° 23′ 48″,	=	10·0516163

Hence K = 21° 36′ 12″

4. To find MK in triangle AMK

L cosec K 21° 36′ 12″,	=	10·4339415
L sin A 40°,	=	9·8080675
L AM 1305·98,	=	3·1159385
L MK 2280·06,	=	3·3579475

It is evident that if the sides MB, BK had been found instead of MA, AK, the distance MK could have been found in a similar manner in the triangle MBK.

EXERCISES

1. Find the distance between two objects situated as in last example, from these measurements:—

BAK = 58° 20′, ABM = 53° 30′,
BAM = 95° 20′, ABK = 98° 45′, and
the base AB = 375 yards; ∴ MK = 599·742.

2. Find the distance between the two objects M and K from these data:—

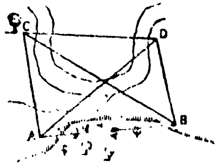
In triangle MAB,	In triangle KAB,
A = 120° 40′,	A = 60° 12′,
B = 45° 30′,	B = 98° 15′,
AB = 1248 feet;	∴ MK = 3581·2 feet.

240. **Problem X.—To find the distance between two objects, having given the angles formed at each of them by lines**

drawn to the other, and to two given stations, the distance between which is also given.

Let C and D be the two objects, and A and B the two stations which may be invisible from each other.

Measure the angles at C and D, formed by lines joining them with the two stations, and with each other.



The angles at C and D are known, and were the distance CD known, AB could be found by calculation by the last problem. CD, however, is unknown; assume it equal to some number, as 1000, and compute by the preceding problem the value of AB on this supposition; then the length of CD will be found by this proportion—the computed value of AB is to its real value as the assumed value of CD to its real value.

The distance between A and B, supposing CD = 1000, can be first found by means of Prob. IX.

Exercises

1. Required the distance between the two objects C and D from these measurements:—

Angle ACB = 36° 15′ 5″,	ADB = 45° 1′ 3″,
BCD = 33° 7′ 40″,	ADC = 30° 2′ 0″,
AB = 1410·4 ;	∴ CD = 2080·88.

2. Find the distance between the objects C and D from the following measurements:—

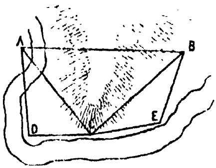
In triangle ACD,	In triangle BCD,
C = 110° 50′,	C = 43° 30′,
D = 38° 45′,	D = 115° 21′,
AB = 1540 ;	∴ CD = 661·78.

241. **Problem XI.—To find the distance between two inaccessible objects, so situated that a base cannot be obtained from the extremities of which both objects are visible, but which are both visible from one point.**

Let A and B be the two objects, and C the point from which both are visible.

Measure two bases DC, CE, and the angles at their extremities.

In triangle ADC, angle $A = 180° - (C + D)$; hence find AC thus:—$\sin A : \sin D = CD : CA$. In the triangle BCE, find BC in a similar manner. Then in the triangle ABC are given AC, CB, and angle C; hence find the angles at A and B by $AC + CB : AC \sim CB = \tan \frac{1}{2}(A + B) : \tan \frac{1}{2}(A \sim B)$; then find A and B; and AB is found thus:—$\sin A : \sin C = BC : AB$.



After finding AC and CB in triangle ABC, AB may also be found independently of the angles A and B, for $AB^2 = AC^2 + CB^2 \pm 2AB . BC . \cos C$.

EXAMPLE.—Find the distance between the two objects A and B, and their distances from C, from these measurements :—

In triangle ADC,	In triangle BCE,
CD = 456 links,	CE = 524,
C = 44° 20′,	C = 50° 24′,
D = 87° 56′,	E = 89° 40′,
ACB = 88° 50′.	

1. In triangle ADC,

$$A = 180° - (C + D) = 180° - 132° 16' = 47° 44'.$$

To find AC

L cosec A 47° 44′,	=	10·1307551
L sin D 87° 56′,	=	9·9997174
L CD 456,	=	2·6589648
L AC 615·797,	=	2·7894373

2. In triangle BCE,

$$B = 180° - (C + E) = 180° - 140° 4' = 39° 56'.$$

To find BC

L cosec B 39° 56′,	=	10·1925354
L sin E 89° 40′,	=	9·9999927
L CE 524,	=	2·7193313
L BC 816·318,	=	2·9118594

3. To find AB in triangle ABC

By Art. 189, $AB^2 = AC^2 + CB^2 - 2AC \cdot BC \cdot \cos C$.

$L\ AC^2 = 2L\ AC$, . .	$= 5{\cdot}5788746 = L\ 379206$	
$L\ BC^2 = 2L\ BC$, . .	$= 5{\cdot}8237188 = L\ 666375$	
Hence $AC^2 + BC^2$, . .	$=$	1045581
$L\ 2$, ...	$= 0{\cdot}3010300$	
$L\ AC \cdot BC = L\ AC + L\ BC$,	$= 5{\cdot}7012967$	
$L \cos C\ 88° 50'$, . .	$= 8{\cdot}3087941$	
	$-\ 10{\cdot}$	
	$4{\cdot}3111208 = L\ 20470{\cdot}14$	
Hence AB^2, . . .	$=$	1025110·86

And $AB = \sqrt{1025110{\cdot}86} = 1012{\cdot}48$.

EXERCISES

1. Find the distance of the two inaccessible objects A and B, and their distances from the station C, from these data :—

In triangle ADC,	In triangle BCE,
$CD = 424$,	$CE = 640$,
$C = 40° 10'$,	$C = 56° 10'$,
$D = 85° 25'$,	$E = 84° 30'$,
$ACB = 89° 20'$;	

$AB = 1126{\cdot}1$, $AC = 519{\cdot}685$, and $BC = 1005{\cdot}08$.

2. Required the distance of the two inaccessible objects, A and B, from these data :—

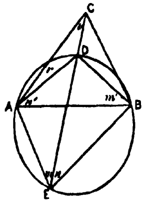
In triangle ADC,	In triangle BCE,
$C = 40° 20'$,	$C = 36° 25'$,
$D = 112° 40'$,	$E = 118° 15'$,
$CD = 1256$,	$CE = 1480$,
and angle $ACB = 108° 24'$;	$\therefore AB = 4550{\cdot}92$.

242. **Problem XII.—Given the distances between three objects, and the angles subtended by them at a station, to find the relative position of the station, and its distance from the objects.**

CASE I.—When the station is out of the triangle formed by lines joining the given objects, and the middle object is beyond the line joining the other two.

Let A, B, C be the three objects, E the station, and *m*, *n* the given angles.

Describe the triangle ABC with the given distances, and make the angles m', n' respectively equal to the given angles m, n. Then, through ABD, describe a circle ABE; draw CD, and produce it to E, and this point will be the station. Draw AE and BE.



In triangle ACB the there sides are given; hence angle A can be found.

In triangle ADB the angles and AB are given; hence AD can be found.

In triangle ADC, AC and AD are given, and angle A = CAB − DAB; hence angle v can be found.

In triangle ACE the angles and AC are known; hence AE and CE can be found; then in triangle ABE the sides AB, AE and the angles are known; and hence BE can be found.

Example.—Let the distances AB, BC, and CA be respectively = 1727, 1793, and 1540, and the angles subtended at the station E by BC and AB respectively = 25° 40′ and 53° 24′; what is the distance between the station and each of the objects?

Angle $n = 25° 40'$, and $m = 53° 24' - 25° 40' = 27° 44'$.

1. To find angle A in triangle ABC

$a = 1793$, $b = 1540$, $c = 1727$, $s = 2530$.

L s 2530,	=	3·4031205
L $(s-a)$ 737,	=	2·8674675
10 − L b 1540,	=	6·8124793
10 − L c 1727,	=	6·7627077
		2)19·8457750
L cos ½A 33° 8′ 33″,	=	9·9228875

∴ A = 66° 17′ 6″

2. To find AD in triangle ADB

Angle D = 180 − $(m' + n')$ = 180° − 53° 24′ = 126° 36′.

L cosec D 126° 36′,	=	10·0953832
L sin B 27° 44′,	=	9·6677863
L AB 1727,	=	3·2372923
L AD 1001·06,	=	3·0004618

3. To find angle C in triangle ADC

$$r = A - n' = 66° \ 17' \ 6'' - 25° \ 40' = 40° \ 37' \ 6''.$$

$$C + D = 180° - r = 139° \ 22' \ 54''.$$

Ar. co. L(AC + AD) 2541·06,	. . . .	=	6·5949850
L(AC − AD) 538·94,	. . . .	=	2·7315404
L tan ½(C + D) 69° 41′ 27″,	. . .	=	10·4316890
L tan ½(C ~ D) 29° 48′ 58″,	. . .	=	9·7582144

∴ $C = 39° \ 52' \ 29''$

4. To find AE and CE in triangle ACE

L cosec m 27° 44′,	. . .	.	10·3322137
L sin r 39° 52′ 29″,	. . .		9·8069333
L AC 1540,	. . . . .	.	3·1875207
L AE 2121·62,	. . . .		3·3266677

$$\text{Angle } A = 180 - (m + r) = 180° - 67° \ 36' \ 29'' = 112° \ 23' \ 31''.$$

L cosec m 27° 44′,	. . .		10·3322137
L sin A 112° 23′ 31″,	. . .		9·9659537
L AC 1540,	. . . . .	=	3·1875207
L CE 3059·76,	. . . . .	=	3·4856881

5. To find BE in triangle ABE

$$\text{Angle } A = CAE - CAB = 112° \ 23' \ 31'' - 66° \ 17' \ 6'' = 46° \ 6' \ 25''.$$

L cosec E 53° 24′,	. . .		10·0953832
L sin A 46° 6′ 25″,	. . . .	=	9·8577155
L AB 1727,	. . . . . .	=	3·2372923
L BE 1550·21,	. . . . .	=	3·1903910

The distances, therefore, are AE = 2121·62, CE = 3059·76, and BE = 1550·21.

Exercise

A, B, and C are three conspicuous objects in three towns. The distance of A from B = 125·6 furlongs, B from C = 130·4, and C from A = 112 furlongs; and at a station E, the distances AB and AC subtend angles = 48° 58′ and 25° 52′; required the distances of the station from the three objects.

AE = 165·357, BE = 123·25, and CE = 234·462 furlongs.

Case 2.—When the station is outside the triangle, and the middle object is on the same side of the line joining the other two.

Let the middle object C be between the station E and the line AB; then the points E and D will be both outside the triangle ABC, and on opposite sides of it, and the solution will be analogous to that of the first case.*

EXERCISE

Let the distances of the objects be AB = 106, AC = 65·5, and BC = 53·25, angle BEC = n = 13° 30′, and AEC = m 29° 50′; what are the distances of E from A, B, and C?

= 131·06, 151·428, and 107·42.

CASE 3.—When the station is inside the triangle.

Let D be the station, then the angles ADC, BDC being given, their supplements ADE, BDE are also given.

Make angles ABE, BAE respectively = ADE and BDE; then describe a circle about ABE; draw CE, and it will cut the circle in the station D.

If the station D is now marked E, and E is changed to D, the method described in the preceding case is exactly applicable to this; excepting that now angle CAD = CAB + DAB, and angle CAE = 180° − (ACE + AEC), and angle BAE = CAB CAE.

EXERCISE

The distances between three objects, taken in order, are BC = 5340, AC = 6920, and AB = 4180 feet; and the angles, subtended by these distances at a point inside the triangle formed by them, are respectively BEC = 128° 40′, AEC = 140°, and AEB = 91° 20′; what are the distances of the objects from the station?

AE = 3577·1, CE = 3786·2, and BE = 2080.

ADDITIONAL EXERCISES IN MENSURATION OF HEIGHTS AND DISTANCES

1. From the bottom of a tower a horizontal line was measured = 230 links, and at its extremity the angle of elevation of the top of the tower was = 43° 30′; required its height? . = 218·262 links.

2. At a horizontal distance of 170 feet from the bottom of a steeple, the angle of elevation of its top was = 52° 30′; what was the height of the steeple? . . . . . . = 221·548 feet.

3. Find the height of a precipice, its angle of elevation at two stations in a horizontal line with its base, and in the same vertical plane with its top, being = 39° 30′ and 34° 15′, and the distance between the stations = 145 feet. . . . . . = 567·293 feet.

* The student may draw a diagram to enable him to understand this case.

4. In order to find the height of a steeple, measurements were taken as in the preceding example; the base was = 90 feet, and the angles of elevation were = 28° 34′ and 50° 9′; required its height, and the distance of the nearer station from it.

Height = 89·818 feet, and distance = 74·9666.

5. From the top of a tower 136·5 feet high, the angle of depression of the root of a tree at a distance on the same plane was = 22° 40′; what was the distance of the tree from the bottom of the tower? . . . . = 326·848 feet.

6. From the summit of a hill, 360 feet high above a plain, the angles of depression of the top and bottom of a tower standing on the same plain were = 41° and 54°; required the height of the tower. . . . . = 132·63 feet.

7. From the summit of a lighthouse 85 feet high, standing on a rock, the angle of depression of a ship was = 3° 38′, and at the bottom of the lighthouse the angle of depression was = 2° 43′; find the horizontal distance of the vessel and the height of the rock.

= 5296·47 and 251·319 feet.

8. In order to find the distance between two objects, A and B, and their distances from a station C, the following measurements were taken, as in Prob. XI.—namely, CD = 200 yards, CE = 200 yards, angle ACD = 89°, ADC = 53° 30′, BCE = 54° 30′, BEC = 88° 30′, and ACB = 72° 30′; what are the distances?

AB = 356·86, AC = 264·096, and BC = 332·214 yards.

9. The distances between three objects, A, B, C, are known namely, AB = 12 miles, BC = 7·2 miles, and AC = 8 miles; and at a station between A and B, in the line joining them, from which the three objects were visible, the distance AC subtended an angle of 107° 56′; required the distances of this station from the three objects. . BD = 6·9984, DA = 5·0016, and DC = 4·8908 miles.

10. Three conspicuous objects, A, B, C, whose distances are AB = 9, BC = 6, and AC = 12 miles, were observed from a station D, from which B appeared to be the middle object, and lay beyond the line joining A and C; at this station the distances AB, BC subtended respectively angles of 33° 45′ and 22° 30′; what is the distance of the station from the objects?

AD = 10·663, BD = 15·641, and CD = 14·0107 miles.

11. From the top of a mountain I observe two milestones on the level ground in a straight line from one another, and I find their angles of depression to be 5° and 15° respectively. Determine the height of the mountain. . . . . . . = 228·64 yards.

12. The cone of dispersion of a shrapnel-shell bursting 100 yards

short of an object is found to be 30°. What is the front covered? Neglect height of burst above horizontal plane. . =53·72 yards.

13. A castle is situated on the top of a hill whose angle of inclination to the horizon is 30°; the angle subtended by the castle to the foot of the hill is found to be 15°, and on ascending 485 feet up the hill the castle is found to subtend an angle of 30°. Find the height of the castle, and the distance of its base from the foot of the hill.

Height of castle=280·02 feet. Distance of its base from foot of hill=765·01 feet.

14. A and B are two stations on a hillside; the inclination of the hill to the horizon is 30°; the distance between A and B is 500 yards. C is the summit of another hill in the same horizontal plane as A and B, and on a level with A; but at B its elevation above the horizon is 15°. Find the distance between A and C.

=1366·025 yards.

15. A man standing at a certain station on a straight sea-wall observes that the straight lines drawn from that station to two boats lying at anchor are each inclined at 45° to the direction of the wall, and when he walks 400 yards along the wall to another station he finds that the former angles of inclination are changed to 15° and 75° respectively. Find the distance between the boats, and the perpendicular distance of each from the sea-wall.

=156·4 yards; 556·4 yards.

16. If the ratio of two sides of a triangle is $2+\sqrt{3}$, and the included angle is 60°, find the other angles. . =105° and 15°.

MENSURATION OF SURFACES

243. The **length and breadth of a surface** are called its **dimensions.**

The **dimensions of a surface** are straight lines, and are therefore measured by some lineal unit—as an **inch,** a **foot,** a **yard** (Art. 218).

244. The unit of measure of surfaces, called the **unit of superficial measure,** or the **superficial unit,** is the square of the lineal unit.

Thus, if an inch is the lineal unit, a square inch—that is (Art. 44), a square whose sides are each one inch—is the

superficial unit; when the lineal unit is a foot, the superficial unit is a square foot; so if the former unit is a yard, the latter unit is a square yard.

245. The quantity of surface of a figure is called its **area** or **superficial content**, and is the number of superficial units it contains.

If a surface is = 20 square feet—that is, 20 times a square whose side is one foot—its content is = 20 square feet. The superficial content of any plane figure is not immediately found, however, by applying to it the superficial unit; as, for instance, a square foot, in the way that a line is measured by directly applying to it the lineal unit; for this method would be very tedious, and incapable of much accuracy; but the content can be computed by certain rules, given in the following problems, with the greatest precision, when the dimensions of the figure are accurately known.

246. It is necessary to make a distinction between some expressions relating to superficial measure that on first consideration appear to be equivalent. Thus, 2 square inches and 2 inches square are very different; for the former expression can mean only one square inch taken twice, whereas the latter means a square described on a line 2 inches long, so that its sides are each 2 inches, and its content, as will be found by Problem IV., is 4 square inches. So 10 square inches are very different from 10 inches square, which, according to the same problem, contains 100 square inches.

247. Problem I.—To find the area of a rectangle when its length and breadth are given.

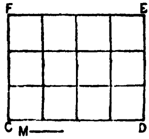
RULE.—Multiply the base by the perpendicular height, and the product is the area.

Let $\mathcal{R}$ = the area, b = the base, and h = the height; then $\mathcal{R} = bh$, or $L\mathcal{R} = Lb + Lh$.

Hence $$b = \frac{\mathcal{R}}{h}, \text{ and } h = \frac{\mathcal{R}}{b}.$$

If CE is a rectangle, and M the lineal unit—as, for example, a foot—and if the base CD contains M 4 times, and the side DE

contains it 3 times, the number of squares described on M that are contained in CE is just $=4\times 3=12$ square feet.



For, by laying off parts on CD, DE equal to M, and drawing through the points of division lines parallel to the sides of the figure, it will evidently be divided into 3 rows of squares, each containing 4 squares; that is, $3\times 4=12$ squares or square feet.

If the side CD contained $4\frac{1}{2}$ inches, and DE 3 inches, it would similarly be found that the number of square inches in the figure would be $=4\frac{1}{2}\times 3=\frac{9}{2}\times 3=13\frac{1}{2}$ square inches; or $4{\cdot}5\times 3=13{\cdot}5$ square inches; and whatever is the length of the sides, the area is found always in the same manner.

EXAMPLES.—1. How many square inches are in a leaf of paper which is $=10$ inches long and $6\frac{1}{2}$ broad?

$$\mathcal{A}=bh=10\times 6\frac{1}{2}=65 \text{ square inches.}$$

2. How many square feet are there in a table which is $=10$ feet 5 inches long and 3 feet 8 inches broad?

$$b=10 \text{ ft. } 5 \text{ in.}=10\tfrac{5}{12} \text{ ft., and } h=3 \text{ ft. } 8 \text{ in.}=3\tfrac{2}{3} \text{ ft.}$$
$$\mathcal{A}=bh=10\tfrac{5}{12}\times 3\tfrac{2}{3}=\tfrac{125}{12}\times\tfrac{11}{3}=\tfrac{1375}{36}=38{\cdot}194 \text{ sq. feet}$$
$$=38 \text{ sq. feet } 28 \text{ sq. inches.}$$

For $\qquad {\cdot}194\times 144=28.$

Or $b=125$ inches, and $h=44$ inches.

$$\mathcal{A}=bh=125\times 44=5500 \text{ sq. in.}=\tfrac{5500}{144}=38 \text{ sq. ft. } 28 \text{ sq. in.}$$

3. Find the number of square yards in the ceiling of a room which is $=24$ feet 9 inches long and 15 feet 6 inches broad.

$$b=24 \text{ ft. } 9 \text{ in.}=24{\cdot}75, \text{ and } h=15 \text{ ft. } 6 \text{ in.}=15{\cdot}5.$$
$$\mathcal{A}=bh=24{\cdot}75\times 15{\cdot}5=383{\cdot}625 \text{ sq. ft.}=383 \text{ sq. ft. } 90 \text{ sq. in.}$$
$$=42 \text{ sq. yd. } 5 \text{ sq. ft. } 90 \text{ sq. in.}$$

Or $L\mathcal{A}=Lb+Lh=L\,24{\cdot}75+L\,15{\cdot}5=1{\cdot}3935752+1{\cdot}1903317$
$=2{\cdot}5839069$; $\therefore\ \mathcal{A}=383{\cdot}625$ sq. ft. $=383$ sq. ft. 90 sq. in.

4. Required the area of a rectangular field whose length is $=24{\cdot}5$ chains and breadth $=8{\cdot}5$ chains.

$$\mathcal{A}=bh=24{\cdot}5\times 8{\cdot}5=208{\cdot}25 \text{ sq. chains}=20{\cdot}825 \text{ acres}$$
$$=20 \text{ acres } 3 \text{ roods } 12 \text{ sq. poles.}$$

Or $\mathcal{A}=2450\times 850$ sq. links $=2082500$ sq. links $=20{\cdot}825$ ac.; for 10 square chains or 100,000 square links make one acre.

EXERCISES

1. Required the number of square inches in a sheet of paper which is $=20$ inches long and 15 inches broad. $=300$ square inches.

2. How many square feet are in a rectangular table, the length of which is = 10 feet 6 inches and breadth = 4 feet 3 inches?

=44 square feet 90 square inches.

3. Required the number of square feet in a rectangular board whose length is = 12 feet 6 inches and breadth = 9 inches.

=9·375 square feet.

4. What is the number of square feet in a piece of carpeting = 14 feet 6 inches long and 4 feet 9 inches broad?

=68 square feet 126 square inches.

5. How many square yards of painting are there in the ceiling of a room whose length is = 24 feet and breadth = 15 feet 6 inches?

=41 square yards 3 square feet.

6. Required the number of square yards in a floor = $16\frac{1}{3}$ yards long and $10\frac{1}{4}$ yards broad. . . . . =168·3 square yards.

7. Find the area of a rectangle = 27 feet 3 inches long and 1 foot 6 inches broad. . . . =40 square feet 126 square inches.

8. Find the number of square yards in a rectangular piece of ground = 162 feet 3 inches long and 32 feet 5 inches broad.

=584 square yards 3 square feet 87 square inches.

9. What is the number of square yards of painting in the side and end walls of a room, the circumference of the room being = 103 feet 2 inches and height = 10 feet?

=114 square yards 5 square feet 96 square inches.

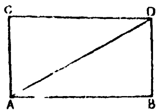
10. Find the area of a rectangular field whose length is = 33·4 chains and breadth = 7·5 chains. . =25 acres 8 square poles.

11. How many acres are in a rectangular field, the length of which is = 2750 links and breadth = 190 links?

=5 acres 36 square poles.

248. **Problem II. To find the area of a rectangle when the base and diagonal are given.**

RULE.—Find the other side (Art. 182), and then find the area by last problem.



Let the diagonal AD = d;

then $h = \sqrt{(d^2 - b^2)}$, or $h = \sqrt{(d+b)(d-b)}$.

Then $Æ = bh = b\sqrt{(d+b)(d-b)}$;

or $\mathrm{L}.Æ = \mathrm{L}b + \frac{1}{2}\{\mathrm{L}(d+b) + \mathrm{L}(d-b)\}$.

EXAMPLE.—Find the area of a rectangle whose base and diagonal are respectively = 100 and 125 feet.

$$h = \sqrt{(d^2 - b^2)} = \sqrt{(125^2 - 100^2)} = \sqrt{(15625 - 10000)}$$
$$= \sqrt{5625} = 75,$$

and $Æ = bh = 100 \times 75 = 7500$ sq. feet = 833 sq. yards 3 sq. feet.

Or $$L\text{Æ} = Lb + \tfrac{1}{2}\{L(d+b) + L(d-b)\}$$
$$= L\,100 + \tfrac{1}{2}\{L\,225 + L\,25\} = 2 + \tfrac{1}{2}(2\cdot3521825 + 1\cdot3979400)$$
$$= 2 + \tfrac{1}{2} \times 3\cdot7501225 = 3\cdot8750612 = L\,7500.$$
Hence $\text{Æ} = 7500$ sq. feet $= 833$ sq. yards 3 sq. feet.

EXERCISES

1. Find the area of a rectangle whose base and diagonal are respectively $= 21$ and 35 feet. . . . . $= 588$ square feet.

2. How many acres are contained in a rectangle whose diagonal is $= 320$ yards and base $= 240$ yards?
$= 10$ acres 1 rood $39\cdot28$ square poles.

3. Find the area of a rectangular field whose base and diagonal are $= 480$ and 720 links. . $= 2$ acres 2 roods $12\cdot152$ square poles.

249. Problem III.—To find the area of a rectangle when a side, or the diagonal, and the inclination of the diagonal and a side are given.

RULE.—Find the other side or the other two sides by Art. 180, and then find the area by Prob. I.

Let angle DAB $= v$ (fig. to Prob. II.), and let the other symbols remain as before.

1. To find h when b and v are given.

Taking natural tangents (Chambers's Math. Tables, Art. 29),
$$1 : \tan v = b : h, \text{ and } h = b \tan v;$$
then $$\text{Æ} = bh = b^2 \tan v \quad \ldots \quad \ldots \quad \ldots \quad [1],$$
or $$L\text{Æ} = 2Lb + L \tan v - 10 \quad \ldots \quad \ldots \quad \ldots \quad [2].$$

2. To find b and h when d and v are given.

By natural sines, $b = d \cos v$, and $h = d \sin v$;
then $$\text{Æ} = bh = d^2 \sin v \,.\, \cos v = \tfrac{1}{2}d^2 \sin 2v \text{ (Art. 204, } a) \quad \ldots \quad [3],$$
or $$L2\text{Æ} = 2Ld + L \sin 2v - 10 \quad \ldots \quad \ldots \quad [4].$$

Use the formula [1] or [2] when b and v are given, and [3] or [4] when d and v are given.

EXAMPLES.—1. Find the area of a rectangle, the base of which is 36 feet, and the inclination of the base and diagonal $= 32° 25'$.

By [1], $\text{Æ} = b^2 \tan v = 36^2 \tan 32° 25' = 36^2 \times \cdot6350274$
$= 1296 \times \cdot6350274 = 822\cdot9955$ sq. feet.

Or by [2], $L\text{Æ} = 2Lb + L \tan v - 10 = 2L\,36 + L \tan 32° 25' - 10$
$= 3\cdot1126050 + 9\cdot8027925 - 10 = 2\cdot9153975$;
hence $\text{Æ} = 822\cdot995$ sq. feet.

2. Find the area of a rectangular field, the diagonal of which is $= 475$ links, and its inclination to the longer side $= 36° 45'$.

Here $d=475$, and $v=36° 45'$, or $2v=73° 30'$.

By [3], $\mathcal{A}\!\!R=\frac{1}{2}d^2 \sin 2v=\frac{1}{2}\times 475^2\times \cdot 9588197$

$=\frac{1}{2}\times 225625\times \cdot 9588197=108166\cdot 85$ sq. links.

Or $\quad$ $\text{L } 2\mathcal{A}\!\!R=2\text{L}d+\text{L }\sin 2v-10=5\cdot 3533872+9\cdot 9817370-10$

$=5\cdot 3351242=\text{L } 216334$;

And $\quad$ $\mathcal{A}\!\!R=108167$ sq. links $=1$ ac. 13 sq. poles.

EXERCISES

1. What is the area of a rectangle, the base of which is $=14\cdot 4$ yards, and the inclination of the base and diagonal $=35° 40'$?

$=148\cdot 82$ square yards.

2. Find the number of acres in a rectangular field, its diagonal being $=470$ links, and its inclination to the longer side $=42° 45'$.

$=1$ acre $16\cdot 17$ square poles.

250. **Problem IV.—To find the area of a square when the side of it is given.**

RULE.—The square of the side is the area, or twice the logarithm of the side is the logarithm of the area.

Let $\quad$ $s=$ a side of a square,

then $\quad$ $\mathcal{A}\!\!R=s^2$, or $\text{L}\mathcal{A}\!\!R=2\text{L}s$.

EXAMPLES.—1. Required the area of a square whose side is $=25$ feet

$\mathcal{A}\!\!R=s^2=25^2=625$ sq. feet.

2. What is the area of a square whose side is $=425$ links?

$\mathcal{A}\!\!R=s^2=425^2=180625$ sq. links $=1$ ac. 3 ro. 9 sq. poles.

Or $\quad$ $\text{L}\mathcal{A}\!\!R=2\text{L}s=5\cdot 2567778=\text{L } 180625$.

EXERCISES

1. What is the area of a square, the side of which is $=11$ feet 6 inches? . . . . Area $=132\frac{1}{4}$ square feet.

2. What is the number of square yards in a square whose side is $=16$ feet 8 inches?

$=30$ square yards 7 square feet 112 square inches.

3. How many square yards are contained in a square, the side of which is $=31$ feet? . . . $=106$ square yards 7 square feet.

4. What is the area of a square whose side is $=12$ feet 6 inches?

$=156$ square feet 36 square inches.

5. Find the number of square yards in a square court, the side of which is $=160$ feet 6 inches. . . . $=2862\frac{1}{4}$ square yards.

6. How many acres are in a square field whose side is $=705$ links? . . . . . $=4$ acres 3 roods $35\cdot 24$ square poles.

7. Find the number of acres in a farm of a square form, the side of which is $=1$ mile. . . . . $=640$ acres.

251. **Problem V.—To find the area of a square when its diagonal is given.**

RULE.—The area is equal to half the square of its diagonal; or the logarithm of twice the area is equal to twice the logarithm of the diagonal.

For (Eucl. I. 47), $d^2=2s^2=2A\!\!R$,

or $$A\!\!R=\tfrac{1}{2}d^2, \text{ and } L\,2A\!\!R=2Ld.$$

EXAMPLES.—1. Find the area of a square whose diagonal is $=45$ feet.

$$A\!\!R=\tfrac{1}{2}d^2=\tfrac{1}{2}\times45^2=\tfrac{1}{2}\times2025=1012\tfrac{1}{2} \text{ sq. feet.}$$

2. Find the area of a square field, its diagonal being $=524$ links.

$$A\!\!R=\tfrac{1}{2}d^2=\tfrac{1}{2}\times524^2=\tfrac{1}{2}\times274576=137288 \text{ sq. links.}$$

Or $$L\,2A\!\!R=2\,Ld=5\cdot4386626;$$

hence $2A\!\!R=274576$, and $A\!\!R=137288$ sq. lk. $=1$ ac. 1 ro. 19·66 sq. po.

EXERCISES

1. What is the area of a square whose diagonal is $=25$ yards? $=312\tfrac{1}{2}$ square yards.

2. How many square yards in a courtyard, the diagonal of which is $=124$ feet? . . . . . $=854$ square yards 2 square feet.

3. How many acres are in a square field, the diagonal of which is $=786$ links? . . . . . $=3$ acres 14·2368 square poles.

252. **Problem VI.—To find the area of a parallelogram when its base and altitude are given.**

RULE.—Multiply the base by the height, and the product is $=$the area.

This is evident from Eucl. I. 35.

$$A\!\!R=bh, \text{ or } LA\!\!R=Lb+Lh.$$

EXAMPLES.—1. Find the area of a parallelogram whose base and perpendicular breadth are respectively $=24$ and 18 feet.

$$A\!\!R=bh=24\times18=432 \text{ sq. feet}=48 \text{ sq. yards.}$$

2. Find the area of a field in the form of a parallelogram whose base and height are respectively $=428$ and 369 links.

$$A\!\!R=bh=428\times369=157932 \text{ sq. links.}$$

Or $$LA\!\!R=Lb+Lh=2\cdot6314438+2\cdot5670264=5\cdot1984702;$$

hence $A\!\!R=157932$ sq. links $=1$ acre 2 roods 12·6912 sq. poles.

EXERCISES

1. What is the area of a parallelogram whose length is $=25$ feet 3 inches, and height $=13$ feet? . . . $=328\frac{1}{4}$ square feet.

2. How many square yards in a parallelogram whose length is $=45$ feet, and breadth $=24$ feet? . . $=120$ square yards.

3. What is the area of a parallelogram whose base and height are $=625$ and 240 links? . . . . . . $=1$ acre 2 roods.

4. How many acres are contained in a farm of the form of a parallelogram, the length and breadth of which are $=48$ and 28 chains? . . . . . $=134$ acres 1 rood 24 square poles.

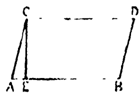
5. What is the area of a field in the form of a parallelogram whose length and perpendicular breadth are $=2102$ and 1284 links?
$=26$ acres 3 roods 38·3488 square poles.

The examples under Prob. I. are performed by the same rule, for parallelograms and rectangles whose lengths and perpendicular breadths are equal have equal areas. (Eucl. I. Prop. 35, 36.)

253. **Problem VII.—To find the area of a parallelogram, when there are given two of its adjacent sides and the contained angle.**

RULE.—The area is equal to the continued product of the two sides and the natural sine of the contained angle; or,

The logarithm of the area is equal to the sum of the logarithms of the two sides and of the sine of the contained angle diminished by 10.



Let $AB=b$, $AC=s$; if these and the contained angle i are given, then

$$\mathcal{A} = bs \sin i,$$

or $$L\mathcal{A} = Lb + Ls + L \sin i - 10.$$

For if $CE=h$, then $h = s \sin i$.

But (Art. 252), $$\mathcal{A} = bh = bs \sin i \quad \ldots \quad \ldots \quad [1],$$

from which $$L\mathcal{A} = Lb + Ls + L \sin i - 10 \quad \ldots \quad \ldots \quad [2].$$

EXAMPLE.—What is the number of square feet in a parallelogram whose length is $=42$ feet 6 inches, the adjacent side $=21$ feet 3 inches, and the contained angle $=53° 30'$?

Here $b = 42$ feet 6 inches $= 42·5$,
$s = 21$ feet 3 inches $= 21·25$.

Hence $\mathcal{A} = bs \sin i = 42·5 \times 21·25 \times ·8038569 = 725·983$ sq. feet.

Or $L\mathcal{A} = Lb + Ls + L \sin A - 10 = 1·6283889 + 1·3273589$
$+ 9·9051787 - 10 = 2·8609265$;

hence $\mathcal{A} = 725·983$ sq. feet.

Exercises

1. Find the area of a rhomboid, two of whose adjacent sides are $=18$ feet and 25 feet 6 inches, and the contained angle $=58°$.
$=389·254$ square feet.

2. What is the area of a field of the form of a parallelogram, two of whose adjacent sides are $=1200$ and 640 links, and the contained angle $=30°$? . . $=3$ acres 3 roods 14·4 square poles.

3. What is the area of a rhombus whose sides are $=42$ feet 6 inches, and the acute angles $=53° \ 20'$? $=1448·835$ square feet.

4. Find the area of a field in the form of a rhombus, the sides of which are $=420$ links, and the acute angles $=54° \ 30'$.
$=1$ acre 1 rood 29·776 square poles.

5. Find the area of a field in the form of a parallelogram, two of whose adjacent sides are $=750$ and 375 links, and the included angle $=60°$. . . . . $=2$ acres 1 rood 29·7 square poles.

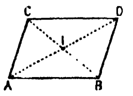
6. What is the area of a rhomboidal field whose sides are $=5070$ and 1040 links, and the acute angles $=30°$?
$=26$ acres 1 rood 18·24 square poles.

7. Find the area of a field in the form of a parallelogram, two of whose adjacent sides are $=1245$ and 864 links, and the contained angle $=65° \ 40'$. . . . $=9$ acres 3 roods 8·192 square poles.

254. Problem VIII.—Given the diagonals of a parallelogram and their inclination, to find its area.

Rule.—Half the continued product of the diagonals and the natural sine of the contained angle is equal to the area; or,

Add together the logarithms of the two diagonals and the logarithmic sine of the contained angle, and from their sum subtract 10; the remainder will be the logarithm of twice the area.



Let the diagonals AD, CB be denoted by d and d', and their inclination or angle DIB by i; then

$$A = \tfrac{1}{2}dd' \sin i \quad \ldots \quad [1],$$

or

$$L\,2A = Ld + Ld' + L \sin i - 10 \quad \ldots \quad [2].$$

Examples.—1. Find the area of a square whose diagonal is $=20$ feet 3 inches.

Here $d = d' = 20·25$ feet, and $i = 90°$;

hence $A = \tfrac{1}{2}dd' \sin 90° = \tfrac{1}{2}d^2 \times 1 = \tfrac{1}{2} \times 20·25^2 = 205·03125$ sq. feet.

Or $L\,2A = 2Ld + L \sin i - 10 = 2·6128500 + 10 - 10$
$= 2·6128500 = L\,410·0625$, and $A = 205·03125$ sq. feet.

2. What is the area of a parallelogram whose diagonals are $=1245$ and 1040 links, and the contained angle $=28° \ 45'$?

$$\begin{aligned}\text{Æ}&=\tfrac{1}{2}dd' \sin i=\tfrac{1}{2}\times 1245\times 1040\times \cdot 4809888\\ &=311392 \text{ sq. links,}\end{aligned}$$

$$\begin{aligned}\text{L } 2\text{Æ}&=\text{L}d+\text{L}d'+\text{L} \sin i-10=3\cdot 0951694+3\cdot 0170333\\ &\quad +9\cdot 6821349-10=5\cdot 7943376=\text{L } 622784,\end{aligned}$$

and $\text{Æ}=311392\cdot$ sq. links $=3$ acres $18\cdot 23$ sq. poles.

The diagonals AD, BC bisect each other; and hence AI = ID, and therefore (Eucl. I. 38) the triangles AIC, IDC are equal, and also AIB and IDB; but (Eucl. I. 34) the diagonal AD bisects the parallelogram, and therefore these four triangles are equal. But (Prob. X.) the area of the triangle DIB is $=\frac{1}{2}\text{BI}\,.\,\text{ID} \sin i$ $=\frac{1}{2}\cdot\frac{d}{2}\cdot\frac{d'}{2} \sin i$; hence the area of the four triangles is four times this quantity, or $\quad \text{Æ}=\frac{1}{2}dd' \sin i$;

hence $\quad \text{L } 2\text{Æ}=\text{L}d+\text{L}d'+\text{L} \sin i-10.$

Exercises

1. Find the number of square yards of pavement in a square court, the diagonal of which is = 36 feet 8 inches.

   = 74·69 square yards.

2. How many square yards are contained in a rectangular field whose diagonals are each = 96 feet, and contain an angle = 30°?

   = 256 square yards.

3. Find the area of a rhombus whose diagonals are 75 and 60 feet. . . . . = 250 square yards.

4. What is the number of square feet in a rhomboidal piece of ground, the diagonals of which are = 90 and 50 feet, and their inclination = 60°? . . . . . . = 1948·557 square feet.

5. How many acres are contained in a rhomboidal field whose diagonals are inclined at an angle of 36° 40′, and their lengths = 875 and 480 links? . . = 1 acre 1 rood 0·615 square pole.

255. Problem IX.—To find the area of a triangle when its base and altitude are given.

Rule.—The area is equal to half the product of the base and height; or,

The logarithm of twice the area is equal to the sum of the logarithms of the base and height.

Let b = the base, and h = the altitude;

then $\quad \text{Æ}=\frac{1}{2}bh$, and $\text{L } 2\text{Æ}=\text{L}b+\text{L}h.$

The truth of the rule is evident from the rule in Prob. VI., and the fact that a triangle is half of a parallelogram of the same base and altitude. (Eucl. I. 41.)

EXAMPLES.—1. Required the number of square feet in a triangle whose base is $=25$ feet and altitude $=36$ feet.

$$Ꭿ = \tfrac{1}{2}bh = \tfrac{1}{2} \times 25 \times 36 = 25 \times 18 = 450 \text{ sq. feet.}$$

2. Find the number of square yards in a triangular field, one of whose sides is $=240$ feet, and the perpendicular upon it from the opposite angle $=125$ feet.

$$Ꭿ = \tfrac{1}{2}bh = \tfrac{1}{2} \times 240 \times 125 = 15000 \text{ sq. feet} = 1666\tfrac{2}{3} \text{ sq. yards.}$$

3. How many acres are contained in a triangular field, one of its sides being $=1248$ links, and the perpendicular upon it from the opposite corner $=945$ links?

$$Ꭿ = \tfrac{1}{2}bh = \tfrac{1}{2} \times 1248 \times 945 = 589680 \text{ sq. links}$$
$$= 5 \text{ acres } 3 \text{ roods } 23{\cdot}488 \text{ sq. poles.}$$

Or $\quad L\,2Ꭿ = Lb \times Lh = 3{\cdot}0962146 + 2{\cdot}9754318$

$$= 6{\cdot}0716464 = L\ 1179360,$$

and $\quad Ꭿ = 589680 = 5 \text{ acres } 3 \text{ roods } 23{\cdot}488 \text{ sq. poles.}$

EXERCISES

1. What is the area of a triangle whose base is $=128$ feet, and height $=40$ feet? . . . . $=2560$ square feet.

2. What is the area of a triangle whose base is $=21$ feet 6 inches, and height $=14$ feet 6 inches? $=155$ square feet 126 square inches.

3. Required the number of square yards in a triangle, the base of which is $=49$ feet 6 inches, and the perpendicular $=42$ feet 9 inches. $=117{\cdot}5625$ square yards.

4. Find the area of a triangle whose base is $=60$ feet, and perpendicular $=10{\cdot}25$ feet. . . . . . $=307{\cdot}5$ square feet.

5. The length of one side of a triangular field is $=160$ yards, and the perpendicular on it from the opposite angle is $=140$ yards; required its area. . . . . . . $=11200$ square yards.

6. Required the area of a right-angled triangle whose base is $=225$ feet, and perpendicular $=160$ feet. $=2000$ square yards.

7. How many acres are contained in a triangular field whose base and height are $=1225$ and 425 links?

$=2$ acres 2 roods $16{\cdot}5$ square poles.

8. Required the area of a triangular field whose base is $=10$ chains, and height $=726{\cdot}184$ links.

$=3$ acres 2 roods $20{\cdot}9472$ square poles.

9. One side of a triangular court is $=97$ feet, and the perpendicular on it from the opposite angle is $=61$ feet; required the expense of paving it, at 2s. 3d. the square yard. . . . $=$£36, 19s. $7\tfrac{1}{2}$d.

256. Problem X.—To find the area of a triangle when two of its sides and the contained angle are given.

Rule.—The area is equal to half the product of the two sides and the natural sine of the contained angle; or,

The logarithm of twice the area is equal to the sum of the logarithms of the two sides, and of the sine of the contained angle, diminished by 10.

Or if b = base, s = a side, and i = included angle, then

$$\text{Æ} = \tfrac{1}{2}bs \,.\, \sin i.$$

Or $$\text{L}\, 2\text{Æ} = \text{L}b + \text{L}s + \text{L} \sin i - 10.$$

The rule is evident from Prob. VII., for the area of the triangle ABC is half that of the parallelogram AD.

Examples.—1. Find the area of a triangle which has two sides = 125 and 80 feet, and the contained angle = 28° 35′.

$$\text{Æ} = \tfrac{1}{2}bs \sin i = \tfrac{1}{2} \times 125 \times 80 \times \cdot 4784364 = 2392 \cdot 182 \text{ feet.}$$

2. How many acres are contained in a triangular field, two of whose sides are = 625 and 640 links, and the contained angle = 40° 25′?

$$\text{Æ} = \tfrac{1}{2}bs \sin i = \tfrac{1}{2} \times 625 \times 640 \times \cdot 6483414 = 129668 \text{ sq. links}$$
$$= 1 \text{ acre } 1 \text{ rood } 7 \cdot 469 \text{ sq. poles.}$$

Or $$\text{L}\, 2\text{Æ} = \text{L}b + \text{L}s + \text{L} \sin i - 10 = 2 \cdot 7958800 + 2 \cdot 8061800 + 9 \cdot 8118038 - 10 = 5 \cdot 4138638 = \text{L}\, 259336;$$

hence $$\text{Æ} = 129668 \cdot 5 = 1 \text{ acre } 1 \text{ rood } 7 \cdot 469 \text{ sq. poles.}$$

Exercises

1. Required the area in square yards of a triangle, two of whose sides are = 50 feet and 42 feet 6 inches, and the contained angle = 45°. . . . . = 83·47788 square yards.

2. How many square yards are contained in an isosceles triangle, the equal sides being = 50·49 feet, and the contained angle = 45°? . . . . = 100 square yards 1·29 square feet.

3. Find the area of a triangle, two of whose sides are = 80 and 90 feet, and the contained angle = 28° 57′ 18″. = 1742·84 square feet.

4. How many square yards are contained in a triangle, two of whose sides are = 42½ and 75 yards, and the included angle = 50°? = 1220·88 square yards.

5. How many square yards are contained in a triangular field, two of whose sides are = 204½ and 146¾ yards, and the contained angle = 30°? . . . . = 7498⅓ square yards.

6. How many acres are contained in a triangular field, two of

whose sides are = 1500 and 6400 links, and the contained angle $=39° \, 36'$? . . . . . = 30 acres 2 roods 15·416 square poles.

257. Problem XI.—To find the area of a triangle when the three sides are given.

RULE.—Find half the sum of the three sides, and also the difference between the half-sum and each of the sides; then find the continued product of the half-sum and the three differences, and the square root of the product will be the area; or,

Add together the logarithms of the half-sum and of the three differences, and half the sum will be the logarithm of the area.

Let the three sides be denoted by a, b, and c, and half their sum by s; that is, $s=\frac{1}{2}(a+b+c)$;

then $$\text{Æ}=\sqrt{\{s(s-a)(s-b)(s-c)\}}.$$

Or $$\text{L}\,\text{Æ}=\tfrac{1}{2}\{\text{L}s+\text{L}(s-a)+\text{L}(s-b)+\text{L}(s-c)\}.$$

EXAMPLE.—Find the area of a triangular field whose three sides are = 4236, 2544, and 3650 links.

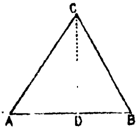
$$\begin{array}{llrl} \text{Let} & a = 4236 & \text{then} & s = 5215 \\ & b = 2544 & & s-a = 979 \\ & c = 3650 & & s-b = 2671 \\ & & & s-c = 1565 \\ & 2s = 10430 & & \end{array}$$

And $$\text{Æ}=\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{5215\times979\times2671\times1565}$$
$$=\sqrt{21341514430775}=4619687.$$

$$\begin{array}{lllr} \text{Or} & \text{L}s\ 5215, & . \ . \ . \ . \ . \ . & = 3{\cdot}7172543 \\ & \text{L}(s-a)\ 979, & . \ . \ . \ . \ . \ . & = 2{\cdot}9907827 \\ & \text{L}(s-b)\ 2671, & . \ . \ . \ . \ . \ . & = 3{\cdot}4266739 \\ & \text{L}(s-c)\ 1565, & . \ . \ . \ . \ . \ . & = 3{\cdot}1945143 \\ & & & 2)13{\cdot}3292252 \\ & \text{L}\,\text{Æ}\ 4619687, & . \ . \ . \ . \ . & = 6{\cdot}6646126 \end{array}$$

Hence area = 46 acres 31·5 sq. poles.

For the demonstration, see Eucl. II. 13, Note.



Or if the sides and angles of the triangle ABC be denoted, as in Art. 210, and CD $=h$, it is proved in Trigonometry (Art. 215, c) that

$$\sin A=\frac{2}{bc}\sqrt{\{s(s-a)s(-b)(s-c)\}}.$$

But if CD $=h$, then $h=b\sin A$,

and $\text{Æ}=\frac{1}{2}\text{AB}\,.\,\text{CD}=\frac{1}{2}ch=\frac{1}{2}bc\,.\,\sin A$;

hence, substituting for sin A the above value,

$$\text{Æ}=\sqrt{\{s(s-a)(s-b)(s-c)\}}.$$

Exercises

1. What is the number of square yards in a triangle whose sides are $=90$, 120, and 150 feet? . . . . . $=600$ square yards.
2. Find the area of a triangle whose sides are $=200$, 150, and 250 feet. . . . . $=15000$ square feet.
3. How many square yards are in a triangular field whose sides are $=126$, 247, and 296 yards? . . . $=15328$ square yards.
4. Find the number of square yards in a triangular court whose sides are $=45$, 42, and 39 yards. . . . $=756$ square yards.
5. What is the area of a triangular field whose sides are $=1000$, 1500, and 2000 links? . . $=7$ acres 1 rood 1·89 square poles.
6. Find the area of a triangular field whose sides are $=1200$, 1800, and 2400 links. . $=10$ acres 1 rood 33·128 square poles.
7. What is the area of a triangular field whose sides are $=2569$, 5025, and 4900 links? . . $=61$ acres 1 rood 39·68 square poles.

258. When the triangle is equilateral, if $b=$ its base, then $s=\frac{3}{2}b$, $s-a=\frac{1}{2}b$, $s-b=\frac{1}{2}b$, $s-c=\frac{1}{2}b$;

hence $$A=\sqrt{\tfrac{3}{2}b\left(\frac{b}{2}\right)^3}=\sqrt{3\left(\frac{b}{2}\right)^4}=\tfrac{1}{4}b^2\sqrt{3}=\cdot433b^2, \text{ nearly.}$$

And if $A=$ the area, $b=\frac{2}{3}\sqrt{A\sqrt{27}}=1\cdot52\sqrt{A}$, nearly.

Find the area of an equilateral triangle whose side is $=16$.

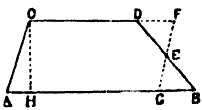
$$A=\cdot433b^2=\cdot433\times16^2=\cdot433\times256=110\cdot848.$$

Find the area of a field in the form of an equilateral triangle, the side of which is $=12\cdot5$ yards. . . . . $=67\cdot658$ square yards.

259. Problem XII.—To find the area of a trapezium.

Rule.—Multiply the sum of the parallel sides by the perpendicular distance between them, and half the product is the area; or,

Add together the logarithms of the sum of the parallel sides and of the perpendicular distance between them, and the sum is the logarithm of twice the area.



Let ABDC be a trapezium, and let the parallel sides AB, CD be denoted by b and s, and their perpendicular distance CH by h;

then $$A=\tfrac{1}{2}(b+s)h,$$

or $$\text{L } 2A=\text{L}(b+s)+\text{L}h.$$

Examples.—1. What is the area of a trapezium whose parallel sides are $=34$ and 26 feet, and their perpendicular distance $=25$ feet?

$$A=\tfrac{1}{2}(b+s)h=\tfrac{1}{2}(34+26)25=\tfrac{1}{2}\times60\times25=750 \text{ sq. feet.}$$

2. What is the area of a trapezium of which the parallel sides are $=1025$ and 836, and their perpendicular distance $=650$ links?

$$Æ = \tfrac{1}{2}(b+s)h = \tfrac{1}{2} \times 1861 \times 650 = 1861 \times 325 = 604825 \text{ sq. links.}$$

Or $\quad \text{L } 2Æ = \text{L } 1861 + \text{L } 650 = 3{\cdot}2697464 + 2{\cdot}8129134$

$$= 6{\cdot}0826598 = \text{L } 1209650;$$

and hence $\quad Æ = 604825$ sq. links $= 6$ acres $7{\cdot}72$ sq. poles.

If DB be bisected in E, and FG be drawn parallel to AC, then GB will be equal to DF, and triangle DEF to BEG (Euclid I. 15, 29, and 26); and hence AG is half the sum of AB and CD, and the area of the parallelogram AF is equal to that of the trapezium. But the area of AF is $=$ AG . CH; hence the rule is evident.

Exercises

1. Find the area of a trapezium whose parallel sides are $=30$ and 40 feet, and perpendicular breadth $=15$ feet. . $=525$ square feet.

2. How many square yards of paving are contained in a court of the form of a trapezium, the parallel sides being $=45$ and 63, and their perpendicular distance $=25$? . . . $=150$ square yards.

3. How many square feet are there in a trapezium whose parallel sides are $=643{\cdot}02$, $428{\cdot}48$, and perpendicular distance $342{\cdot}32$?
$=183397{\cdot}95$ square feet.

4. Find the area of a trapezium whose parallel sides are $=41$ and $24{\cdot}5$ feet, and perpendicular distance $=43$. $=1408{\cdot}25$ square feet.

5. How many square feet are contained in a trapezium whose parallel sides $=24$ feet and 36 feet 8 inches, and the perpendicular distance between them $=21$ feet? . . . $=637$ square feet.

6. How many square yards are contained in a trapezium whose parallel sides are $=54$ and 60 feet, and their perpendicular distance $=30$ feet? . . . . $=190$ square yards.

7. The parallel sides of a trapezium are $=45$ and 50 feet, and their perpendicular distance $=25$ feet; how many square yards does it contain? . . . . . . $=131$ square yards $8{\cdot}5$ square feet.

8. The parallel sides of a trapezoidal field are $=2482$ and 1644 links, and its perpendicular breadth is $=1030$ links; what is its area? . . . . . . $=21$ acres $39{\cdot}824$ square poles.

9. Find the area of a trapezoidal field whose parallel sides are $=1500$ and 2450 links, and breadth 770 links.
$=15$ acres $33{\cdot}2$ square poles.

10. What is the area of a trapezoidal field whose parallel sides are $=750$ and 975 links, and perpendicular breadth $=700$ links?
$=6$ acres 6 square poles.

260. Problem XIII.—To find the area of any quadrilateral when its diagonals and their inclination are given.

Rule.—Multiply half the product of the diagonals by the natural sine of their inclination; or,

Add together the logarithms of the two diagonals and of the sine of the contained angle; from the sum subtract 10, and the remainder will be the logarithm of twice the area.

Let d and d' be the diagonals, and i the included angle;

then $$A = \tfrac{1}{2} dd' \sin i,$$

or $$L\,2A = Ld + Ld' + L\sin i - 10.$$

Example.—Find the number of square yards in a quadrilateral whose diagonals are $=420$ and 325 feet, and the contained angle $=40°\ 25'$.

$$A = \tfrac{1}{2} dd' \sin i = \tfrac{1}{2} \times 420 \times 325 \times \cdot 6483414 = 44249\cdot 3 \text{ sq. feet.}$$

Or $$L\,2A = Ld + Ld' + L\sin i - 10 = 2\cdot 6232493 + 2\cdot 5118834$$
$$+ 9\cdot 8118038 - 10 = 4\cdot 9469365 = L\,88498\cdot 6;$$

and $A = 44249\cdot 3$ sq. feet $= 4916$ sq. yards $5\cdot 3$ sq. feet.

Let ABDC be the given quadrilateral, and AD, BC its diagonals. Through its angular points draw lines parallel to its diagonals, and they will form the parallelogram EG, which is evidently double of the quadrilateral; for the parallelogram IG is double of the triangle AIB (Eucl. I. 34), and so of the other four parallelograms that compose EG; also angle F = DIB = i.

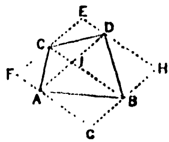
Now, the area of EG (Article 253) is = EF . FG . sin F;

hence $\quad$ ABDC $= \tfrac{1}{2}$AD . CB . sin i,

for $\quad$ AD = EF, and BC = FG (Eucl. I. 34).

Hence $$A = \tfrac{1}{2} dd' \sin i;$$

or $$L\,2A = Ld + Ld' + L\sin i - 10.$$



Exercises

1. What is the area of a quadrilateral whose diagonals are $=50$ and 40 feet, and the included angle $=60°$? . . . . $=866\cdot 0254$ square feet.

2. How many square yards are contained in a court, the diagonals of which are $=180$ and 210 feet, and the contained angle $30°$? $=1050$ square yards.

3. Find the number of acres contained in a quadrilateral field whose diagonals are $=1500$ and 2000 links, and their inclination $=48°$. . . . . $=11$ acres $23\cdot 55$ square poles.

4. How many acres are contained in a quadrilateral field whose diagonals are $=30$ and 40 chains, and the contained angle $=60°$? $=51$ acres 3 roods $33\cdot 84$ square poles.

261. Problem XIV.—To find the area of a quadrilateral that can be inscribed in a circle; that is, one whose opposite angles are supplementary.

RULE.—From half the sum of the four sides subtract each side separately; find the continued product of the four remainders, and the square root of this product is the area; or,

Add together the logarithms of the four remainders, and half their sum is the logarithm of the area.

Let a, b, c, d denote the sides, and s half their sum; then
$$s=\tfrac{1}{2}(a+b+c+d),$$
and
$$\text{Æ}=\sqrt{\{(s-a)(s-b)(s-c)(s-d)\}};$$
or
$$\text{L, Æ}=\tfrac{1}{2}\{\text{L}(s-a)+\text{L}(s-b)+\text{L}(s-c)+\text{L}(s-d)\}.$$

EXAMPLE.—What is the area of a quadrilateral inscribed in a circle whose four sides are $=24$, 26, 28, and 30 chains?

$$\begin{array}{lcr}
a & = & 24 \\
b & = & 26 \\
c & = & 28 \\
d & = & 30 \\
 & & 2)\overline{108} \\
s & = & 54
\end{array}
\qquad\qquad
\begin{array}{lcr}
s-a & = & 30 \\
s-b & = & 28 \\
s-c & = & 26 \\
s-d & = & 24
\end{array}$$

and area $=\sqrt{30\times28\times26\times24}=\sqrt{524160}=723\cdot989$ sq. chains
$=72$ acres 1 rood $23\cdot824$ sq. poles.

EXERCISES

1. The four sides of a quadrilateral inscribed in a circle are $=40$, 75, 55, and 60 feet; required its area. . $=3146\cdot4265$ square feet.

2. How many acres are contained in a quadrilateral field whose opposite angles are supplementary, its sides being $=600$, 650, 700, and 750 links? . . . $=4$ acres 2 roods $3\cdot988$ square poles.

262. Problem XV.—To find the area of a quadrilateral when one of its diagonals and the perpendiculars on it from the opposite angles are given.

RULE.—Multiply the diagonal by the sum of the perpendiculars, and half the product is the area; or,

Add the logarithms of the diagonal and of the sum of the perpendiculars; the sum will be the logarithm of twice the area.

Let d be the diagonal, and p, p' the two perpendiculars on it; then
$$\text{Æ}=\tfrac{1}{2}d(p+p');$$
or
$$\text{L }2\text{Æ}=\text{L}d+\text{L}(p+p').$$

EXAMPLE.—How many acres are contained in a quadrilateral field, a diagonal of which is $=1245$ links, and the perpendiculars on it from the opposite angles $=675$ and 450 links?

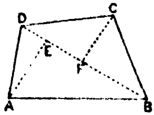
$$\mathcal{A}=\tfrac{1}{2}d(p+p')=\tfrac{1}{2}\times 1245(675+450)=\tfrac{1}{2}\times 1245\times 1125=700312{\cdot}5 \text{ sq. lk.}$$

Or $\quad L\,2\mathcal{A}=Ld+L(p+p')=L\,1245+L\,1125$

$$=3{\cdot}0951694+3{\cdot}0511525=6{\cdot}1463219=L\,1400625,$$

and $\quad \mathcal{A}=700312{\cdot}5$ sq. links $=7$ acres $0{\cdot}5$ sq. pole.

Let ABCD be the quadrilateral, DB its diagonal, and CF, AE the two perpendiculars on it; then (Art. 255),



triangle $ADB=DB\times\tfrac{1}{2}AE$,

and triangle $DCB=DB\times\tfrac{1}{2}CF$;

hence $ABCD=\tfrac{1}{2}\times DB(AE+CF)$.

EXERCISES

1. How many square yards are contained in a quadrilateral, one of its diagonals being $=60$ yards, and the perpendiculars upon it $=12{\cdot}6$ and $11{\cdot}4$ yards? . . . . . . $=720$ square yards.

2. Find the area of a quadrilateral, one of its diagonals and the perpendiculars on it being respectively $=168$, 42, and 56 feet. $=8232$ square feet.

3. Find the number of square yards in a quadrilateral which has a diagonal $=70$ feet, and the perpendiculars upon it $=28$ and 35 feet. $=245$ square yards.

4. How many square yards in a quadrilateral, one of its diagonals being $=40$ feet, and the perpendiculars on it $=21{\cdot}6$ and 13 feet? $=76$ square yards 8 square feet.

5. Find the number of acres in a quadrilateral field, one of whose diagonals is $=4025$, and the perpendiculars on it $=1225$ and 1505 links. . . . . . . $=54$ acres 3 roods $30{\cdot}6$ square poles.

263. **Problem XVI.—To find the area of a quadrilateral when the four sides and the inclination of the diagonals are given.**

RULE.—Add the squares of each pair of opposite sides together; subtract the less sum from the greater; then multiply the difference by the tangent of the angle formed by the diagonals, and one-fourth of this product is the area; or,

Add the logarithm of the remainder to the logarithmic tangent of the inclination of the diagonals, and the sum diminished by 10 will be the logarithm of four times the area.

Let the sides be denoted by a, b, c, d, and the inclination of the

diagonals by i; then if a and d are the opposite sides whose squares exceed those of the other two,

$$\text{Æ} = \{(a^2+d^2)-(b^2+c^2)\} \,.\, \tfrac{1}{4} \tan i;$$

or
$$\text{L } 4\text{Æ} = \text{L}\{(a^2+d^2)-(b^2+c^2)\} + \text{L } \tan i - 10.$$

EXAMPLE.—Find the area of a quadrilateral figure, two of whose opposite sides are = 10 and 12 chains, the other two sides = 9 and 18, and the inclination of the diagonals = 84° 25′.

$$a^2+d^2 = 81+324 = 405,\ b^2+c^2 = 100+144 = 244;$$

hence
$$a^2+d^2-(b^2+c^2) = 405-244 = 161,$$

and
$$\text{Æ} = \tfrac{1}{4} \times 161 \tan 84° 25' = \tfrac{1}{4} \times 161 \times 10{\cdot}229428 = \tfrac{1}{4} \times 1646{\cdot}9379$$
$$= 411{\cdot}7345.$$

Or
$$\text{L } 4\text{Æ} = \text{L } 161 + \text{L } \tan 84° 25' - 10 = 2{\cdot}2068259 + 11{\cdot}0098513 - 10$$
$$= 3{\cdot}2166772 = \text{L } 1646{\cdot}94;$$

and
$$\text{Æ} = 411{\cdot}735 \text{ sq. chains} = 41 \text{ acres } 27{\cdot}76 \text{ sq. poles.}$$

EXERCISES

1. Find the area of a quadrilateral, two of whose opposite sides are = 500 and 400 links, the other two = 450 and 350 links, and the inclination of the diagonals = 80°. . = 1 acre 32·8 square poles.

2. What is the area of a quadrilateral field, two of whose opposite sides are = 450 and 900 links; the other two = 600 and 500 links; and the inclination of its diagonals = 78° 40′?

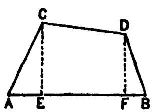
= 5 acres 3·29 square poles.

264. **Problem XVII.—To find the area of any quadrilateral.**

RULE.—Divide the quadrilateral into triangles, or triangles and trapeziums, calculate the areas of these component figures by former rules, and the sum of these partial areas will be the area of the whole figure.

EXERCISES

1. Find the area of the quadrilateral ABDC, the lines AE, EF, FB being = 40, 64, and 28 feet, and the perpendiculars CE, DF = 50 and 40 feet.

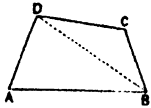


Calculate the area of AEC by Prob. IX., that of CEFD by Prob. XII., and of DFB also by Prob. IX.; and the sum is = 4440 square feet, the area of ABDC.

2. What is the number of acres in a quadrilateral field ABDC, the distances AE, AF, AB being = 420, 1160, and 1380 links, and the perpendiculars CE, DF = 840 and 680 links?

= 8 acres 21·76 square poles.

3. Given the four sides AB, BC, CD, DA of a quadrilateral field = 650, 425, 470, and 580; the angle A = 85° 40′ and C = 112° 15′; to find its area.



Find the area of the triangle ADB by Prob. X., and also that of DCB, and the sum of their areas is the area required.

= 2 acres 3 roods 8·6 square poles.

4. Find the area of the quadrilateral ABCD, its sides AB, BC, CD, and AD being = 720, 540, 520, and 600 links, and the angles A and C = 72° 40′ and 102° 20′. = 3 acres 1 rood 29·36 square poles.

5. Required the area of the quadrilateral figure ABCD, the sides AB, BC, CD, and AD being = 1600, 1150, 1500, and 1650 links, and the diagonal AC = 1800 links.

Find the areas of the two triangles ABC and ACD separately by Prob. XI., and their sum will be the area of the quadrilateral.

= 20 acres 2 roods 24·2 square poles.

6. Find the area of the quadrilateral ABCD from these data:—

AB = 548 links,	CD = 751 links,
BC = 715 „	AD = 821 „

and the diagonal AC = 967 links.

= 4 acres 3 roods 27·67 square poles.

7. Find the area of the quadrilateral field ABCD, having given

AB = 205 links,	CD = 1000 links,
BC = 700 „	AD = 600 „

and the diagonal AC = 800 links. = 3 acres 10·37 square poles.

8. How many acres are contained in a quadrilateral field, from these measurements:—

AB = 15 chains,	CD = 14 chains,
BC = 13 „	AD = 12 „

and the diagonal AC = 16 chains?

= 17 acres 1 rood 0·396 square pole.

9. Find the area of the quadrilateral field ABCD, having given AB = 2000, AD = 1500, and AC the diagonal = 2390 links; and each of the angles BAC, DAC = 30°. = 20 acres 3 roods 26 square poles.

In this example, find the areas of the triangles BAC, CAD separately by Prob. X.

10. Find the area of the quadrilateral ABCD from these measurements:—

AB = 468 links,	Angle ABC = 73°,
BC = 395 „ ,	„ BCD = 87° 30′;
CD = 410 „ ,	

Find the side DB and angle B in triangle DCB by Trigo-

nometry; then angle ABD = ABC − DBC is known. Hence the areas of the two triangles ABD, DBC can now be found by Prob. X., as in the preceding ninth exercise.

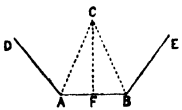
= 1 acre 1 rood 19·6 square poles.

11. Find the area of the quadrilateral field ABCD, the four sides AB, BC, CD, DA being respectively = 750, 700, 650, and 600 links, and the angle A = 83° 30′.

In the triangle ADB find the angle at D or B, and then the side DB (Art. 187); next find the area of triangle ADB by Prob. X., and that of DBC by Prob. XI. . = 4 acres 1 rood 39·4 square poles.

265. Problem XVIII.—To find the interior and central angle of any regular polygon.

RULE.—From double the number of the sides of the polygon subtract 4; multiply the remainder by 90; divide the product by the number of sides, and the quotient is the number of degrees in the interior angle.



Divide four right angles, or 360°, by the number of sides, and the quotient is the central angle.

Let i = one of the interior angles DAB, c = one of the central angles at C, and n = the number of sides of the polygon.

Then $$i = \frac{90^\circ}{n}(2n-4) = \frac{180^\circ}{n}(n-2), \text{ and } c = \frac{360^\circ}{n}.$$

EXAMPLE.—Find the interior and central angles of a regular pentagon.

$$i = \frac{180^\circ}{n}(n-2) = \frac{180^\circ}{5}(5-2) = 108^\circ;\ c = \frac{360^\circ}{n} = \frac{360^\circ}{5} = 72^\circ.$$

EXERCISES

1. Find the interior and central angles of a regular hexagon.

Interior = 120°, and central = 60°.

2. What is the number of degrees contained in the interior and central angles of a regular heptagon?

Interior = 128° 34′ 17$\frac{1}{7}$″, and central = 51° 25′ 42$\frac{6}{7}$″.

3. Find the number of degrees in the interior and central angles of a dodecagon. . . . . Interior = 150°, and central = 30°.

266. Problem XIX.—To find the apothem of a regular polygon, its side being given.

RULE.—Multiply half the side of the polygon by the tangent of half its interior angle, and the product is the apothem; or,

Add the logarithm of half the side to the logarithmic tangent of half the interior angle, and the sum, diminished by 10, is the logarithm of the apothem.

Let p = the apothem CF,
s = one of the sides AB,
i = the interior angle DAB;

then $p = \frac{1}{2}s \,.\, \tan \frac{1}{2}i$, and $Lp = L\,\frac{1}{2}s + L \tan \frac{1}{2}i - 10$.

In the right-angled triangle AFC (fig. to Prob. XVIII.)

$$1 : \tan CAF = AF : FC = \tfrac{1}{2}s : p;$$

therefore $p = \frac{1}{2}s \,.\, \tan \frac{1}{2}i$, or $Lp = L\,\frac{1}{2}s + L \tan \frac{1}{2}i - 10$.

EXAMPLE.—Find the apothem of a regular hexagon whose side is = 120.

$$p = \tfrac{1}{2}s \,.\, \tan \tfrac{1}{2}i = 60 \tan 60° = 60 \times 1{\cdot}7320508 = 103{\cdot}923048.$$

Or $$Lp = L\,60 + L \tan 60° - 10$$
$$= 1{\cdot}7781513 + 10{\cdot}2385606 - 10 = 2{\cdot}0167119;$$

hence $$p = 103{\cdot}923.$$

EXERCISES

1. Find the apothem of a regular pentagon whose side is = 10. = 6·8810.

2. What is the length of the apothem of a regular heptagon whose side is 80? . . . . = 83·0608

267. **Problem XX.—Given a side of a regular polygon and its apothem, to find its area.**

RULE.—Find the continued product of the side, the number of sides, and the apothem and half this product is the area; or,

Add together the logarithms of the side, the number of sides, and the apothem, and the sum is the logarithm of twice the area.

Let s, n, and p denote the same quantities as in the two preceding problems;

then $\mathcal{A}\!\!R = \frac{1}{2}nps$, or $L\,2\mathcal{A}\!\!R = Ls + Ln + Lp$.

The area of the triangle ABC (fig. to Prob. XVIII.) is = $\frac{1}{2}$AB . FC = $\frac{1}{2}sp$. And there are as many triangles equal to ABC as the polygon has sides; hence its area is = $n \,.\, \frac{1}{2}sp = \frac{1}{2}nsp$.

EXAMPLE.—The side of a regular hexagon is = 10, and its apothem is = 8·66; what is its area?

$$\mathcal{A}\!\!R = \tfrac{1}{2}nsp = \tfrac{1}{2} \times 6 \times 10 \times 8{\cdot}66 = 259{\cdot}8.$$

EXERCISES

1. The side of a regular pentagon is = 5, and its apothem is = 3·44; what is the area? . . . . = 43.

2. Find the area of a park in the form of a regular octagon whose side is = 12 chains, and apothem = 14·485 chains.

= 69 acres 2 roods 4·48 square poles.

268. Problem XXI.—To find the area of a regular polygon when only a side is given.

RULE.—Find the interior angle, and then the apothem by Prob. XVIII. and XIX.; then find the area by last problem.

Or, by substituting the value of p in the expression for the area, we have

$$\text{Æ} = n\left(\frac{s}{2}\right)^2 \tan \tfrac{1}{2}i = n\left(\frac{s}{2}\right)^2 \cot \tfrac{1}{2}c.$$

Whence $$\text{LÆ} = \text{L}n + 2\text{L}\ \tfrac{1}{2}s + \text{L}\tan \tfrac{1}{2}i - 10.$$

EXAMPLE.—Find the area of a regular hexagon whose side is = 10.

$$\text{LÆ} = \text{L}n + 2\ \text{L}\tfrac{1}{2}s + \text{L}\tan \tfrac{1}{2}i - 10 = \text{L}\,6 + 2\text{L}\,5 + \text{L}\tan 60° - 10 = \cdot 7781513$$
$$+ 1\cdot 3979400 + 10\cdot 2385606 - 10 = 2\cdot 4146519;$$

hence $$\text{Æ} = 259\cdot 808.$$

EXERCISES

1. Find the area of a regular pentagon whose side is = 30 feet.
= 1548·4275 square feet.

2. What is the number of square yards in a regular heptagon whose side is = 20 yards? . . . = 1453·564 square yards.

3. How many acres are contained in a field of the form of a regular octagon whose side is = 5 chains?
= 12 acres 11·37 square poles.

By means of the preceding problems regarding regular polygons, the following Table may be constructed:—

Name of Polygon	No. of Sides	Apothem when Side = 1	Area when Side = 1	Interior Angle	Central Angle
Triangle	3	0·2886751	0·4330127	60° 0′	120° 0′
Square	4	0·5	1·	90	90
Pentagon	5	0·6881910	1·7204774	108	72
Hexagon	6	0·8660254	2·5980762	120	60
Heptagon	7	1·0382607	3·6339124	128 $34\frac{2}{7}$	51 $25\frac{5}{7}$
Octagon	8	1·2071068	4·8284271	135	45
Nonagon	9	1·3737387	6·1818242	140	40
Decagon	10	1·5388418	7·6942088	144	36
Undecagon	11	1·7028436	9·3656399	147 $16\frac{4}{11}$	32 $43\frac{7}{11}$
Dodecagon	12	1·8660254	11·1961524	150	30

269. Problem XXII.—To find the area of a regular polygon of not more than twelve sides, by the preceding Table.

RULE.—Multiply the tabular area for the corresponding polygon, whose side is $=1$, by the square of the side of the given polygon, and the product will be the required area; or,

To the logarithm of the tabular area add twice the logarithm of the given side, and the sum is the logarithm of the required area.

Let $\quad\quad A\!\!\!R' =$ the tabular area;

then $\quad\quad A\!\!\!R = s^2 A\!\!\!R'$,

and $\quad\quad LA\!\!\!R = 2\ Ls + LA\!\!\!R'$.

The areas of similar polygons are to one another as the squares of their sides (Eucl. VI. 20); hence $A\!\!\!R' : A\!\!\!R = 1^2 : s^2$; therefore $A\!\!\!R = s^2 A\!\!\!R'$.

EXAMPLE.—Find the area of a regular heptagon whose side is $= 15$ feet.

$$A\!\!\!R = s^2 A\!\!\!R' = 15^2 \times 3{\cdot}6339 = 225 \times 3{\cdot}6339 = 817{\cdot}6275 \text{ sq. feet.}$$

Or $\quad LA\!\!\!R = L\ 3{\cdot}6339 + 2\ L\ 15 = 0{\cdot}5603730 + 2{\cdot}3521826 = 2{\cdot}9125556$, and

$$A\!\!\!R = 817{\cdot}628 \text{ square feet.}$$

EXERCISES

1. How many square yards are contained in a regular hexagon whose side is $= 50$ feet? . . . . $= 721{\cdot}688$ square yards.
2. Required the area of a regular pentagon whose side is $= 50$ feet. . . . . $= 4301{\cdot}1935$ square feet.
3. What is the area of a regular pentagon whose side is $= 45$ feet? . . . . $= 3483{\cdot}9667$ square feet.
4. What is the area of a regular hexagon whose side is $= 40$ yards? . . . . $= 4156{\cdot}92192$ square yards.
5. Find the area of a pentagon whose side is $= 60$ feet. $= 6193{\cdot}71864$ square feet.
6. Find the area of a regular octagon whose side is $= 80$ yards. $= 30901{\cdot}93$ square yards.
7. How many square yards are contained in a regular decagon whose side is $= 12$ feet? . . . . $= 123{\cdot}10734$ square yards.
8. How many acres are contained in a farm of the form of a regular decagon whose side is $= 2050$ links? $= 323$ acres 1 rood $15{\cdot}86$ square poles.

270. Problem XXIII.—Given the diameter of a circle, to find the circumference.

RULE.—Multiply the diameter by $3{\cdot}1416$, and the product is the circumference; or,

Add the constant logarithm 0·4971509 to that of the diameter, and the sum is the logarithm of the circumference.

Let d, r, and c denote the diameter, radius, and circumference of a circle, and $\pi = 3·1416$;

then $$c = 3·1416d = \pi d, \text{ or } c = 2 \times 3·1416r = 2\pi r.$$

Or $$Lc = 0·4971509 + Ld.$$

When greater accuracy is required, the number 3·14159 may be used instead of 3·1416; or, for still greater accuracy, the number 3·1415926536. This number is nearly the length of the circumference of a circle whose diameter is 1. When less accuracy is required, the ratio of 1 to $3\frac{1}{7}$, or 7 to 22, or of 113 to 355, may be taken for the ratio of the diameter to the circumference of a circle.

EXAMPLE.—Required the circumference of a circle whose diameter is = 25 feet.

$$c = \pi d = 3·1416 \times 25 = 78·54 \text{ feet.}$$

EXERCISES

1. Find the circumference of a circle whose diameter is = 28 feet. = 87·9648 feet.

2. What is the circumference of a circle whose diameter is = 24 feet 3 inches? . . . . = 76 feet 2·2 inches.

3. Find the circumference of a circle whose diameter is = 120 feet. . . . . = 376·992 feet.

4. If the mean diameter of the earth be = 7912 miles, what is its mean circumference? . . . . = 24856 miles.

271. Problem XXIV.—Given the circumference of a circle, to find the diameter.

RULE.—Divide the circumference by 3·1416, or multiply it by ·3183, and the result is the diameter; or,

From the logarithm of the circumference subtract the constant logarithm 0·4971509, and the remainder is the logarithm of the diameter.

For $$d = \frac{c}{\pi} = \frac{c}{3·1416}, \text{ or } d = ·3183c.$$

EXAMPLE.—Find the diameter of a circle whose circumference is = 45 feet.

$$d = ·3183c = ·3183 \times 45 = 14·3235 = 14 \text{ feet } 3·882 \text{ inches.}$$

EXERCISES

1. Find the diameter of a circle whose circumference is = 177 feet. . . . . = 56·3391 feet.

2. What is the diameter of a circle whose circumference is $=32$ feet? . . . . $=10\cdot1856$ feet.

3. What is the diameter of a wheel whose rim is $=11$ feet? $=3\cdot5013$ feet.

4. What is the diameter of a circular pond whose circumference is $=200$ feet? . . . . $=63\cdot66$ feet.

5. What is the diameter of a circular plantation whose circumference is $=1250$ yards? . . . . . . $=397\cdot875$ yards.

272. **Problem XXV.—To find the area of a circle when the diameter and circumference are given.**

RULE.—Multiply the diameter by the circumference, and one-fourth of the product will be the area; or,

Add the logarithm of the diameter to that of the circumference, and the sum is the logarithm of four times the area.

Or $\quad A\!\!R=\frac{1}{4}cd$, and $LA\!\!R=Ld+Lc-\cdot6020600$.

EXAMPLE.—Find the area of a circle whose diameter is $=12\cdot732$ feet, and circumference $=40$ feet.

$$A\!\!R=\tfrac{1}{4}cd=\tfrac{1}{4}\times40\times12\cdot732=127\cdot32 \text{ sq. feet.}$$

EXERCISES

1. Find the area of a circle whose diameter is $=21$, and circumference $=65\cdot973$. . . . . $=346\cdot358$.

2. What is the area of a circle whose diameter is $=20$, and circumference $=62\cdot8318$? . . . . $=314\cdot159$.

3. Find the area of a circle whose diameter is $=226$ links, and circumference $=710$. . . . . . . $=40115$ square links.

4. Find the area of a circular plantation whose diameter is $=640$ links, and circumference $2010\cdot6$. . $=3$ acres $34\cdot71$ square poles.

273. **Problem XXVI.—To find the area of a circle when the diameter is given.**

RULE.—Multiply the square of the diameter by $\cdot7854$, or the square of the radius by $3\cdot1416$, and the product is the area.

$$A\!\!R=\cdot7854d^2=\tfrac{1}{4}\pi d^2, \text{ or } A\!\!R=3\cdot1416r^2=\pi r^2;$$

and
$$d=\sqrt{\frac{A\!\!R}{\cdot7854}}=\sqrt{\frac{4A\!\!R}{\pi}}.$$

EXAMPLE.—What is the area of a circle whose diameter is $=120$ feet?

$$A\!\!R=\cdot7854d^2=\cdot7854\times120^2=\cdot7854\times14400=11309\cdot76 \text{ sq. feet.}$$

Or $\quad A\!\!R=\pi r^2=3\cdot1416\times60^2=3\cdot1416\times3600=11309\cdot76$ sq. feet.

For by last problem, $A\!\!R=\frac{1}{4}cd$; and by Prob. XXIII., $c=3\cdot1416d$; therefore $A\!\!R=\frac{1}{4}\times3\cdot1416dd=\cdot7854d^2$.

And since $d=2r$, or $d^2=4r^2$; therefore $A\!\!R=3\cdot1416r^2=\pi r^2$.

EXERCISES

1. What is the area of a circle whose diameter is = 60 feet? = 2827·44 square feet
2. Find the area of a circle whose diameter is = 35 feet. = 962·115 square feet.
3. Find the area of a circle whose diameter is = 397·885 feet. = 124338·4 square feet.
4. What is the area of a circle whose diameter is = 50 yards? = 1963·5 square yards.
5. *Find the area of a circle whose diameter is = 450 links.* = 1 acre 2 roods 14·5 square poles.
6. How many square yards are contained in a circle whose diameter is = 350 feet? . . . = 10690·16 square yards.

274. **Problem XXVII.—To find the area of a circle when the circumference is given.**

RULE.—Multiply the square of the circumference by ·0795775, and the product is the area.

Or $$A\!\!R=\cdot0795775c^2=\frac{c^2}{4\pi}.$$

If no great accuracy be required, $A\!\!R=\cdot0796c^2$.

EXAMPLE.—What is the area of a circle whose circumference is = 20 feet 3 inches?

$$A\!\!R=\cdot0795775c^2=\cdot0795775\times410\tfrac{1}{16}=32\cdot63 \text{ square feet.}$$

By the former problems, $A\!\!R=\frac{1}{4}\pi d^2$, and $c=\pi d$, hence $d=\frac{c}{\pi}$; and therefore $A\!\!R=\frac{1}{4}\pi\times\frac{c^2}{\pi^2}=\frac{c^2}{4\pi}=\frac{c^2}{4\times3\cdot14159}=\frac{c^2}{12\cdot56636}=\cdot0795775c^2$.

EXERCISES

1. Find the area of a circle whose circumference is = 25 feet. = 49·73594 square feet.
2. What is the area of a circle whose circumference is = 15·708? = 19·635.
3. Find the area of a circular field whose circumference is = 50 chains. . . . . . = 19 acres 3 roods 23·1 square poles.
4. Find the area of a circle whose circumference is = 200 yards. = 3183·1 square yards.

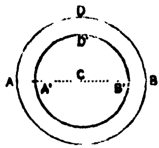
5. What is the number of square yards in a circle whose circumference is 25·1328 yards? . . . . =50·2656 square yards.

275. Problem XXVIII.—To find the area of a circular annulus or ring.

RULE.—Multiply the sum of the diameters by their difference, and this product by ·7854, and the result will be the area; or,

Multiply the sum of the circumferences by their difference, and this product by ·0795775, and the result will be the area; or,

Multiply the sum of the circumferences by the difference of the diameters, and one-fourth of the product will be the area.



Let d and d' be the diameters AB, A'B' of the greater and less circle, and c, c' their circumferences; then

$$Æ = ·7854(d+d')(d-d')$$
$$Æ = ·0796(c+c')(c-c')$$
$$Æ = \tfrac{1}{4}(c+c')(d-d').$$

EXAMPLES.—1. Find the area of a circular annulus contained between two concentric circles whose diameters are $=10$ and 12.

$$Æ = ·7854(10+12)(12-10) = ·7854 \times 22 \times 2 = 34·5576.$$

2. Find the area of a circular annulus, the circumferences of the containing circles being $=30$ and 40.

$$Æ = ·0796(c+c')(c-c') = ·0796 \times 70 \times 10 = 55·72.$$

3. Find the area of a circular annulus, the diameters of the containing circles being $=50$ and 60, and their circumferences $=157·08$ and 188·496.

$$Æ = \tfrac{1}{4}(c+c')(d-d') = \tfrac{1}{4} \times 345·576 \times 10 = 863·94.$$

If $Æ'$, $Æ''$ be the areas of the greater and less circles, and $Æ$ that of the annulus, then is $Æ = Æ' - Æ'' = ·7854d^2 - ·7854d'^2 = ·7854(d^2 - d'^2) = ·7854(d+d')(d-d')$; since $d^2 - d'^2 = (d+d')(d-d')$.

Again: $Æ' - Æ'' = ·0796c^2 - ·0796c'^2 = ·0796(c^2 - c'^2) = ·0796(c+c')(c-c') = \frac{1}{4\pi}(c+c')(c-c')$.

Also $Æ = \frac{1}{4\pi}(c+c')(c-c') = \tfrac{1}{4}(c+c') \times \frac{1}{\pi}(c-c') = \tfrac{1}{4}(c+c')(d-d')$.

EXERCISES

1. What is the area of a circular annulus, the diameters of the containing circles being $=30$ and 40 feet? =549·78 square feet.

2. The circumferences of two concentric circles are $=62\cdot832$ and $37\cdot6992$; required the area of the annulus contained by them.
$=201\cdot063$.

3. The diameters of two concentric circles are $=20$ and 32, and their circumferences are $=62\cdot832$ and $100\cdot531$; what is the area of the annulus contained between them? . . . . $=490\cdot089$.

4. The diameters of two concentric circles are $=19$ and $43\cdot5$ feet; what is the area of the included annulus? $=1202\cdot64$ square feet.

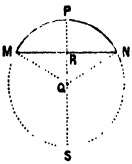
5. The circumferences of two circles are $=62\cdot832$ and $94\cdot248$ feet; what is the area of the contained annulus? $=392\cdot7$ square feet.

276. **Problem XXIX.—Of the chord, height, and apothem of an arc of a circle, any two being given, to find the radius of the circle.**

Let $MN=c$, $PR=h$, and the apothem $RQ=p$.

1. When PR and RQ, or h and p, are given, then $PQ=QR+RP$, or $r=p+h$.

2. In the triangle MQR, when MR and RQ are given, MQ can be found by Trigonometry.



Thus, $MQ^2=MR^2+RQ^2$, or $r^2=\frac{1}{4}c^2+p^2$.

3. When MR and RP, or c and h, are given, then (Eucl. III. 35) $RS.PR=MR^2$, or $(2r-h)h=\frac{1}{4}c^2$;

hence $2r-h=\dfrac{c^2}{4h}$; hence $r=\dfrac{c^2+4h^2}{8h}$, and $d=\dfrac{c^2+4h^2}{4h}$.

277. If the chord of MP, half the arc, is given, and the height PR, then $PS.PR=MP^2$; or if chord $MP=c'$, then since $PS=2r$, $2rh=c'^2$;

therefore $$r=\frac{c'^2}{2h}.$$

EXAMPLES.—1. Given the apothem and height of an arc $=3\cdot5$ and $8\cdot7$, to find the radius of the circle.

$$r=p+h=8\cdot7+3\cdot5=12\cdot2.$$

2. Given the chord $=20$ and apothem $=12$ of an arc, to find the radius of the circle.

$$r^2=\tfrac{1}{4}c^2+p^2=\tfrac{1}{4}\times 20^2+12^2=100+144=244,$$

and $$r=\sqrt{244}=15\cdot6204994.$$

3. Given the height and chord of an arc $=4$ and 30 respectively, to find the radius of the circle.

$$r=\frac{c^2+4h^2}{8h}=\frac{900+64}{32}=\frac{964}{32}=30\cdot125.$$

4. The height of an arc is $=4$, and the chord of its half is $=20$; find the radius.

$$r=\frac{c'^2}{2h}=\frac{20^2}{8}=\frac{400}{8}=50.$$

Exercises

1. What is the radius of a circle, the height of an arc of which is $=5{\cdot}6$, and apothem $=8{\cdot}4$? . . . . $=14$.
2. What is the radius of a circle, the chord of an arc of which is $=12$, and the apothem $=10$? . . . . . . $=11{\cdot}6619$.
3. The chord of an arc is $=36$, and its apothem is $=25$; find the radius of the circle. . . . . $=30{\cdot}8058$.
4. The height and chord of an arc are $=10$ and 24 respectively; find the radius of the circle. . . . . $12{\cdot}2$.
5. Find the radius of a circle, the chord and height of an arc of it being $=24$ and 4. . . . . $=20$.
6. The height of an arc is $=2$, and its chord is $=15$; find the diameter of the circle. . . . . $=30{\cdot}125$.
7. What is the diameter when the height is $=1$ and the chord $=12$? . . . . 37.
8. Find the radius when the chord is $=40$ and the height $=5$. . . . . $42{\cdot}5$.
9. What is the radius of an arc whose height is $=6$, and the chord of its half $=15$? . . . . $=18{\cdot}75$.

278. **Problem XXX.—Of the chord, height, and apothem of an arc, and the radius of the circle, any two being given, to find the number of degrees in the arc.**

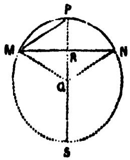
1. In the triangle MRQ, when any two of its sides r, p, and $\frac{1}{2}c$ are given, the angle MQR, or $\frac{1}{2}n°$, can be found by Trigonometry.

2. When QR and RP—that is, p and h—are given, then, since $p+h=r$, in this case r and p, or MQ and QR, are given, and this case is reduced to the former.

3. When MQ and PR—that is, r and h—are given, then, since $QR=QP-PR$, or $p=r-h$; therefore r and p are again known, and this case is reduced to the first.

4. When any two sides of the triangle MPR are given—that is, any two of the quantities $\frac{1}{2}c$, c', and h—angle M, which is $=\frac{1}{2}n$, can be found by Trigonometry.

EXAMPLES.—1. The chord of an arc is $=40$, and the radius of the circle $=60$; how many degrees does it contain?



$$MQ : MR = 1 : \sin Q.$$

Or $$r : \tfrac{1}{2}c = 1 : \sin \tfrac{1}{2}n;$$

hence, $$\sin \tfrac{1}{2}n = \frac{c}{2r} = \frac{40}{120} = \tfrac{1}{3} = \cdot\dot{3}$$
$$= \sin 19^\circ\ 28'\ 16''.$$

Or $$L \sin \tfrac{1}{2}n = L\ \tfrac{1}{2}c + 10 - Lr$$
$$= 1\cdot3010300 + 10 - 1\cdot7781513 = 9\cdot5228787;$$

and $$\tfrac{1}{2}n = 19^\circ\ 28'\ 16'', \text{ and } n = 38^\circ\ 56'\ 32''.$$

2. Find the number of degrees in a circular arc whose apothem and height are $=24$ and 6.

Here $p=24$, $h=6$; therefore $r=p+h=30$;

hence $$\cos \tfrac{1}{2}n = \frac{p}{r} = \frac{24}{30} = \frac{4}{5} = \cdot8 = \cos 36^\circ\ 52'\ 12'',$$

and $$n = 73^\circ\ 44'\ 24''.$$

3. The radius of a circle is $=25$, and the height of an arc of it is $=5$; required the number of degrees in it.

$$p = r - h = 25 - 5 = 20,$$

and $$\cos \tfrac{1}{2}n = \frac{p}{r} = \frac{20}{25} = \frac{4}{5} = \cdot8 = \cos 36^\circ\ 52'\ 12'',$$

and $$n = 73^\circ\ 44'\ 24''.$$

4. The chord of an arc is $=36$, and its height is $=4$; how many degrees are contained in it?

$$\tan \tfrac{1}{4}n = \frac{h}{\tfrac{1}{2}c} = \frac{2h}{c} = \frac{8}{36} = \frac{2}{9} = \cdot\dot{2} = \tan 12^\circ\ 31'\ 43''\cdot7,$$

and $$n = 50^\circ\ 6'\ 54''\cdot8.$$

EXERCISES

1. The chord of an arc is $=36$, and the radius of the circle is $=54$; what is the number of degrees in the arc? . $=38^\circ\ 56'\ 32''$.

2. The apothem and height of an arc are $=50$ and 12; required the number of degrees in it. . . . . . $=72^\circ\ 29'\ 55''\cdot2$.

3. What is the number of degrees in an arc whose height is $=12$, the radius of the circle being $=56$? . . . . . $=76^\circ\ 25'\cdot6$.

4. How many degrees are contained in an arc whose chord is $=40$, and height $=5$? . . . . $=56^\circ\ 8'\cdot7$.

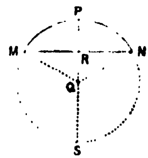
5. The chord of half an arc is $=20$, and the height of the arc is $=2$; how many degrees are contained in it? . . . . $=22^\circ\ 57'\cdot2$.

279. Problem XXXI.—To find the length of a circular arc, the number of degrees in it and the diameter being given.

RULE.—Multiply the number of degrees in it by the diameter, and the product by ·008727; the result will be the length of the arc.

Let	n = the number of degrees in the arc,
	l = the length of the arc;
then	$l = \cdot 008727nd$.
Or	$l = \cdot 0174533nr$, where r = radius.
For	$c = 3\cdot 1416d$, and $360^\circ : n^\circ = 3\cdot 1416d : l$;
hence	$l = \dfrac{3\cdot 1416nd}{360} = \cdot 008727nd$.

280. When, instead of the diameter being given, the chord or apothem is given, the radius can be found. For in the triangle MQR a side is then given, and angle $Q = \frac{1}{2}n$, to find the radius MQ. Also, when the chord and apothem are given, two sides of the triangle MQR are given; and hence the radius and number of degrees can be found by Trigonometry.



EXAMPLES.—1. Find the length of a circular arc containing 30°, the diameter being = 50.

$$l = \cdot 008727nd = \cdot 008727 \times 30 \times 50 = 13\cdot 0905.$$

2. The number of degrees in the arc of a circle, whose radius is = 25, is 25° 30′; what is the length of the arc?

$$l = \cdot 0174533nr = \cdot 0174533 \times 25\cdot 5 \times 25 = 11\cdot 126.$$

EXERCISES

1. What is the length of a circular arc of 45°, the diameter being = 12? . . . . = 4·71258.

2. What is the length of a circular arc of 32°, the radius of the circle being = 20? . . . . = 11·17.

3. Find the length of a circular arc containing 120° 40′, the radius of the circle being = 50. . . . . = 105·3.

4. Required the length of an arc of a circle whose diameter is = 125, and the number of degrees in the arc = 54°·6, or 54° 36′. = 59·559.

5. Find the length of an arc whose chord is = 12, and radius = 18. = 12·234.

6. What is the length of an arc whose chord is $=25{\cdot}4$, and the chord of its half $=15{\cdot}3$? . . . . $=32{\cdot}461$.

7. Find the length of an arc whose chord is $=4{\cdot}8$, and that of its half $=2{\cdot}443$. . . . . $=4{\cdot}9159$.

281. The lengths of arcs may also be easily computed by means of a Table containing the lengths of arcs of any number of degrees belonging to a circle whose radius is $=1$. Such a Table can be calculated by this problem. The rule by this method is:—

Multiply the tabular length of the arc of the same number of degrees by the radius of the given arc, and the product will be its length.

Let l' = length of arc in Table; then $l=rl'$.

Thus, for the first example given above, where $n=30°$, and $d=50$, it is found that $l'=\cdot5235988$;

hence $\qquad l=rl'=25\times\cdot5235988=13{\cdot}08997$.

And, for the second example, where $n=25°$, $l'=\cdot4363325$, and for $30'$, $l'=\cdot0087266$; hence, for $25°\ 30'$, l' is the sum of these two, or $=\cdot4450591$;

hence $\qquad l=rl'=25\times\cdot4450591=11{\cdot}12648$.

282. Problem XXXII.—To find the area of a circular sector.

RULE.—Multiply the length of the arc of the sector by the radius, and half the product will be the area.

Or $\qquad$ $A\!\!R=\frac{1}{2}lr$.

For the area of the whole circle is equal to the product of its circumference into the radius divided by two; and hence the area of the sector is also the product of its arc into the radius divided by two.

EXAMPLES.—1. The length of a circular arc is $=24$, and the diameter of the circle is $=30$; find its area.

$$A\!\!R=\tfrac{1}{2}lr=\tfrac{1}{4}\times24\times30=180.$$

2. The number of degrees in a circular arc is $=30$, and the radius is $=25$; what is its area?

$$l=\cdot0174533nr=\cdot0174533\times30\times25=13{\cdot}08997,$$

and $\qquad A\!\!R=\frac{1}{2}lr=\frac{1}{2}\times13{\cdot}08997\times25=163{\cdot}62468$.

283. When the number of degrees in the arc is given, as in the last example, the formula may be a little improved.

Thus, if in $A\!\!R=\frac{1}{2}lr$, the value of l found in Art. 279 be substituted, it becomes $A\!\!R=\frac{1}{2}\times\cdot0174533nrr=\cdot008727nr^2$.

The last example, calculated by this formula, gives

$$\mathcal{A}R = \cdot008727nr^2 = \cdot008727 \times 30 \times 25^2 = 163\cdot625.$$

When the radius and the length of the arc, or the number of degrees in it, are not given, they must be found by preceding problems.

Exercises

1. The length of a circular arc is $=50$, and its radius is $=30$; what is the area of the sector? . . . . $=750$.

2. The length of a circular arc is $=10\cdot75$, and its radius $=12\cdot5$; what is the area of the sector? . . . . . . $=67\cdot1875$.

3. The number of degrees in a circular arc is $=40°$, and the diameter is $=60$; find the area of the sector. . . $=314\cdot172$.

4. What is the area of a sector, the arc of which contains $50° 42'$, the radius of the circle being $=28$? . . . . . $=346\cdot8877$.

5. What is the area of a sector, the length of its arc being $=78\cdot14$, and the diameter of the circle $=70$? . . . . . $=1367\cdot45$.

6. What is the area of a sector whose radius is $=18$, and its chord $=12$? . . . . $=110\cdot106$.

7. Find the area of a sector whose arc contains $27°$, its radius being 6 feet. . . . . $=8\cdot4826$ square feet.

8. Find the area of a sector whose arc contains $36°$, its radius being $=50$. . . . . $=785\cdot4$.

9. Find the area of a circular sector, the chord of the arc being $=8$, and that of half the arc $=5$. . . . . $=22\cdot344$.

10. What is the area of a sector whose chord is $=30$, and height $=4$? . . . . $=473\cdot015$.

11. The height of the arc of a sector is $=2\cdot5$, and the chord of its half is $=5$; what is its area? . . . . $=26\cdot18$.

284. **Problem XXXIII.—To find the area of a circular segment.**

Rule I.—Find the area of the sector that has the same arc as the segment; find also the area of the triangle whose vertex is the centre and whose base is the chord of the segment; then the area of the segment is the difference or sum of these two areas, according as the segment is less or greater than a semicircle.

Example.—The chord and height of a segment are $=24$ and 6; find its area.

By Art. 278, $\tan \frac{1}{4}n = \frac{2h}{c} = \frac{12}{24} = \cdot5 = \tan 26° 33' 54''$;

and hence $n = 106° 15' 36'' = 106°\cdot26$.

Also (Art. 276), $2r - h = \frac{c^2}{4h} = \frac{144}{6} = 24$, and $r = 15$.

Then (Art. 283), sector $= \cdot 008727 n r^2 = \cdot 008727 \times 106 \cdot 26 \times 225 = 208 \cdot 64$;
also, triangle $= \frac{1}{2} cp = \frac{1}{2} \times 24 \times 9 = 108$;
hence segment $=$ sector $-$ triangle $= 208 \cdot 64 - 108 = 100 \cdot 64$.

285. When either the chord or apothem is unknown, and the radius is either given or found, and also the number of degrees in the arc, the area of the triangle is to be found by Prob. X.

Exercises

1. Find the area of a circular segment, its chord being $=40$, and height $=4$. . . . . $=107 \cdot 56$.
2. The chord of a segment is $=20$, and its height $=5$; what is its area? . . . . $=69 \cdot 896$.
3. Find the area of a segment whose chord is $=24$ feet, and height $=9$. . . . . $=159 \cdot 1$ square feet.
4. What is the area of a segment whose chord is $=30$, and diameter $=50$? . . . . $=102 \cdot 188$.
5. Required the area of a segment whose chord is $=16$, and diameter $=20$. . . . . $=44 \cdot 7293$.
6. The chord of a segment is $=24$, and the radius $=20$; what is its area? . . . . $=65 \cdot 401$.
7. What is the area of a segment, the arc of which is a quadrant, and the diameter $=12$ feet? . . . . . $=10 \cdot 274$ square feet.
8. Find the area of a segment whose arc contains 280°, the diameter being $=10$ feet. . . . . . $=73 \cdot 3966$ square feet.
9. The height of a segment is $=18$, and the diameter of the circle $=50$; what is the area of the segment? . . . . $=636 \cdot 376$.
10. The diameter of a circle is $=100$ feet, and the height of a segment of it is $6 \cdot 5$; what is its area? . $=216 \cdot 597$ square feet.

286. The area may also be found by means of a Table containing the areas of segments of a circle, whose diameter is $=1$, and whose heights are all the numbers between 0 and $\cdot 5$ carried to any number of decimal places, as to two or four, or any other number, according to the degree of accuracy required. Such a Table can be calculated by means of the preceding rule. The rule by this method is :—

Rule II.—Divide the height by the diameter, the quotient is the height of the similar segment when the diameter is $=1$; take the tabular area corresponding to this height, and multiply it

by the square of the diameter, and the product is the area of the given segment.

EXAMPLE.—For the above example, $h' = \frac{h}{d} = \frac{6}{30} = \cdot 2.$

The tabular area is then $\mathcal{R}' = \cdot 111824$;

and $\mathcal{R} = d^2\mathcal{R}' = 30^2 \times \cdot 111824 = 100 \cdot 6416.$

The exercises given above may be performed in the same manner to exemplify this rule.

Before this method can be used, h must be known.

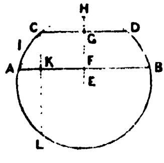
When r and c are known, then $p^2 = r^2 - \frac{1}{4}c^2$, from which p is found, then $h = r - p$.

When the chord of half the arc is given and the diameter, then (Art. 277) $2rh = c'^2$, where c' is the chord of half the arc; and from this,

$$h = \frac{c'^2}{2r}.$$

When the chord of the arc and that of half the arc are given, or c and c'; then in triangle MPR (fig. to Prob. XXX.), $PR^2 = MP^2 - MR^2$, or $h^2 = c'^2 - \frac{1}{4}c^2$.

287. **Problem XXXIV.—To find the area of a circular zone—that is, the figure contained by two parallel chords and the intercepted arcs.**



Find the area of the trapezium ACGF, and of the segment AIC, and double their sum will be the area of the zone ABDC; or,

Find the areas of the two segments AHB, CHD (Art. 285), and their difference will be the area of the zone ABDC.

Let the chord $AB = c$, $CD = c'$, and $AC = c''$; and the distance $GF = b$.

When b, c, and c' are given, the diameter d will be found thus—

Let $CL = m$, then $m = b + \frac{(c+c')(c-c')}{4b}$ ... ... [1],

and $d^2 = m^2 + c'^2$ ... ... [2].

For CK . KL = AK . KB (Eucl. III. 35); hence

$$KL = \frac{AK \cdot KB}{CK};$$

that is, $KL = \frac{\frac{1}{2}(c-c') \cdot \frac{1}{2}(c+c')}{b} = \frac{(c-c')(c+c')}{4b}.$

But LCD being a right angle, LD is the diameter, and

$$DL^2 = CL^2 + CD^2.$$

Also $c''^2 = b^2 + \frac{1}{4}(c-c')^2$ ... ... [3].

For $AC^2 = CK^2 + AK^2$, and $AK = \frac{1}{2}(c-c')$.

The diameter d and c'' being known, the area of the segment AIC can be found by Prob. XXXIII.; and if t = the area of the trapezium ACGF, it is

$$=\tfrac{1}{2}(AF+CG)CK, \text{ or } t=\tfrac{1}{4}(c+c')b \quad \ldots \quad [4].$$

The diameter and the chords c and c' being known, the areas of the segments AHB, CHD can be found by Prob. XXXIII.

Hence, if a, a', and a'' denote the areas of the segments, whose chords are c, c', and c'', and $\mathcal{R}$ that of the zone, then

$$\mathcal{R}=2(t+a''), \text{ or } \mathcal{R}=a-a' \quad \ldots \quad \ldots \quad [5].$$

288. Instead of finding the areas of the segments by the first rule of Prob. XXXIII., they may be found by the second—that is, by means of a Table. The heights, however, of the segments must be known before the rule can be applied. For the methods of finding h, see end of Art. 286.

When the zone contains the centre of the circle, the areas of the two segments on its opposite sides may be found, and their sum being taken from the area of the whole circle, will give that of the zone.

EXAMPLE.—Find the area of a circular zone, the parallel chords of which are = 90 and 50, and the distance between them = 20.

The areas of the segments may be calculated by either of the two rules of the last problem. They are calculated here by the second rule.

Here $c=90$, $c'=50$, and $b=20$.

By [1], $m=b+\dfrac{(c+c')(c-c')}{4b}=20+\dfrac{140\times 40}{80}=20+70=90$;

by [2], $d^2=m^2+c'^2=90^2+50^2=10600$, and $d=102{\cdot}956$.

Let p, p' and h, h' be the apothems and heights of these two segments, then (Prob. XXXIII.)

$$p'^2=r^2-(\tfrac{1}{2}c')^2=2650-25^2=2025, \text{ and } p'=45;$$

hence $\qquad h'=r-p'=51{\cdot}478\ -45=6{\cdot}478.$

Also $\qquad h\ =b+h'=20+\ \ 6{\cdot}478=26{\cdot}478.$

The tabular height for h' is $=\dfrac{h'}{d}=\dfrac{6{\cdot}478}{102{\cdot}956}={\cdot}06292.$

" " for h is $=\dfrac{h}{d}=\dfrac{26{\cdot}478}{102{\cdot}956}={\cdot}2572.$

Tabular area for ·0629 is = ·020642

" " for ·2572 is = ·159811

Difference, = ·139169

Hence $\quad \mathcal{R}={\cdot}139169d^2={\cdot}139169\times 10600=1475{\cdot}19.$

Exercises

1. The chords of a circular zone are = 30 and 48, and the distance between them is = 13; required its area. . . . = 534·19.

2. The chords of a zone that contains within it the centre of the circle are = 30 and 40, and their distance is = 35; what is its area? = 1581·7475.

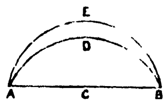
3. The diameter of a circle is = 25, and two parallel chords in it, on the same side of the centre, are = 20 and 15; find the area of the zone contained by them. . . . . = 44·343.

289. **Problem XXXV.—To find the area of a lune that is, the space contained between the arcs of two circles that have a common chord.**

Rule.—Find the areas of the two segments that stand on the same side of the chord, and their difference is the area of the lune.

Exercises

1. The length of the common chord AB is = 40, the heights CE and CD = 10 and 4; what is the area of the lune AEBD? . . . . = 172·05.



2. The chord is = 30, and the heights = 3 and 15; find the area of the lune . . = 292·954.

3. The chord is = 48, and the heights are = 7 and 18; what is the area? . . . . = 408·609.

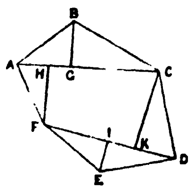
290. **Problem XXXVI.—To find the area of any irregular polygon.**

Rule.—Divide the polygon, by means of diagonals, into triangles, or into triangles and trapeziums, and find the areas of these component figures by former problems, and the sum of their areas will be the area required.

1. Find the area of a hexagonal figure from these measurements :—

AC	= 525 links
BG	= 160 "
DF	= 490 "
FH	= 210 "
EI	= 100 "
CK	= 300 "

= 1 acre 3 roods 32·2 poles.



2. Required the area of the irregular hexagon ABCDEF from these data :—

The side	AB = 690 links	the side	FA = 630 links
" "	BC = 870 "	the diagonal	AE = 1210 "
" "	CD = 770 "	" "	AD = 1634 "
" "	DE = 510 "	" "	BD = 1486 "
" "	EF = 670 "		

= 11 acres 18·46 square poles.

In this example the polygon is divided into triangles of which the three sides are known; and their areas are found by Prob. XI.

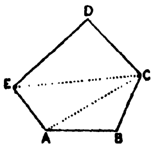
3. Find the area of the figure ABCDEF from these measurements :—

The side	AB = 2000 links	the angle	BAC = 40°
" "	AF = 1800 "	" "	CAD = 43°
the diagonal	AC = 2500 "	" "	DAE = 40° 30′
" "	AD = 2750 "	" "	EAF = 48° 20′
" "	AE = 3450 "		

= 93 acres 2 roods 2·67 square poles.

The areas of the triangles in the preceding question are to be found by Prob. X.

4. Find the area of the field ABCDE from these data :—



The side AB	= 450 links
" " BC	= 365 "
" " CD	= 324 "
" " DE	= 428 "
the angle ABC	= 110° 14′
" " BCD	= 84° 30′
" " CDE	= 140° 24′

= 2 acres 21·27 square poles.

Divide the polygon into triangles by means of the diagonals AC and CE. In triangle ABC, calculate the angle C and side AC (Art. 187); and similarly in triangle EDC, calculate angle C and side EC; then, if the sum of these two angles be subtracted from the whole angle BCD, the remainder is angle ACE. Two sides and a contained angle are then known in each of the three triangles; and hence their areas can be found (Art. 256).

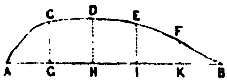
291. When all the sides but one of any polygon are known, and

also all the angles except the two at the extremities of that side, the area may be calculated in a manner similar to the method used in the solution of the preceding example.

292. Problem XXXVII.—To find the area of any curvilineal space by means of equidistant ordinates.

Let ACDB be the given space.

Draw the perpendiculars or ordinates GC, HD, &c.; then, if the curves AC, CD, DE, &c. are sufficiently short, they may be considered as straight lines without any material error, and then the figure will be divided into triangles and trapeziums, whose areas can be found as formerly.



I. When the curve meets the base at both extremities, and the base is divided into a number of equal parts, and ordinates are drawn from the points of division, multiply the sum of the ordinates by the common distance between them, and the product is the area.

Or, if the common distance $=l$, and the sum of the perpendiculars $=s$, then $\mathcal{A}\!\!R = ls$.

For let the perpendiculars be a, b, c, d, taken in order; then the areas of the triangle AGC, of the trapeziums, and of triangle FKB are

$$=\tfrac{1}{2}al+\tfrac{1}{2}(a+b)l+\tfrac{1}{2}(b+c)l+\tfrac{1}{2}(c+d)l+\tfrac{1}{2}dl\,;$$

that is, l is multiplied twice by $\frac{1}{2}a$, twice by $\frac{1}{2}b$, &c., or by the sum of a, b, c, and d.

When the figure is bounded by two perpendiculars, as by CG and KF, let them be denoted by a and z, and the sum of all the perpendiculars by s', as above; then if

$$s=s'-\tfrac{1}{2}(a+z),\ \mathcal{A}\!\!R=ls.$$

Example.—Let the perpendiculars of the figure ABD be $=10$, 12, 13, and 11, and the equal divisions of $AB=9$, what is its area?

$$s=10+12+13+11=46\,;$$

hence $$\mathcal{A}\!\!R=ls=9\times 46=414.$$

Exercises

1. The perpendiculars are $=12$, 20, 26, 30, and 24, and the common distance is $=14$; find the area. . . . . . $=1568$.

2. What is the area of the figure CGKF, terminated by the perpendiculars CG and FK, the four perpendiculars being $=14$, 15, 16, and 18, and the common distance $=12$? . . . . $=564$.

II. When the surface is terminated at its two extremities by ordinates, divide the base into an even number of equal parts; find the sum of the first and last ordinates; also the sum of the even ordinates—that is, the second, fourth, &c.—and also the sum of the remaining ordinates; then add together the first sum, four times the second, and twice the third; and the resulting sum, multiplied by one-third of the common distance of the ordinates, will give the area.

Let A = the sum of the first and last ordinates,

B = " " even ordinates, the second, fourth, &c.,

C = " " remaining ordinates,

and D = the common distance between the ordinates;

then the area $= \frac{1}{3}(A + 4B + 2C)D$.

For twice the area by last case $=(A + 2B + 2C)D$; and supposing the second ordinate to be equal to half the first and third, the fourth equal to half the third and fifth, and so on, the area will equal $2BD$; and adding these two, gives

$$3\mathcal{R} = (A + 4B + 2C)D;$$

hence the

$$\mathcal{R} = \tfrac{1}{3}(A + 4B + 2C)D.$$

EXAMPLE. Find the area of a surface, the ordinates being in order $=10$, 11, 14, 16, and 16, and the common distance between them $=5$.

Here $A = 26$, $B = 27$, and $C = 14$,

and area $= \frac{1}{3}(26 + 108 + 28) \times 5 = \frac{5}{3} \times 162 = 270$.

EXERCISES

1. What is the area of a surface, the common distance between the ordinates being $=10$, and the ordinates in order $=20$, 22, 28, 32, and 32? . . . . $=1080$.

2. Find the area of a field, one side of it being $=198$ links, and seven ordinates to it measured at equal distances to the opposite curvilineal boundary being in order $=60$, 75, 80, 82, 76, 63, and 50. $=14322$ square links.

3. One side of a field is $=60$, and five equidistant ordinates are measured perpendicular to it, extending to the curvilineal boundary, which are $=30$, 33, 42, 48, and 48; what is the area of the field? . . . . $=2430$.

4. Find the area of a field, one side of it being = 990 links, and seven equidistant ordinates from it to the opposite curvilinear boundary being = 300, 375, 400, 410, 380, 315, and 250.

= 3 acres 2 roods 12·88 square poles.

LAND-SURVEYING

293. Land-surveying is the method of measuring and computing the area of any small portion of the earth's surface—as a field, a farm, an estate, or district of moderate extent.

294. The quantity of surface to be ascertained in any case by this species of surveying is comparatively so limited that the spherical form of the earth is seldom taken into consideration.

295. The surfaces to be measured are divided into triangles and trapeziums, as in Articles 290 and 292 in 'Mensuration of Surfaces.' Various instruments are used for obtaining the measurements necessary for the computation of the areas, and for the construction of plans of the surfaces. The most common instruments are the chain, the surveying-cross, a theodolite, and a plane table.

296. The **chain**, called also **Gunter's chain**, is 22 yards or 66 feet long, and is composed of 100 equal links, the length of each being 7·92 inches. At every tenth link is a mark made of brass, to assist the eye in reckoning the number of links measured off. An acre consists of 10 square chains, or 100,000 square links. There are 80 chains in a mile, and 640 acres in a square mile.

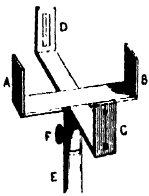
297. Ten iron pins, called **arrows**, with pieces of red cloth attached to them, are used for sticking in the ground

at the end of each chain-length when measuring in the field.

298. **Offset-staffs** are wooden rods ten links long divided into links for measuring offsets (Art. 305).

299. Other staffs, about six feet long, called **picket-staffs** or **station-staffs,** with small red flags attached, are used for marks to be placed at the corners of fields and other places called stations (Art. 303).

300. The **surveying-cross,** or **cross-staff,** consists of two bars of brass placed at right angles, with sights at their extremities, perpendicular to the plane of the bars. There are narrow slits at A and C, to which the eye is applied, and wider openings at B and D, with a fine wire fixed vertically in the middle of them. The cross is supported on a staff E, about $4\frac{1}{2}$ feet high, which at the lower end is pointed and shod with brass, so that it can be easily stuck in the ground.



The sights are placed on the top of the staff, and fixed in any position by a screw F.

301. A simple cross-staff may be made by cutting two grooves with a saw along the diagonals of a square board, to be fixed on the top of the staff.

302. It can easily be ascertained if the sights are at right angles, by directing one pair of them, as AB, to one object, and observing to what object the other pair, CD, are then directed; then by turning the sights till the second object is seen through the first pair of sights AB, if the first object is then visible through the second pair of sights and is exactly in apparent coincidence with the wire, the sights are at right angles; if not, they must be adjusted.

303. The angular points of the large triangles or polygons into which a field is to be divided for the purpose of taking

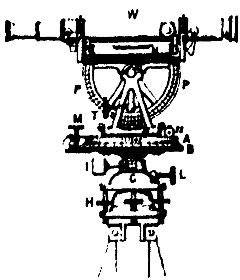
its dimensions are called **stations**, and are denoted by the mark ⊙; thus, $⊙_1$ is the first station, $⊙_2$ the second, and so on.

304. The stations are joined by lines, which are measured by the chain; hence called **chain-lines** or **station-lines.**

305. Lines measured perpendicularly to chain-lines, to the angular points, and other points of the boundary of a field are called **offsets.**

306. The cross-staff is used for finding the position of offsets. The point in the chain-line from which an offset is to be measured to any point in the boundary is found by fixing the staff in the chain-line so that one pair of sights may coincide with it; then, if the point in the boundary coincides with the other sight, the cross is at the proper point for an offset. Thus, the cross being placed at g (fig. to Art. 310), and one pair of sights coinciding with AB, the other will coincide with gC.

307. The **theodolite** is one of the most common and useful angular instruments. It consists of two graduated circles perpendicular to each other, one of which is fixed in a horizontal and the other in a vertical plane, and is used for measuring horizontal and vertical angles.



In the figure, WPB represents a side view of the horizontal circle, and PTP a direct view of the vertical one, which extends to little more than a semicircle. The vertical circle is movable about an axis, coinciding with the centre of the circular arc PTP.

On the vertical circle is fixed a **telescope**, W, furnished with a spirit-level, N; the **telescope moves** vertically about a horizontal axis which

passes through the centre of the vertical arc; and it moves horizontally by turning the upper horizontal plate on which it is supported, the lower plate B remaining fixed.

Both the horizontal and vertical circles are graduated to half-degrees, and by means of verniers, which are applied to them, angles can be read to minutes. Two levels are placed on the top of the horizontal plate, and when the instrument is to be used it is placed on a tripod stand, the horizontal circle being brought to a horizontal position by means of adjusting screws, H, and two spirit-levels, *n*, fixed on the circular plate.

308. **To measure a horizontal angle** subtended at the instrument by the horizontal distances of two objects: direct the telescope to one of the objects, and observe the number of degrees at the vernier on the horizontal circle; then turn the vertical circle, which is supported on the upper horizontal plate, till the other object is visible through the telescope, and in apparent coincidence with the intersection of the cross wires, and note the number of degrees on the horizontal circle; then the difference between this and the former number is the required horizontal angle.

309. **To measure a vertical angle**: direct the telescope to the object whose angle of elevation is required; then the arc intercepted between the zero of the arc and that of the vernier is the required angle. An **angle of depression** is similarly measured.

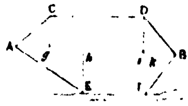
310. **Problem I.—To survey with the chain and cross-staff.**

RULE.—Divide the field into triangles, or into triangles and quadrilaterals, the principal triangles or trapeziums occupying the great body of the field, and the rest of it containing secondary triangles and trapeziums formed by offsets from the chain-lines. Measure the base and height, or else the three sides of each of the principal triangles, then calculate their

areas by the rules in 'Mensuration of Surfaces,' and also the offset spaces, and the sum of all the areas will be that of the entire field.

EXAMPLE.—Find the contents of the adjoining field from these measurements, A being the first and B the second station:—

On chain-line	Offsets
$Ag = 150$	$gC = 141$ to left
$Ah = 323$	hE 180 to right
$Ai = 597$	iD 167 to left
$Ak = 624$	$kF = 172$ to right
$AB = 769$	



The doubles of the areas of the component triangles and trapeziums are found, in order that there may be only one division by 2 — namely, that of their sum.

$gi = Ai - Ag = 447$, $iB = AB - Ai = 172$, and $hk = Ak - Ah = 301$, $Bk = AB - Ak = 145$.

Twice the area of the	triangle $AgC = Ag \cdot gC = 150 \times 141$, ...	21150
	trapezium $CgiD = gi(Cg + Di) = 447 \times (141 + 167)$, .	137676
	triangle $DiB = Bi \times iD = 172 \times 167$, ...	28724
	triangle $AhE = Ah \times hE = 323 \times 180$, . . .	58140
	trapezium $hEFk = hk(hE + kF) = 301(180 + 172)$, .	105952
	triangle $BkF = Bk \times kF = 145 \times 172$, ...	24940
	Twice area	376582

therefore area $= 188291 = 1$ acre 3 roods 21·2656 square poles.

311. Instead of writing the measurements as above, they are usually registered in a tabular form, called a **field-book,** as follows. The beginning of the field-book is at the lower end of the table, as this arrangement suggests more readily the direction of the measurements. The middle column of the field-book contains the lengths measured on the chain-lines, and the columns to the right and left of it contain respectively the **right** and **left** offsets.

The station from which the measurements are begun is called the **first station**; that next arrived at, the **second**; and so on.

The field-book of the measurements of a field similar to that of the last example is given below in the following exercise, in which A is O_1 and B is O_2.

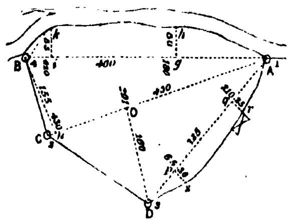
Exercises

1. Find the area of a field, the dimensions of which are given in the following field-book :—

Left Offsets	Chain-line	Right Offsets
	1538 to O_2	
	1248	344
334	1194	
	646	360 to road.
To fence, 282	300	
	From O_1	

=7 acres 2 roods 5·06 square poles.

2. Find the area of the subjoined field from the following measurements :—



Chain-lines	Offsets
AO = 291 links	Bn = 155
An = 430 „	DO = 160
AC = 450 „	ps = 30
Dp = 65 „	qr = 25
Dq = 210 „	gh = 50
DA = 325 „	ik = 55
Ag = 180 „	
Ai = 410 „	
AB = 460 „	

Twice the area of the quadrilateral ABCD $= AC(Bn + DO) = 450(155 + 160)$, $= 141750$

Twice the area of the triangle $Agh = Ag \,.\, gh = 180 \times 50$, ... $= 9000$

Twice the area of the trapezium $gikh = gi(ik + gh) = 230(55 + 50)$, . . $= 24150$

Twice the area of the triangle $Bik = Bi \,.\, ik = 50 \times 55$, ... $= 2750$

Twice the area of the triangle $Dps = Dp \,.\, ps = 65 \times 30$, ... $= 1950$

Twice the area of the trapezium $pqrs = pq(ps + qr) = 145(30 + 25)$, . . $= 7975$

Twice the area of the triangle $Aqr = Aq \,.\, qr = 115 \times 25$, ... $= 2875$

Twice area . . . . . . $= 190450$

therefore area = 95225 square links = 3 roods 32·36 square poles.

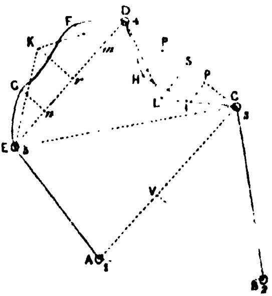
3. Find the area of a field similar to the preceding from the measurements given in the subjoined field-book.

=1 acre 1·3184 square poles.

Left Offsets	Chain-line	Right Offsets
	368 to O_4	
	328	44 to river
	144	40
	From O_1 on R of O_2	
	360 to O_1	
	168	20 to gate
	52	24
	From O_3 on L of O_2	
	360 to O_2	
	344	192 to O_4
To O_3 276	248	
	From O_1	

312. The initial letters R and L are used for **right** and **left**, to denote the direction in which a line is to be measured. Sometimes the marks [and] are used to denote respectively a turning to the right and left. The expression in the above field-book 'From O_3 on L of O_2' means that a chain-line is to be measured from the third station, and that it is situated to the left of the second station, in reference to the direction in which the first chain line, AC, is measured; so 'From O_1 on R of O_2' means that the next chain-line extends from O_1 to a point on the right of O_2 —namely, to O_4.

When the field to be surveyed is not very extensive, or the measurements not complex, they may be marked on a rough sketch of the field instead of in a field-book, as in the figure to Example 2.



313. On the left of the numbers denoting the left offsets, and to the right of those denoting the right offsets, lines are sometimes made, to represent in a general way the form of the boundary to which the offsets are drawn.

4. Find the area of the adjoining field ABCIHDFGE from

the measurements in the following field-book, A, B, C, D, E being respectively the 1st, 2nd, 3rd, 4th, and 5th stations.

Offsets on Left	Chain-lines	Right Offsets
	From O_3 802 to O_5	
	From O_1 760 to O_3	
	444 to O_1	
	From O_5	
	585 to O_5	
	426	112
	136	120
	From O_4	
	474 to O_4	
110	310	
80	120	
	From O_3	
	623 to O_3	
	From O_2	
	547 to O_2	
	From O_1	

=4 acres 3 roods 23·7 square poles.

Find the areas of the principal triangles ABC, ACE, and CDE, in the above exercise, by Article 257 in 'Mensuration of Surfaces;' then find the areas of the triangles and trapeziums composing the offset spaces EGFD and DHIC, the former of which is to be added to the areas of the principal triangles, and the latter to be deducted, in order to give the area of the given field ABCIHDFGE.

The crooked boundary, DHI, may be reduced to a straight line DL, meeting CI produced in L (see Art. 122, 'Descriptive Geometry'), and then a triangle CLD is formed equal to the irregular space CIHD, the area of which is = $\frac{1}{2}$LS . CD. The length of LS can be found by means of the scale used in constructing the figure. The offset space EGFD, with the curvilineal boundary, can also be reduced to a triangle EKD of equal area, which can be calculated like that of triangle CLD. The straight lines EK, KD can be determined with sufficient accuracy by the eye, so as to cut off as much space from the inside of the curved

boundary EGFD as is added on the outside. A ruler made of transparent horn is used for this purpose, or a fine wire stretched on a whalebone bow.

314. The practice of constructing a **plan** of any surface, the dimensions of which are taken, and reducing the crooked and curved boundaries in the manner stated above, is very common with the best surveyors, on account of its expedition and sufficient accuracy. It is also usual to measure on the plan the altitudes of the principal triangles, and to calculate their areas by the simple rule in Article 255 of 'Mensuration of Surfaces.'

Thus, by drawing the perpendicular BV on AC, and measuring it, the area of triangle ABC is $= \frac{1}{2} AC \cdot BV$; and in a similar manner the areas of the other principal triangles are found.

COMPUTATION OF ACREAGE

Divide the area into convenient triangles, and multiply the base of each triangle in links by half the perpendicular in links; cut off 5 figures to the right, and the remaining figures will be acres. Multiply the 5 figures so cut off by 4, and again cut off 5 figures, and the remainder is in roods. Multiply the 5 figures by 40, and again cut off for square poles.

OBSTACLES IN RANGING SURVEY LINES

If it be possible to see over the obstacle, but not to chain over it, lay off AC and BD (fig. 1) equal to each other, and at right

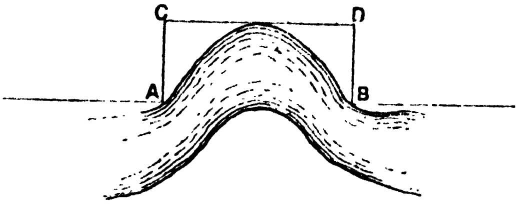


Fig. 1.

angles to the line; then CD = AB. If it be not possible either to chain or see over the obstacle, lay off the lines EF, AC equal to each other, and at right angles to the line (fig. 2) range the points DH in line with EC, and set off the lines DB, HG equal to AC and EF,

and at right angles to the line EH; then B and G are points for ranging the continuation of the line FA, and AB=CD.

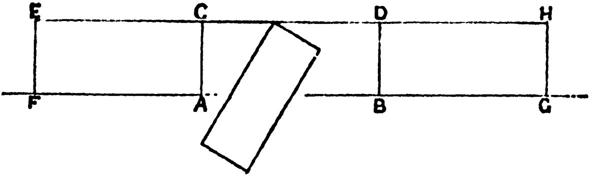


Fig. 2.

TO SET OUT A RIGHT ANGLE WITH THE CHAIN

Take 40 links on the chain for the base, 30 links for the perpendicular, and 50 for the hypotenuse.

Useful Numbers in Surveying

For Converting	Multiplier	Converse
Feet into links, . . .	1·515	·66
Yards into links, . . .	4·545	·22
Square feet into acres, . .	·0000229	43560
Square yards into acres, .	·0002066	4840
Feet into miles, . . .	·00019	5280
Yards into miles, . . .	·00057	1760
Chains into miles, . . .	·0125	80

TO SURVEY WITH THE CHAIN, CROSS, AND THEODOLITE

315. Although it frequently happens that the most expeditious mode of surveying is by the chain and cross, yet in the case of large surveys the theodolite is very advantageously combined with them for measuring angles. When some of the angles of a triangle are known, its area can be found without knowing all its sides; and the tedious process of measuring them all by the chain is thus dispensed with, unless the measuring of offsets or some other cause requires all the sides to be measured. It is often useful to measure more lines and angles than are necessary for determining

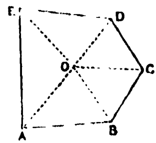
the area, for the purpose of serving as a check to ensure accuracy in the results.

316. **Problem II.—To survey a field by taking a single station within it, and measuring the distances to its different corners, and the angles at the station contained by these distances.**

The field is thus divided into triangles, in each of which two sides and the contained angle are known; and their areas may therefore be found by Article 256 in 'Mensuration of Surfaces;' their sum will be the area of the field.

EXERCISES

1. From a station O within a pentagonal field, the distances to the different corners A, B, C, D, E were measured and found to be respectively 1469, 1196, 1299, 1203, and 1410; and the angles AOB, BOC, &c. contained by them were in order 71° 30′, 55° 45′, 49° 15′, and 81° 30′; required the area of the field.



=39 acres 30·2 square poles.

2. From a station near the middle of a field of six sides, ABCDEF, the distances and angles, measured as in the preceding exercise, were as below:—

AO = 4315 links		Angle AOB = 60° 30′	
OB = 2982	"	" BOC = 47° 40′	
OC = 3561	"	" COD = 49° 50′	
OD = 5010	"	" DOE = 57° 10′	
OE = 4618	"	" EOF = 64° 15′	
OF = 3606	"	" FOA = 80° 35′	

What is the area? . . =412 acres 1 rood 17·3 square poles.

317. **Problem III.—To survey a polygonal field by measuring all its sides but one, and all its angles except the two at the extremities of that side.**

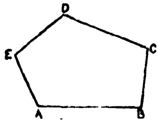
From the data it will always be possible, by applying trigonometrical calculation, to find two sides and the contained angle of each of the component triangles, the areas of which can be calculated as in last problem.

Let ABCDE be the polygonal field; and let the sides AB, BC, CD, DE be given, and also the angles B, C, and D. Join CE and CA; then, in triangle ABC, find AC and angle C; and in

triangle CDE, find CE and angle C; then angle ACE = BCD − (ACB + DCE). There are therefore now known two sides and a contained triangle in each triangle; and hence their areas can be found, the sum of which is that of the given field.

Exercise

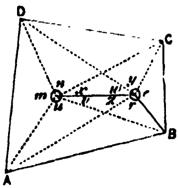
Find the area of the subjoined field ABCDE from these measurements :—



Side AB = 388 Angle B = 110° 30′
" BC = 311 " C = 117° 45′
" CD = 425 " D = 91° 20′
" DE = 548
= 2 acres 2 roods 24·68 square poles.

318. **Problem IV.—To survey a field from two stations in it by measuring the distance between them, and all the angles at each station contained by this distance, and lines drawn from the stations to the corners of the field.**

From the data all the lines drawn from one of the stations to the corners of the field can be calculated by trigonometry; and then the areas of the triangles contained by these lines, and the sides of the figure, can be calculated as in the last problem.



Let ABCD be the given figure, and OQ the stations; measure all the angles at O and Q; then in triangle DOQ the angles are known, and the side OQ; hence find OD; similarly in triangle OQC find OC; then find OB in triangle OBQ; and, lastly, OA in triangle OAQ. Then the areas of the four triangles AOB, BOC, COD, DOA can be found as in the last problem.

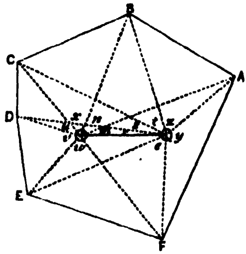
Exercises

1. Find the area of the field ABCD from these measurements :—

Angle m =	120° 40′	Angle w =	36° 10′
" n =	85° 30′	" y =	86° 45′
" x =	25° 50′	" e =	115° 16′
" v =	20° 40′	" r =	94° 30′
and hence u =	107° 20′	and hence z =	27° 19′

and OQ = 1440 links. . . = 61 acres 1 rood 6·448 square poles.

2. Find the area of the field ABCDEF from the subjoined measurements :—



Angles at O	Angles at Q
$AOQ = m = 21° 20'$	$OQD = r = 10° 40'$
$AOB = n = 49° 10'$	$DQC = s = 18° 40'$
$BOC = x = 57° 12'$	$CQB = t = 42° 0'$
$COD = u = 29° 40'$	$BQA = z = 67° 5'$
$DOE = v = 64° 25'$	$AQF = y = 137° 0'$
$EOF = w = 79° 16'$	$FQE = c = 62° 52'$

and the distance $OQ = 500$ links.

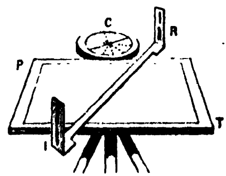
$= 12$ acres 3 roods 1·18 square poles.

319. It is evident that, by the preceding method, a field may be surveyed from two stations situated outside the field, its area computed, and a plan of it made. But in this case the area of some of the triangles will have to be subtracted from the sum of the areas of the others.

SURVEYING WITH THE PLANE-TABLE

320. By means of the plane-table, a plan of a field or estate is expeditiously made during the survey, from which the contents may be computed by the method described in Article 314.

321. This instrument consists of a plain and smooth rectangular board fitted in a movable frame of wood, which fixes the paper on the table, PT, in the adjoining figure. The centre of the table below is fixed to a tripod-stand, having at the top a ball-and-socket joint, so that the table may be fixed in any required position.



The table is fixed in a horizontal position by means of two spirit-levels lying in different directions, or by placing a ball on the table, and observing the position of it in which the ball remains at rest.

The edges of one side of the frame are divided into equal parts, for the purpose of drawing on the paper lines parallel or perpendicular to the edges of the frame ; and the edges of the other side are divided into degrees corresponding to a central point on the board for the purpose of measuring angles.

A magnetic compass-box, C, is fixed to one side of the table for determining the bearings of stations and other objects, and for

the purpose of fixing the table in the same *relative* position in different stations.

There is also an index-rule of brass, IR, fitted with a telescope or sights, one edge of which, called the *fiducial* edge, is in the same plane with the sights, and by which lines are drawn on the paper to represent the direction of any object observed through the sights. This rule is graduated to serve as a scale of equal parts.

322. Problem V.—To survey with the plane-table from a station inside the field.

Place the table at the station O (fig. to Art. 316); adjust it so that the magnetic needle shall point to north on the compass-card, or else observe the bearing of the needle, and fix on some point in the paper on the table for this station; bring the nearer end of the fiducial edge of the index-rule to this point, and direct the sights to the corner A, and draw an obscure line with the pencil or a point along this edge to represent the direction OA; measure OA, and from the scale lay down its length on the obscure line, and then the point A is determined. Draw on the table the lines OB, OC, OD, and OE exactly in the same way. The points A, B, C, D, E being now joined, the plan of the field is finished, and its contents may be computed as explained in Article 314, by measurements taken on the plan.

The angles at O, subtended by the sides of the field, can also be measured at the same time by placing the frame with that side uppermost which contains the angular divisions, and then the contents of the field can be calculated independently of the plan.

323. Problem VI.—To survey with the plane-table by taking stations at all the corners of the field but one, and measuring all its sides.

Let ABCDE (fig. to Art. 317) be the field; place the table at some corner, as A, and mark a point in the paper where most convenient to represent that station; adjust the instrument, as to the direction of the magnetic-needle, as in last problem. Apply the nearer end of the index-rule to this station point, and direct the sights to the station E, and draw an obscure line as before to denote the direction AE; then, in a similar manner, through the station point, draw a line for the direction AB; measure AE and AB, and with the scale lay off these measures on the obscure lines denoting AE and AB. Remove the instrument now to the second station B, and place it so that the needle shall rest at the *same*

point of the compass-card as before. If the index-rule is now laid along the direction of AB, the first station A will coincide with the sights, if the table is properly placed. With the index-rule draw an obscure line through the second station point to represent BC; measure BC, and lay the distance off on the line BC on the paper; remove the table to the third station C, adjust its position as before, and draw a line to represent the direction CD; measure CD, and, by the scale, lay this length off on CD on the paper; and, lastly, place it at D as before, and draw a line to represent DE; this line will meet AE in E, if the work has been correctly performed. The plan of the field is now completed.

324. **Problem VII.—To survey a field from two stations.**

Let O and Q (fig. to Art. 318) be the two stations. Fix the table at O, and adjust it as formerly; assume a convenient point on the paper for the first station O; draw an obscure line to represent OQ; as before, measure OQ, and, with the scale, lay this length off on OQ on the paper, and the point for the station Q is determined. Then draw obscure lines to represent the lines OC, OB, &c., drawn from O to the angles of the field, without measuring these lines, as in Article 322; and having placed the table at Q, and adjusted it, draw obscure lines from Q to represent the lines drawn from Q to the corners of the field; and the intersections of these lines, with the former lines from O, will determine the corners C, B, A, &c., and the plan will be completed.

325. **Problem VIII.—To survey more than one field with the plane-table.**

Having surveyed one of the fields according to any of the methods in the three preceding problems, fix on a station in this field, whose position is known on the paper, and take some station in the adjoining field at a sufficient distance; then, from the former station, draw an obscure line in the direction of the latter, measure the distance between them, and lay it off from a scale on the paper; and thus the new station in the adjoining field is determined on the plan. Place the table in this station, adjust it, and if it is correctly placed, and the index-rule placed on the line joining the two last stations, the sights will coincide with the station in the first field. Proceed to the planning of this second field; then, in a similar manner, plan the next; and so on till the whole survey is finished, and then measure it as before by means of the plan (Art. 314).

When a new sheet of paper is required in consequence of that on the table being filled, some line must be drawn on the latter at the most advanced part of the work, and the edge of the former being applied to it, the station lines must be produced on this sheet. Before drawing the line, the latter sheet must be held in such a position as is most convenient for continuing the next part of the work upon it. The first sheet being removed from the table, and this one, previously moistened, fixed in by means of the frame, the work may be continued after the paper has got dry. When this sheet is filled, another is similarly fixed on the table; and when the survey is completed, the sheets can all be accurately joined by means of the *connecting* lines.

At the beginning of the work, the position of some conspicuous object or mark may be laid down on the paper, and at any stage of the subsequent operation its position may be ascertained; and if it coincide with the first position, it is a proof that the work is correct. If not, some error must have been committed, which must be rectified before proceeding further; with this check, the greatest accuracy may be secured in the survey.

Exercise

From a station within a hexagonal field the distances of each of its corners were measured, and also their bearings; required its plan and area, the measurements being as below.

=12 acres 3 roods 6·448 square poles.

		Distances	Bearings
To first corner	=	708	NE.
" second "	=	957	N ½ E.
" third "	=	783	NW by W.
" fourth "	=	825	SW by S.
" fifth "	=	406	SSE 7° E.
" sixth "	=	589	E by S 3½° E.

This exercise is to be solved like that under Problem II.

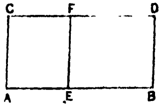
DIVISION OF LAND

326. It frequently becomes a problem in land-surveying to cut off a certain portion from a field. When the field is of a regular form, this process may be frequently effected by a direct method; but in the case of irregular fields, it can be accomplished only by indirect or tentative methods.

327. Problem IX.—To cut off a portion from a rectangular field by a line parallel to its ends.

Find the area of the field; then as its area is to that of the part to be cut off, so is the length of the field to the length of the part.

Let AD be the field, AF the part to be cut off, then



Divide the area of the required part by the breadth of the field, and the quotient will be the length.

Let a = area of AF, and b = the breadth AC, and l = the length AE; then $l = \frac{a}{b}$.

Or, if A = area of the field AD, and L = its length AB, then $A : a = L : l$, and $l = \frac{aL}{A} = AE$.

EXAMPLE.—The area of a rectangular field is = 10 acres 3 roods 20 square poles, its length is = 1500 links, and breadth = 725 links; it is required to cut off a part from it of the contents of 2 acres 28 square poles by a line parallel to its side.

A = 10 acres 3 roods 20 sq. poles = 1087500 links,
a = 2 " 0 " 28 " = 217500 "

hence, $l = \frac{aL}{A} = \frac{217500}{1087500} \times 1500 = 300$ links = AE.

Or, $l = \frac{a}{b} = \frac{217500}{725} = 300$ links = AE.

When any aliquot part is to be cut off from the field, find the same part of the base, and it will be the length of the required part.

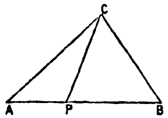
EXERCISE

A rectangular field is = 1250 links long and 320 broad; it is required to cut off a part of it, to contain 1 acre 2 roods 16 square poles, by a line parallel to one of its ends; what is the length of this part? . . . . = 500 links.

328. Problem X.—To cut off any portion from a triangle by a line drawn from its vertex.

Let ABC be the triangle, and APC the part to be cut off, then

Divide the area of the required part by the altitude of the triangle, and the quotient will be half the length of its base.



If a = area of the reqnired part APC, l = AP, and h = altitude of the triangle, then $\frac{1}{2}l=\frac{a}{h}$, or $l=\frac{2a}{h}$.

Or, if A = area of the given triangle ABC, L = its base AB, then A : a = L : l, and $l=\frac{a\text{L}}{\text{A}}$ = AP.

EXAMPLE.—The length of one side of a triangular field is = 2500 links, and the perpendicular upon it from the opposite corner is = 1240 links; it is required to cut off a triangular portion from it, by a line drawn from the same angle to this side, so that its contents shall be = 5 acres 16 poles.

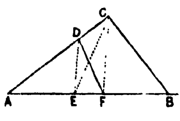
a = 5 acres 0 roods 16 sq. poles = 510000 sq. links;

hence, $l=\frac{2a}{h}=\frac{1020000}{1240}$ = 822·6 links = AP.

EXERCISE

Cut off from a triangular field, as in the preceding exercise, a part containing 2 acres 1 rood 24 square poles, the length of one side of the triangle being = 1280 links, and the perdendicular on it, from the opposite corner, = 1500. . . Length of base = 320 links.

329. Problem XI.—To cut off any portion from a triangular field by a line drawn from a point in one of its sides.



Let ABC be the given triangle, and D the given point.

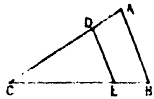
Cut off a part ACE, by last problem, of the required content. Join DE, and through C draw CF parallel to DE; draw DF, and it is the line required.

For triangle DFE = DEC (Eucl. I. 37); and hence triangle ADF = ACE = the required area.

330. Problem XII.—To cut off a part from a triangle by a line parallel to one of its sides.

Let ACB be the given triangle, and AB the side to which the required line is to be parallel.

Let A = area of the given triangle ACB, a = area of the required triangle CDE, S = the side CB, s = the side CE. Then $A : a = S^2 : s^2$, and $s^2 = \frac{aS^2}{A}$, or $s = S\sqrt{\frac{a}{A}}$.



Hence find s, and make CE equal to it, and through E draw DE parallel to AB, and it will be the required triangle.

EXAMPLE.—The area of a triangle ACB is = 5 acres 2 roods 15 square poles, the side CB is = 1525 links; required the length of CE, so that the triangle CDE shall contain 2 acres 1 rood 10 poles.

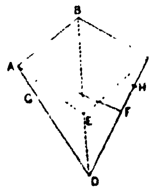
$A = 5$ acres 2 roods 15 sq. poles = 559375 sq. links,
$a = 2$ „ 1 rood 10 „ 231250 „

hence, $s = S\sqrt{\frac{a}{A}} = 1525\sqrt{\frac{231250}{559375}} = 980{\cdot}4$ links = CE.

EXERCISE

The area of a triangle is = 12·96 acres, its side CB is = 1200 links; required the length of CE, so that the triangle CDE shall contain 3·24 acres. . . . . = 600 links.

331. Problem XIII.—To cut off from a quadrilateral any portion of surface by a line drawn from one of its angles, or from a point in one of its sides.



Let ABCD be the quadrilateral.

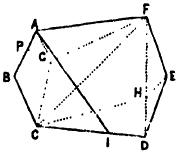
1. Let A be the angle from which the line is to be drawn.

Draw the diagonal DB, and cut off a part, DE, from it that has the same proportion to DB as the required part has to the quadrilateral; draw AE, EC; then AECD is equal to the required area. Rectify the crooked boundary AEC by drawing AF (Prac. Geom.), and it is the required line which cuts off the part AFD = AECD = the given area.

2. When it is required to draw the dividing line from a point, G, in one side.

Draw AF, by the preceding case, then join GF, and through A draw a line parallel to GF, cutting CD in H, and a line joining G and H will be the required line.

332. **Problem XIV.—To cut off any part of the area of a given polygon by a line drawn from any of its angles, or from a point in one of its sides.**



Let ABCDEF be the given polygon.

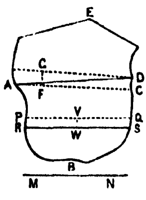
1. Let A be the point from which the line is to be drawn.

Draw the diagonals FB, FC, FD. Cut BF in G, so that A : a = BF : BG, A and a being the areas of the polygon and of the part required. Join AG and GC. Cut DF in H, so that A : a = DF : DH, and join CH and HE. Then the crooked boundary AGCHE evidently cuts off an area equal to that required; for the triangle AGB is the same part of ABF that a is of A, and BGC the same part of FBC, and so on. Hence, rectify the crooked boundary AGCHE by drawing from A the straight line AI, and AICB is the required part.

2. When the line is to be drawn from a point P in one of the sides.

Draw AI, as in the first case, then from P draw another line to be determined, as GH in the preceding problem.

333. **Problem XV.—To cut off any proposed portion from a field with curvilineal boundaries by a line from a point in its boundary, or by a line parallel to a given line.**



Let ABCE be the given field.

1. When the line is to be drawn from a point in the boundary A.

Draw a trial line AC, and measure the area of the part cut off, AEDC. If it is too great, divide the excess in square links by the length of AC in links, and make the GF perpendicular to AC equal to twice the quotient; draw GD parallel to AC; join AD, and AD is the required line.

For the area of the triangle ADC, considering CD as a straight line, is $= \frac{1}{2}$AC . GF, and therefore equal to the excess.

2. When the dividing line is to be parallel to a given line MN.

Draw a trial line PQ parallel to MN, to cut off a portion PBQ equal to the required part, and measure it. If it is too small, find the defect in square links, and divide it by the length of PQ in links, and make the VW perpendicular to PQ equal to the

quotient; and through W draw RS parallel to PQ, and it will be the required line when PR and QS are either parallel or equally inclined to PQ. When they are not so, a small correction may require to be made, by drawing RS a little nearer to or a little farther from PQ.

INCLINED LANDS

334. When the surface of a field is inclined, it is not that surface, but the surface of its projection on a horizontal plane, that is laid down on the plan as its area. This *projection* is just the quantity of surface on a horizontal plane, determined by drawing perpendiculars upon it from every point in the boundary of the field, or, in other words, by projecting its boundaries on a horizontal plane; and a plan of this projection only is made: it is impossible to construct a plan of a curved surface on one plane.

The area of the horizontal projection can easily be computed by measuring the angle of acclivity of the field at different places. Thus, if ABCD is a vertical section of the field, then if AB is measured, and the angle of elevation A, the horizontal projection AE of AB is AE = AB cos A when rad. = 1. Thus, if AB = 1200 links,

and angle A = 15° 40′, AE = 1200 × ·9628490 = 1155·4188, or 1155 links, is the length of AE on the plan, which must also be taken for its length in computing the area. So if BC and angle CBF be measured, BF, or its projection EG, can be found; then AG = AE + EG, is the projection of AB and BC. In the same way the other dimensions of the projection can be found; and if a theodolite is used for measuring any of the angles contained by lines measured on the field, these being horizontal angles on the instrument, are just the angles of the projection, and are to be used unaltered for constructing the plan.

CHAINING ON SLOPES

A = Angle of slope with horizon.
L = Length of line chained on the slope.
l = length of line reduced to the horizontal.
l = LK.
K = cos A.

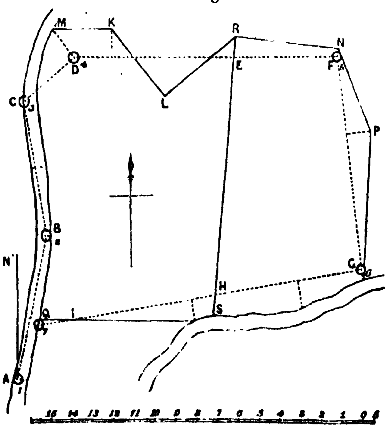
TABLE SHOWING VALUES OF K

A	K	A	K	A	K
°		°		°	
5	·996	19	·945	33	·839
6	·994	20	·94	34	·829
7	·992	21	·933	35	·819
8	·99	22	·927	36	·809
9	·988	23	·92	37	·799
10	·985	24	·913	38	·788
11	·982	25	·906	39	·777
12	·978	26	·899	40	·766
13	·974	27	·891	41	·755
14	·97	28	·883	42	·743
15	·966	29	·875	43	·731
16	·961	30	·866	44	·719
17	·956	31	·857	45	·707
18	·951	32	·848		

SURVEY OF A ROAD AND ADJOINING FIELDS

335. Construct a plan of a road and adjoining fields from the subjoined field-book and following sketch :—

Plan from following Field-book



Scale of Chains

Left Offsets	Chain-lines	Right Offsets
Along the road	at O_7	48 to corner of field.
Cross	1404	fence to outside.
92	840	
s—	725	—s.
120	300	
To river 57 broad 40	0	
	From O_6 to R	
	925 to O_6	
140	340	
	96° 40′	
	From O_5 to R	
42	1256 to O_5	
s—100	780	— — s
Cross	640	fence to inside.
Cross	300	fence to outside.
120	180	
	138° 15′	
	From O_4 to R	
To road . . . 160	328 to O_4	
Cross the	78	road to inside of field.
	122° 40′	
	From O_3 to R	
5	605 to O_3	56
25	320	35
	156° 15′	
	From O_2 to L	
54	625 to O_2	6
4	240	58 to O_7
20	0	42
	12° 10′ NE.	
	From O_1	

The chain-lines in this field-book are the sides of a polygon, and the lengths of all these sides except one, and all its interior angles except the two at the extremities of the unknown side, are given; and these are sufficient for constructing it, or for calculating its contents.

The first angle 12° 10′ gives the *bearing* of the first chain-line—that is, its inclination to the meridian; and as the inclination of each of the successive chain-lines to the preceding is given, the bearings of all the rest are given. The direction of the meridian can therefore be drawn through the first station, as it will lie to the left of the first chain-line, making with it an angle of 12° 10′, as AN′ in the plan. Then any line, NS, on any convenient part of the plan parallel to AN′, will be the direction of the meridian.

The second chain-line lies to the left of the first; and hence the angle of the polygon is here a re-entrant angle, and $=360° - 156° 15' = 203° 45'$.

When any boundary-line crosses a chain-line, as KL in the plan, an oblique line is drawn on the right and left opposite to the number which denotes the distance of the point of intersection from the station at the beginning of the line. Thus, opposite to 300 and 640, between O_4 and O_5, oblique lines are drawn for this purpose. When any internal boundary, fence, or other important line is passed, as at E in DF, a straight line is drawn on both sides of the corresponding distance in the chain-line; and when the line is straight to the end, these lines are marked S, as in the field-book opposite to 780, between O_4 and O_5; and opposite to 725, between O_6 and O_7; and these two points determine the line. Thus, RS is determined by the points E and H. When an offset is not at right angles, but in another direction, as along a fence, the mark ⁀ is placed over it, as 100 at 780 between O_4 and O_5.

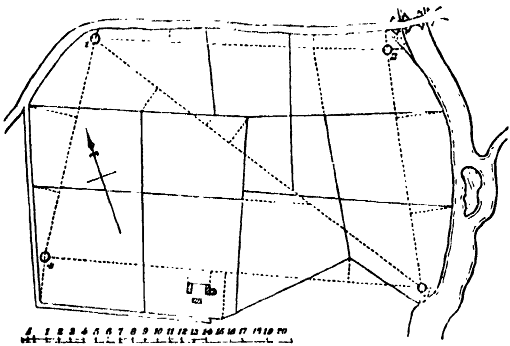
SURVEY OF A SMALL FARM

336. Construct the plan and compute the area of a small farm from the following field-book. Area = 66 ac. 3 ro. 13·46 sq. po.

337. In the following field-book, the expression 'in line with fence,' with dots ... on the left or right of the chain-lines, means that the point arrived at in the chain-line is in the same line with some straight fence on the left or right, which does not extend so far as the chain-line. In the subjoined plan the continuous lines represent the fences, and the dotted lines represent the lines that are measured, and also the extension of the lines of some of the fences.

Left Offsets	Chain-lines	Right Offsets
	3208 to O_3	
s———	2464	———s.
In line with	2145	...fence.
s———	2020	———s.
In line with...	1904	fence.
To corner where 240	1324	three fences meet.
To end of fence 88	1080	
	620	91 to end of fence.
60 broad Diagonal	From O_1 to O_3	
120	1776 to O_1	
To road 42 broad to end of s— 408	1140	———s.
to end of s— 156	525	———s.
44	0	
	From O_4	
To road 42 broad 352	3040 to O_4	
s———	2208	———s.
To N.W. corner of stables 92	1868	
To NE. corner of barn 94	1704	
365	1560	
s———	1475	———s.
Cross fence ↘	925	to inside.
	584	183
Cross fence ↗	252	to outside.
104	0	
	From O_3	
To river 160 broad 102	1896 to O_3	
to end of s—— 387	1300	———s.
to end of s—— 401	534	———s.
150	0	
	From O_2	
112	2340 to O_2	
100	1760	...in line with fence.
To road 60 broad s———	1581	———s.
s———	880	———s.
100	600	
64	0	
	E by S 6° 24′ S From O_1	

Plan from preceding Field-book



EXTENSIVE SURVEYS WITH THE THEODOLITE

338. In large surveys with the theodolite, as that of an estate or the mapping of a district, extensive chain-lines are run through the country, joining a series of successive stations conveniently chosen for observing the important or conspicuous objects within the limits of the survey; the bearing of each station-line is also observed—that is, its inclination to the meridian (see Art. 335)—and the bearings of each of the distant and important points or objects of the survey are observed at least at two different stations. Offsets are also measured, in the usual way, for determining the positions of objects not far distant from the chain-line.

Let ABCD (fig. to Art. 335) represent a portion of a chain-line, and AN′ the direction of the meridian passing through the first station A, this direction being determined either by means of the magnetic compass, or more accurately by astronomical methods, as by the position of the pole-star when on the meridian, or by means of the computed culminations of any other star (Astronomical Prob. XVIII.). The theodolite is first placed in the first station, A, and the horizontal circle is brought to a level position by means of the adjusting screws; the index or zero of the vernier is now brought to 360° on the limb of the horizontal circle, which is now to be fastened to the other part of the head by means of the clamping-screw, and the whole head is now turned till 360° is in the direction of the meridian line, which is determined by directing the telescope till the centre of the cross wires coincides with the

picket placed at N'. The head is now fixed in this position by means of the locking-screw below the head of the tripod. The upper part of the head to which the vernier is attached is now set free by unclamping the screw, the theodolite is directed to the picket at the second station, and the upper part is again clamped; and the degree opposite to the vernier, which is under the eye-glass of the telescope, is noted, as it measures the bearing of the second station from the first. The first station-line, AB, is now measured, and the instrument removed to the second station at B; then, after being levelled, and the locking-screw unscrewed, the whole head is turned till the telescope is directed back on the first station A, when the whole head is again locked. In this position of the head, the division 360° is evidently again situated in the meridian line passing through the second station, B, on the south of this station, for the angle contained by this portion of the meridian through B, and the line AB, is equal to the alternate angle A. The clamping-screw is now unscrewed, and the telescope is directed to the third station, and the angle noted as before, which in this instance measures the bearing of the second station-line, BC, from the meridian drawn through B towards the south. The upper part of the head is now clamped, and the second station-line measured, the instrument being placed in the third station, and the telescope directed back to the second as before. This process is continued throughout the survey.

The measurements of lines and angles thus obtained are sufficient for determining, on a plan or map, the position of the chain-lines and stations. The usual mode of plotting these lines is this: a straight line is drawn in the direction in which the meridian line is intended to lie on the plan, and the central point of a protracting scale, or of a protractor (the former of which is divided into 180°, and the latter is a complete circle divided into 360°), is placed on some convenient point of this line with the degree 0° coinciding with the line; fine marks are then made on the paper at the various divisions of the protractor, corresponding to the bearings observed; then the protractor being removed, lines are drawn from the assumed central point through these marks, and are produced in both directions. These lines will evidently be parallel to the directions of the various station-lines joining the successive stations. In order to plot these lines on the plan, a convenient point is chosen for the first station, and through it a line is drawn parallel to the first line of bearing, and in this instance between the N and E, because AB (fig. to Art. 335) is in

this direction; the length of this first station-line is then laid off, and the second station is thus determined. Proceed in the same manner with the second station-line, and the third station will be determined. In this manner the chain-lines and all the stations are plotted.

In order to lay down the positions on the plan of any important points or objects in the country, the bearings of each from at least two stations are to be observed; these bearings are also to be drawn through the formerly assumed central point by means of the protractor; and then lines being drawn from each of the two stations parallel to these bearing-lines (or rhumb-lines) respectively, their intersections determine the positions of the corresponding points. In this manner the positions of the tops of hills, of conspicuous buildings, of a succession of points on the banks of rivers, and of other objects are determined.

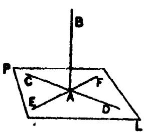
The bearings will sometimes exceed 180°, but this is no inconvenience when a complete or circular protractor is used; but if the horizontal circle of the theodolite is graduated into 180° twice instead of 360°, or if both an ocular and objective vernier are attached to the instrument—that is, verniers under the eye and object-end of the telescope—the bearings can all be got in angles not exceeding 180°.

MENSURATION OF SOLIDS

339. In Solid Geometry the magnitudes have three dimensions—namely, length, breadth, and thickness. They do not, therefore, exist in one plane, but they can be represented by diagrams drawn on a plane.

DEFINITIONS

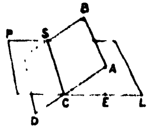
340. When a straight line is at right angles to every line it meets in a plane, it is said to be **perpendicular** to the plane; and if it be at right angles to two straight lines in the plane, it can be proved to be at right angles to every straight line that meets it in that plane.



Let PL be a plane, CD and EF any two straight lines in it, and AB a line perpendicular to both these lines; then AB is perpendicular to the plane.

341. The **inclination** of a **straight line** and a **plane** is the acute angle contained by that line and a line drawn from the point in which the former meets the plane to the foot of the perpendicular to the plane, from any point in the first line.

Thus, if AC is a line, and PL a plane, and AE a perpendicular on the plane, the angle ACE is the inclination of the line AC to the plane PL.



342. The **inclination** of one **plane to another** is the acute angle formed by two lines, one in each plane, drawn from any point in their line of common section, and perpendicular to this line. This angle is called a **dihedral** angle.

Let PL and BD be two planes, and CS their line of common section, and CL, CA lines in these planes perpendicular to CS; then ACE is the inclination of the planes.

343. One plane is **perpendicular** to another when its angle of inclination to it is a right angle.

344. **Parallel planes** are such as do not meet though produced.

345. A **straight line** and **plane** are said to be **parallel** if they do not meet though produced.

346. A **solid** is a figure that has length, breadth, and thickness.

347. A **solid angle** is formed by more than two plane angles in different planes meeting at a point.

348. The **boundaries** of solids are surfaces. A surface no part of which is plane is called a **curve surface.**

349. Any solid contained by planes is called a **polyhedron.**

350. When the solid is contained by four planes it is called a **tetrahedron**; by six, a **hexahedron**; by eight, an **octahedron**; by twelve, a **dodecahedron**; and by twenty, an **icosahedron.**

351. The planes containing a polyhedron are called its **sides** or faces, and the lines bounding its sides, its **edges.**

352. Two polyhedrons are said to be **similar** when they are contained by the same number of similar sides, similarly situated, and containing the same dihedral angles.

353. A polyhedron is said to be **regular** when its sides are equal and regular figures of the same kind, and its solid angles equal.

There are only five regular polyhedrons, of 4, 6, 8, 12, and 20 sides, which are named, as in the definition in Article 350. The first is contained by equilateral triangles, the second by squares,

the third by equilateral triangles, the fourth by pentagons, and the fifth by equilateral triangles.

354. A **prism** is a solid contained by plane figures, of which two are opposite, equal, similar, and having their sides parallel; and the others are parallelograms.

The two parallel similar sides are called the **ends,** or **terminating planes,** either of which is called the base; the other sides are called the **lateral** sides, and constitute the **lateral** or **convex surface.** The edges of the lateral surface are called **lateral edges,** and those of the terminating planes are called **terminating edges.** The **altitude** of a prism is the perpendicular distance of its terminating planes. The prism is said to be **triangular, rectangular, square,** or **polygonal** according as the ends are triangles, rectangles, squares, or polygons. When the lateral edges are perpendicular to the base, it is said to be a **right** prism; in other cases it is said to be **oblique.**

355. A right prism, having regular polygons for its terminating planes, is said to be **regular.**

The line joining the centres of the ends of a regular prism is called its **axis.**

356. A **parallelepiped** is a solid contained by six quadrilateral figures, every opposite two of which are parallel.

It can be proved that these sides are parallelograms. A parallelepiped is a prism having parallelograms for its terminating planes. The other terms applied to a prism, respecting the sides, edges, and altitude, are applicable to the parallelepiped.

357. A **cube** is a solid contained by six equal squares.

358. A **pyramid** is a solid having any rectilineal figure for its base, and for its other sides triangles, having a common vertex outside the base, and for their bases the sides of the base of the solid. The **altitude** of a pyramid is a perpendicular from its vertex on the plane of the base, and the **apothem** is a perpendicular from the vertex on a side of the base.

The pyramid is said to be **triangular, quadrilateral, polygonal,** &c., according as its base is a triangle, a quadrilateral, a polygon, &c.

359. When the base is regular, a line joining its centre and the vertex is called the **axis** of the pyramid.

360. When the axis of a pyramid having a regular base is perpendicular to the base, it is called a **regular** pyramid.

361. A **cone** is a solid contained by a circle as its base, and a

curve surface, such that any straight line drawn from a certain point in it, called its **vertex,** to any point in the circumference of the base, lies wholly in that surface.

362. The line joining the vertex and centre of the base of a cone is called its **axis**; and when the axis of a cone is perpendicular to its base, it is called a **right** cone. Other cones are said to be **oblique.**

The axis of a right cone is also its altitude. A line from the vertex of a right cone to any point in the circumference of its base is called its **slant side.** A right cone may be described by the revolution of a right-angled triangle about one of the sides of the right angle.

363. A **cylinder** is contained by two equal and parallel circles and a convex surface, such that any straight line that joins two points in the circumferences of these circles, and is parallel to the axis, lies wholly in the curve surface.

The circles are called the **bases, ends,** or **terminating planes** of the cylinder; the line joining their centres, its **axis.**

364. When the axis of a cylinder is perpendicular to the plane of one of its bases, it is called a **right** cylinder.

365. A **wedge** is a solid having a rectangular base, and two opposite sides terminating in an edge.

366. A **prismoid** is a solid whose ends are any dissimilar parallel plane figures, having the same number of sides.

When the ends of a prismoid are rectangles, it is said to be **rectangular.**

367. A **sphere,** or **globe,** is a solid such that every point in its surface is equidistant from a certain point within it, and may be generated by the revolution of a semicircle about its diameter.

The point within the sphere is called its **centre**; any line drawn from the centre to the circumference, a **radius**; and any line through the centre, terminated at both extremities by the surface, a **diameter.**

A cylinder **circumscribing** a sphere is a cylinder of the same diameter as the sphere, whose ends touch the sphere, and whose axis passes through its centre.

368. Circles of the sphere, whose planes pass through the centre, are called **great** circles; other circles of the sphere are called **small** circles.

369. A **segment of a sphere** is a portion of it cut off by a **plane**; and a **segment** of a cone, pyramid, or solid with a **plane**

base is a portion of it cut off from the top by a plane parallel to the base.

370. A **frustum** of a solid is a portion contained between the base and a plane parallel to it when its base is plane; or between two parallel planes when the solid has no plane base. The frustum of a sphere is also called a **zone**; and when the ends of a spherical zone are equidistant from the centre, it is called a **middle** zone.

371. A **sector** of a sphere is composed of a segment and a cone having the same base and its vertex in the centre of the sphere; or it is the difference between these two solids, according as the segment is greater or less than a hemisphere.

372. The **unit** of measure for solids is a **cube,** the length of whose edge is the lineal unit. Thus, if the lineal unit is 1 inch, a cube whose edges are each 1 inch is the unit of measure for solids, or, in other words, a **cubic inch** is the **cubic unit.** So, if the lineal unit is a foot, the cubic unit is a cubic foot; and so on.

373. The number of cubic units contained in a solid, or in a vessel, is called its **volume.**

The volume of a solid is also called its **solidity,** or **solid contents,** or **cubic contents**; and that of a vessel is called its **cubic contents,** or **capacity.**

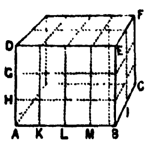
374. Problem I.—To find the solidity of a rectangular parallelepiped.

RULE.—Find the continued product of the length, breadth, and height, and the result is the solidity.

Let l, b, and h be the length, breadth, and the height, and V the volume or solid contents, then

$$V = lbh; \therefore l = \frac{V}{bh}, \ b = \frac{V}{lh}, \text{ and } h = \frac{V}{lb}.$$

Let AF be a right rectangular parallelepiped. Let its length AB be 4 lineal units, as 4 inches, its breadth BC, 2 inches, and its height AD, 3 inches. The solid can evidently be divided into three equal portions by planes through G and H parallel to the base AC; and into four equal portions by means of planes through K, L, M, parallel to the side BF; and into two equal portions by a plane through I parallel to BD. Each of the small cubes into which the solid is now divided is a cubic inch; the number of cubic



inches in the lowest portion HC is 4 × 2, or 8, and in the second and

uppermost portion there are as many; and in them all, therefore, there are $4 \times 2 \times 3$, or 24; that is, to find the cubic contents of the solid, find the continued product of the length, breadth, and height.

EXAMPLES.—1. Find the number of cubic feet in a parallelepiped whose length is $= 15$ feet, breadth $= 12$ feet, and height $= 5$ feet 6 inches.

$$V = lbh = 15 \times 12 \times \tfrac{11}{2} = 990 \text{ cubic feet.}$$

2. How many solid feet are contained in a square parallelepiped, each side of its base being $= 1$ foot 4 inches, and its height $= 5$ feet 6 inches?

$$V = lbh = 1\tfrac{1}{3} \times 1\tfrac{1}{3} \times 5\tfrac{1}{2} = \tfrac{4}{3} \times \tfrac{4}{3} \times \tfrac{11}{2} = \tfrac{88}{9} = 9\tfrac{7}{9} \text{ cubic feet}$$
$$= 9 \text{ cubic feet } 1344 \text{ cubic inches.}$$

EXERCISES

1. Find the solid contents of a block of granite $= 25$ feet long, 4 broad, and 3 thick. . . . . . . 300 cubic feet.

2. The length of a square parallelepiped is $= 15$ feet, and each side of its base $= 1$ foot 9 inches; what are its contents?

45·9375 cubic feet.

3. Find the number of cubic yards in a rectangular block of sandstone, the length of which is $= 16$ feet, its breadth $= 9$ feet, and height $= 6$ feet 9 inches. . . . . . 36 cubic yards.

4. What is the number of cubic feet in a log of wood 10 feet long, 1 foot 6 inches broad, and 1 foot 4 inches thick?

20 cubic feet.

5. Find the contents of a parallelepiped whose length, breadth, and thickness are respectively $= 30·5$ feet, 9·5 feet, and 2 feet.

$= 579·5$ cubic feet.

6. Find the solidity of a block of marble whose length, breadth, and thickness are respectively $= 10$ feet, $5\tfrac{3}{4}$ feet, and $3\tfrac{1}{2}$ feet.

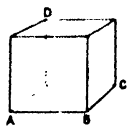
201·25 cubic feet.

375. Problem II. To find the solidity of a cube.

RULE.—Find the cube of one of its edges, and the result is the solidity.

Let $e =$ an edge of a cube,

then $V = e^3$.



The reason of the rule is evident, since a cube is just a parallelepiped whose length, breadth, and height are equal.

EXAMPLE.—How many cubic feet are contained in a block

of granite of the form of a cube, one of its edges being = 2 feet 6 inches?

$$V = e^3 = (2\tfrac{1}{2})^3 = (\tfrac{5}{2})^3 = \tfrac{125}{8} = 15{\cdot}625 \text{ cubic feet.}$$

EXERCISES

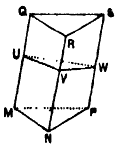
1. Find the solidity of a cube whose edge is = 4 feet. = 64 cubic feet.

2. How many cubic feet are contained in a cube whose edge is = 7 feet 6 inches? . . . . = 421·875 cubic feet.

3. The edge of a cube is = 12 feet 9 inches; required its volume. = 2072·671875 cubic feet.

4. Find the contents of a cube whose edge is = 6·5 yards. = 274·625 cubic yards.

376. **Problem III.—To find the solidity of a prism, or of any parallelepiped.**



RULE.—Multiply the area of the base by the height, and the product will be the solidity.

Let b denote the base, and h the height,

then $V = bh$.

EXAMPLE.—Find the solidity of a regular triangular prism, a side of its base being = 5 feet, and its length = 20 feet.

By Art. 268, area of base = $\cdot433 \times 5^2 = 10{\cdot}825$;

hence $V = bh = 10{\cdot}825 \times 20 = 216{\cdot}5$ cubic feet.

EXERCISES

1. What is the solidity of a triangular prism whose length is = 10 feet 6 inches, one side of its base being = 14 inches, and the perpendicular on it from the opposite angle = 15 inches? = 7·65625 cubic feet.

2. Find the solidity of a regular triangular prism whose length is = 9 feet, and one side of its base = 1 foot 6 inches. = 8·76825 cubic feet.

3. Find the contents of a square prism whose length is = 20·5 feet, and one side of its base = 2·5 feet. . . . = 128·125 cubic feet.

4. What is the solidity of a regular pentagonal prism whose length is = 25 feet, and a side of its base = 10 feet? = 4301·1935 cubic feet.

5. Find the contents of a regular hexagonal prism whose length is = 18 feet, and a side of its base = 16 inches. = 83·138 cubic feet.

6. What is the solidity of a regular octagonal prism = 20 feet long, and a side of its base = 10 feet? . . = 9656·854 cubic feet.

377. Problem IV.—To find the surface of a cube, parallelepiped, or prism.

RULE I.—**When the prism or parallelepiped is right,** multiply the perimeter of the base by the height of the solid, and the product will be the lateral surface, to which add double the area of the base, and the sum is the whole surface of the solid.

RULE II.—**When the prism or parallelepiped is oblique,** its lateral surface is found by multiplying the perimeter of a section perpendicular to one of the lateral edges by that edge.

The surface of a cube can be found by the first rule; but it is more readily found by taking six times the square of one of its edges.

Let e = one of the lateral edges of a prism or parallelepiped,
p = the perimeter of the base when the solid is right,
p' = " " of a section perpendicular to one of the edges, UVW (fig. to Prob. III.),
b = area of the base,
s = whole surface;
then $s = pe + 2b$, when the solid is right,
and $s = p'e + 2b$, " " is oblique;
then $s = 6e^2$, when the figure is a cube.

The reason of the rule is evident from those for the Mensuration of Surfaces.

EXAMPLES.—1. Find the surface of a cube, one of its edges being = 18 inches.

$$s = 6e^2 = 6 \times (1·5)^2 = 6 \times 2·25 = 13·5 \text{ square feet.}$$

2. What is the surface of an oblique prism = 20 feet long, the perimeter of a section perpendicular to one of its lateral edges being = 25 feet, and its base a rectangle = 6 feet long and 4 broad?

$$s = p'e + 2b = 25 \times 20 + 2 \times 4 \times 6 = 500 + 48 = 548 \text{ square feet.}$$

EXERCISES

1. Find the surface of a cube whose edges are each = 10 feet. = 600 square feet.

2. What is the surface of a cube whose edge is = 2 feet 4 inches? = 32⅔ square feet.

3. Find the number of square yards in the surface of a cube whose edge is = 11 feet. . . = 80 square yards 6 square feet.

4. What is the surface of a right rectangular parallelepiped whose length is $=36$ feet, breadth $=10$ feet, and thickness $=8$ feet?
$=1456$ square feet.

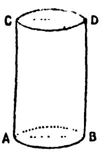
5. The length of a rectangular cistern within is $=3$ feet 2 inches, the breadth $=2$ feet 8 inches, and height $=2$ feet 6 inches; required the internal surface, and also the expense of lining it with lead at 2d. per lb., the lead being 7 lb. weight per square foot.
$=37\frac{11}{18}$ square feet, and £2, 3s. $10\frac{5}{9}$d.

6. Find the surface of a right triangular prism, its length being $=20$ feet, and the sides of its base respectively $=6$, 8, and 10 feet.
$=528$ square feet.

7. What is the surface of a regular pentagonal prism whose length is $=32\frac{1}{2}$ feet, and a side of its base $=6\frac{1}{4}$ feet?
$=1150{\cdot}037$ square feet.

8. What is the surface of an oblique prism, having a regular hexagonal base whose side is $=10$ inches, the lateral edges of the prism being $=20$ feet, and the perimeter of a section perpendicular to them $=4\frac{1}{2}$ feet? . . . . . . $=93{\cdot}6084$ square feet.

378. Problem V.—To find the solidity of a cylinder.



RULE.—Multiply the area of the base by the altitude of the cylinder, and the product will be the solidity.

Or, $V=bh$, where $b={\cdot}7854d^2$, or πr^2 by Art. 273.

EXAMPLE.—What is the solidity of a cylinder whose length is $=21$ feet, the diameter of its base being $=15$ inches?

Here $\quad b={\cdot}7854d^2={\cdot}7854\times(1{\cdot}25)^2=1{\cdot}227$;
and $\quad V=bh=1{\cdot}227\times21=25{\cdot}767$ cubic feet.

EXERCISES

1. Find the solidity of a cylinder the height of which is $=25$ inches, and the diameter of its base $=15$ inches. $=2{\cdot}5566$ cubic feet.

2. What is the volume of a cylinder whose altitude is $=28$ feet, and diameter $=2\frac{1}{2}$ feet? . . . . . . $=137{\cdot}445$ cubic feet.

3. The circumference of the base of an oblique cylinder is $=20$ feet, and its perpendicular height $=19{\cdot}318$; what is its volume?
$=614{\cdot}91$ cubic feet.

4. The circumference of the base of an oblique cylinder is $=40$ feet, its axis $=22$ feet, and the axis is inclined to the base at an angle of 75°; what is its volume? . . $=2705{\cdot}6818$ cubic feet.

379. Problem VI.—To find the surface of a right cylinder.

RULE.—Multiply the circumference of its base into its height, and the product is the convex surface; and double the area of the base being added, gives the whole surface of the cylinder.

Let d, r, and c = diameter, radius, and circumference of base,
h = height of cylinder,
b = its base,
z = convex surface;

then $z = ch = 3{\cdot}1416dh = 2\pi rh$,

and $s = z + 2b = ch + 2\pi r^2 = 2\pi r(h + r)$.

EXAMPLE.—The radius of the base of a right cylinder is = 5 feet, and its height = 20; what is its surface?

$$z = ch = 2\pi rh = 2 \times 3{\cdot}1416 \times 5 \times 20 = 628{\cdot}32;$$
$$2b = 2\pi r^2 = 2 \times 3{\cdot}1416 \times 5^2 = 157{\cdot}08;$$

hence $s = z + 2b = 785{\cdot}4$ square feet;

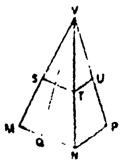
or $s = 2\pi r(h + r) = 2 \times 3{\cdot}1416 \times 5 \times 25 = 785{\cdot}4$ square feet.

The curve surface of a right cylinder is evidently equal to the area of a rectangle whose height is that of the cylinder, and length equal to its circumference.

EXERCISES

1. Find the surface of a right cylinder whose length is = 20 feet, and circumference = 6. . . . . . 125·72958 square feet.
2. What is the convex surface of a right cylinder whose diameter is = 10 inches, and length = 14½ feet? . . . 37·961 square feet.
3. Find the convex surface of a cylinder whose length is = 40 feet, and the diameter of its base = 4 feet. 502·656 square feet.
4. What is the superficies of a right cylinder whose length is = 40 feet 8 inches, and the diameter of its base = 10 feet 6 inches? 1514·6439 square feet.

380. Problem VII.—To find the solidity of a pyramid.



RULE.—Multiply the area of the base of the pyramid by its perpendicular height, and one-third of the product is the solidity.

$$V = \tfrac{1}{3}bh.$$

EXAMPLE.—Find the solidity of a rectangular pyramid, the length and breadth of its base being = 6 and 4 feet respectively, and its altitude = 20 feet.

$$V = \tfrac{1}{3}bh = \tfrac{1}{3} \times 6 \times 4 \times 20 = 160 \text{ cubic feet.}$$

EXERCISES

1. What is the solidity of a square pyramid, each side of its base being $=3$ feet, and its altitude $=10$ feet? . . $=30$ cubic feet.

2. Find the volume of a regular triangular pyramid, a side of its base being $=6$ feet, and its altitude $=60$ feet. $=311{\cdot}769$ cubic feet.

3. Find the solidity of a square pyramid, a side of its base being $=30$ feet, and its apothem $=25$ feet. . . $=6000$ cubic feet.

4. What is the solidity of a pentagonal pyramid, with a regular base, each side of which is $=4$ feet, and the altitude of the pyramid $=30$ feet? . . . . $=275{\cdot}276$ cubic feet.

381. Problem VIII.—To find the surface of a pyramid.

RULE.—When the pyramid is regular, multiply the perimeter of the base by the apothem of the pyramid, and half the product is the convex surface, to which add the area of the base, and the sum is the whole surface.

When the pyramid is irregular, find separately the areas of the lateral triangles, and to their sum add the area of the base.

Let $c=$ the perimeter of the base of a regular pyramid,
$p=$ the apothem of the pyramid $=VQ$ (fig. to last problem),
and z and b, as in Prob. VI.;
then, for a regular pyramid,

$$z=\tfrac{1}{2}pc, \text{ and } s=z+b.$$

For the area of one of the lateral triangles is evidently equal to half the product of the apothem by the base of the triangle; hence the truth of the rule is evident.

EXAMPLE.—Find the surface of a square pyramid, its apothem being $=40$ feet, and each side of its base $=6$ feet.

$$z=\tfrac{1}{2}cp=\tfrac{1}{2}\times 24\times 40=480,$$

and
$$s=z+b=480+6^2=516 \text{ square feet.}$$

EXERCISES

1. What is the surface of a square pyramid, a side of its base being $=5$ feet, and the apothem of the pyramid $=12$ feet? $=145$ square feet.

2. Find the convex surface of a pyramid whose apothem is $=10$ feet, and its base an equilateral triangle whose side $=18$ inches. $22{\cdot}5$ square feet.

3. What is the surface of a regular pentagonal pyramid whose apothem is = 10 feet, and each side of its base = 1 foot 8 inches?
=46·4457 square feet.

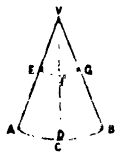
4. The apothem of a regular hexagonal pyramid is = 8 feet, and a side of its base = $2\frac{1}{2}$ feet; what is its surface? =76·24 square feet.

382. Problem IX.—To find the solidity of a cone.

RULE.—Multiply the area of the base by the altitude of the cone, and one-third of the product is the solidity.

Or, $V = \frac{1}{3}bh$, where b is found by Art. 270.

EXAMPLE.—Find the solidity of a right cone, the slant side of which is = 5 feet, and the diameter of its base = 6 feet.



Here $h = VD$, and $VD^2 = AV^2 - AD^2$, or if $AV = p$, $h^2 = p^2 - (\frac{1}{2}d)^2 = 5^2 - 3^2 = 16$, or $h = 4$,
and $b = \cdot7854d^2 = \cdot7854 \times 6^2 = 28\cdot2744$ square feet;
hence $V = \frac{1}{3}bh = \frac{1}{3} \times 28\cdot2744 \times 4 = 37\cdot6992$ cubic feet.

EXERCISES

1. The altitude of a right cone is = 30 feet, and the diameter of its base = 6 feet; what is its volume? . . . =282·744 cubic feet.

2. The diameter of the base of a cone is = 10, and its altitude = 12; what is its solidity? . . . . =314·16.

3. Find the volume of a cone whose altitude is = 10 feet, and the diameter of its base = 2 feet 8 inches. . . =18·617 cubic feet.

4. Find the solidity of a cone, the diameter of whose base is = 3 feet, and its altitude = 30 feet. . . . =70·686 cubic feet.

5. The diameter of the base of a cone is = 3 feet 4 inches, and its slant side = 16 feet; what is its solidity? =46·2856 cubic feet.

6. The circumference of the base of a cone is = 20 feet, and its height = 25; required its volume. . . . =265·258 cubic feet.

383. Problem X.—To find the surface of a right cone.

RULE.—Multiply the circumference of the base of the cone by the slant side, and half the product will be the curve surface, to which add the area of the base, and the sum will be the whole surface.

Or, $z = \frac{1}{2}cp$, and $b = \cdot7854d^2$, or $= \pi r^2$, and $s = z + b$.

It is evident that if the cone (last fig.) AVB be rolled on a plane, the curve surface will be equivalent to a circular sector whose

radius is BV, the slant side of the cone, and its arc the circumference of the base of the cone, from which the rule is evident.

EXAMPLE.—Find the surface of a cone whose base has a diameter of 12 feet, and whose height is $=8$ feet.

Here (last fig.) $AD=r=6$, and $DV=h=8$;
hence $AV^2=p^2=h^2+r^2=64+36=100$, and $p=10$;
therefore, $z=\frac{1}{2}cp=\frac{1}{2}\times 3{\cdot}1416\times 12\times 10=188{\cdot}496$,
and $s=z+b=188{\cdot}496+{\cdot}7854\times 12^2=301{\cdot}5936$ square feet.

EXERCISES

1. What is the surface of a cone, the diameter of its base being $=5$ feet, and its slant height $=18$ feet? . $=161{\cdot}007$ square feet.

2. The slant height of a cone is $=40$ feet, and the diameter of its base $=9$ feet; what is its surface? . $=629{\cdot}1054$ square feet.

3. The diameter of the base of a cone is $=6$ feet, and its slant height $=30$ feet; required its convex surface. $=282{\cdot}744$ square feet.

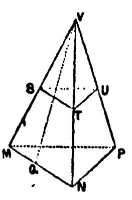
4. The slant height of a cone is $=18\frac{1}{4}$ feet, and the circumference of its base $=10\frac{3}{4}$ feet; find its convex surface.
$=98{\cdot}09375$ square feet.

384. Problem XI.—To find the solidity of a frustum of a pyramid.

RULE I.—Add together the areas of the two ends and their mean proportional, multiply this sum by the altitude of the frustum, and one-third of the product will be the solidity.

RULE II.—Multiply the area of the greater end by one of its sides, and that of the smaller end by its corresponding side; divide the difference of these products by the difference of the sides, and multiply the quotient by the height of the frustum, and one-third of this product will be the solidity.

RULE III.—When the ends are regular polygons, to the sum of the squares of the ends add the product of the ends; multiply the sum by the tabular area corresponding to the polygons, and by a third of the height, and the result will be the solidity.



Let MNPUTS be a frustum of a pyramid, the complete pyramid being VMNP.

Let the heights of the whole pyramid and the smaller one VSTU be h', h'', and that of the frustum h: and let V', V'', V denote the solidities of these three solids respectively; B, b the greater and smaller

ends of the frustum; and E, e two of their corresponding sides, as MN, ST; also,

When the ends are regular polygons, let A′ = the corresponding tabular area (Art. 268), then the three rules above can be expressed thus:—

$$V = \frac{h}{3}(B + b + \sqrt{Bb}) = \frac{h}{3}\left(\frac{BE - be}{E - e}\right) = \tfrac{1}{3}(E^2 + Ee + e^2)A'h.$$

EXAMPLE.—Find the solidity of a frustum of a square pyramid, a side of the ends being = 6 and 4 feet, and the altitude = 10 feet.

By the first rule—

$$\sqrt{Bb} = \sqrt{(36 \times 16)} = \sqrt{576} = 24;$$

hence $V = \frac{h}{3}(B + b + \sqrt{Bb}) = \frac{10}{3}(36 + 16 + 24) = 253\tfrac{1}{3}$ cubic feet.

By the second rule—

$$V = \frac{h}{3}\left(\frac{BE - be}{E - e}\right) = \frac{10}{3}\left(\frac{36 \times 6 - 16 \times 4}{6 - 4}\right) = \frac{10}{3} \times 76 = 253\tfrac{1}{3} \text{ cubic feet.}$$

By the third rule—

$$V = \tfrac{1}{3}(E^2 + Ee + e^2)A'h = \tfrac{1}{3}(6^2 + 6 \times 4 + 4^2) \times 1 \times 10$$
$$= \tfrac{1}{3}(36 + 24 + 16) \times 10 = \tfrac{1}{3} \times 76 \times 10 = 253\tfrac{1}{3} \text{ cubic feet.}$$

If $V' = \tfrac{1}{3}Bh'$, and $V'' = \tfrac{1}{3}bh''$;

then $$V = V' - V'' = \tfrac{1}{3}(Bh' - bh'') \quad \ldots \quad \ldots \quad [1].$$

But the two ends are proportional to the squares of two of their corresponding sides, as they are similar, or of the edges MV, SV, or of the altitudes h', h'';

hence, $$B : b = h'^2 : h''^2; \text{ hence } h'' = h'\sqrt{\frac{b}{B}} \quad \ldots \quad \ldots \quad [2].$$

Also, $$\sqrt{B} : \sqrt{b} = h' : h'',$$

and $$\sqrt{B} - \sqrt{b} : \sqrt{B} = h' - h'' : h' \text{ or } h : h' \quad \ldots \quad \ldots \quad [3].$$

Therefore, $h' = \dfrac{h\sqrt{B}}{\sqrt{B} - \sqrt{b}}$; and hence $h'' = \dfrac{h\sqrt{b}}{\sqrt{B} - \sqrt{b}}$.

Substituting these values of h', h'' in [1], it becomes

$$V = \tfrac{1}{3} \cdot \frac{Bh\sqrt{B} - bh\sqrt{b}}{\sqrt{B} - \sqrt{b}} = \frac{h}{3}\left(\frac{B^{\frac{3}{2}} - b^{\frac{3}{2}}}{B^{\frac{1}{2}} - b^{\frac{1}{2}}}\right) = \frac{h}{3}(B + b + \sqrt{Bb}).$$

This result is the first rule. In order to prove the second, substitute in the proportions [2], [3] above the quantities E and e, instead of $\sqrt{B}$ and $\sqrt{b}$, since they are proportional to them, and the expressions for h' and h'' will then become

$h' = \dfrac{hE}{E - e}$, $h'' = \dfrac{he}{E - e}$; which, being substituted in [1], gives

$$V = \tfrac{1}{3}\frac{BhE - bhe}{E - e} = \frac{h}{3}\left(\frac{BE - be}{E - e}\right).$$

When the ends are regular polygons, then A' being the tabular area, as in Art. 268, $B = A'E^2$, and $b = A'e^2$; hence, substituting these values for B and b, the last expression for s gives

$$V = \frac{h}{3}\left(\frac{A'E^3 - A'e^3}{E - e}\right) = \tfrac{1}{3}\left(\frac{E^3 - e^3}{E - e}\right)A'h = \tfrac{1}{3}(E^2 + Ee + e^2)A'h.$$

Exercises

1. Find the solidity of a frustum of a square pyramid, the sides of its two ends being $= 3$ feet and $2\frac{1}{2}$ feet, and its height $= 5$ feet. $= 37\frac{11}{12}$ cubic feet.

2. Find the solidity of a frustum of a square pyramid, the sides of its ends being $= 10$ and 16 inches, and its length $= 18$ feet. $= 21\frac{1}{2}$ cubic feet.

3. What is the solidity of a frustum of a regular hexagonal pyramid, the sides of its ends being $= 4$ and 6 feet, and its length $= 24$ feet? . . . . $= 1579{\cdot}6303$ cubic feet.

4. Find the solidity of a frustum of a regular octagonal pyramid, the sides of its bases being $= 3$ and 5 feet, and its height $= 10$ feet. $= 788{\cdot}643$ cubic feet.

5. Find the number of solid feet in a piece of timber of the form of a frustum of a square pyramid, the sides of its ends being $= 1$ foot and $2\frac{1}{2}$ feet, and the perpendicular length of one of the sides $= 48$ feet. . . . . $= 155{\cdot}981$ cubic feet.

385. Problem XII.—To find the surface of a frustum of a pyramid.

Rule I.—When the pyramid is regular, add together the perimeters of the two ends; multiply their sum by the lateral length, and half the product will be the lateral surface; to which add the areas of the two ends, and the sum will be the whole surface.

Rule II.—When the pyramid is irregular, the lateral planes are trapeziums, and their areas being separately found by Art. 259, and those of the two ends added, the sum will be the whole surface.

Let P and p be the perimeters of the two ends, and l the lateral length, or apothem, and B and b the areas of the two ends; then the first rule is—

$$s = \tfrac{1}{2}(P + p)l + B + b.$$

Example.—What is the surface of a frustum of a regular triangular pyramid, a side of its ends being $= 3$ and 2 feet, and the lateral length $= 10$ feet?

Here $$P = 3 \times 3 = 9,\ p = 2 \times 3 = 6:$$
hence $$s = \tfrac{1}{2}(P+p)l + B + b = \tfrac{1}{2}(9+6) \times 10 + \cdot 433(3^2 + 2^2)$$
$$= 75 + \cdot 433 \times 13 = 80 \cdot 629 \text{ square feet.}$$

EXERCISES

1. Find the surface of a frustum of a regular square pyramid, the sides of its ends being = 14 and 24 inches, and the lateral length = 2 feet 3 inches. . . . . - 19·61 square feet.

2. Find the surface of a frustum of a regular square pyramid whose lateral length is = 5 feet, the sides of its ends being = 13 and 20 inches. . . . . 31·45138 square feet.

3. What is the surface of a frustum of a regular pentagonal pyramid, its lateral length being 5 feet 10 inches, and the sides of its ends = 10 and 15 inches? . . . 34·26496 square feet.

386. **Problem XIII.—To find the solidity of a frustum of a cone.**

RULE I.—To the squares of the diameters of the two ends add the product of the diameters; multiply the sum by the height of the frustum, and this product by ·2618; the result will be the solidity; or,

RULE II.—To the squares of the circumferences of the two ends add the product of the circumferences; multiply the sum by the height of the frustum, and this product by ·026526; the result will be the solidity.

Let D and d be the diameters of the two ends, C and c their circumferences, and h the height of the frustum (fig. to Prob. IX.); then $$V = \cdot 2618h(D^2 + d^2 + Dd),$$
and $$V = \cdot 026526h(C^2 + c^2 + Cc).$$

EXAMPLE.—What is the solidity of a frustum of a cone whose height is = 5 feet, the diameters of its two ends being respectively = 2 and 3 feet.

$$V = \cdot 2618h(D^2 + d^2 + Dd) = \cdot 2618 \times 5(3^2 + 2^2 + 2 \times 3)$$
$$= 1 \cdot 309 \times 19 = 24 \cdot 871.$$

Let the notation used in Prob. XI. for the frustums of pyramids be similarly applied to the conic frustums in the figure, and by the same reasoning it is proved that
$$V = \tfrac{1}{3}h(B + b + \sqrt{Bb}).$$
But $B = \cdot 7854D^2$, and $b = \cdot 7854d^2$; and hence
$$\sqrt{Bb} = \sqrt{\cdot 7854^2 D^2 d^2} = \cdot 7854Dd;$$
and therefore $V = \cdot 7854(D^2 + d^2 + Dd)\frac{h}{3} = \cdot 2618h(D^2 + d^2 + Dd)$; which is the first rule.

And since $\cdot 7854D^2 = \cdot 0795775C^2$ by Articles 273 and 274, and $\cdot 7854d^2 = \cdot 0795775c^2$, and $\cdot 7854Dd = \cdot 0795775Cc$; by substitution,

$$V = \cdot 0795775\frac{h}{3}(C^2 + c^2 + Cc) = \cdot 026526h(C^2 + c^2 + Cc).$$

EXERCISES

1. What is the solidity of a frustum of a cone whose height is $=10$ feet, and the diameters of its ends $=2$ and 4 feet?

$=73\cdot 304$ cubic feet.

2. Find the solidity of a conic frustum of which the height is $=9$ feet, and the diameters of its ends $=1\frac{1}{2}$ and $2\frac{1}{2}$ feet.

$=28\cdot 8634$ cubic feet.

3. What is the solidity of a conic frustum whose height is $=4\frac{1}{2}$ feet, and the diameters of its two ends $=2$ and 4 feet?

$=32\cdot 9868$ cubic feet.

4. What is the solidity of a conic frustum whose length is $=25$ feet, and the diameters of its two ends $=10$ and 20 feet?

$=4581\cdot 5$ cubic feet.

5. Find the solidity of a conic frustum, its length being $=38$ inches, and the diameters of its ends $=18$ and 32 inches.

$=19140\cdot 72$ cubic inches.

6. How many cubic feet are contained in a ship's mast whose length is $=72$ feet, and the diameters of its ends $=1$ foot and $1\frac{1}{2}$?

$=89\cdot 5356$ cubic feet.

7. How many cubic feet are contained in a cask which is composed of two equal and similar conic frustums, united at their greater ends, its bung diameter being $=14$ inches, its head diameter $=10$ inches, and its length $=20$ inches?

$=1\cdot 32112$ cubic feet.

387. **Problem XIV.—To find the surface of a frustum of a right cone.**

RULE.—Multiply the sum of the circumferences of the two ends by the slant side, and half the product will be the convex surface; to which add the areas of the two ends for the whole surface.

Let C, c be the circumferences of the ends, D and d their diameters, B and b their areas, and p the slant side AE (fig. to Prob. IX.);

then $$s = \tfrac{1}{2}p(C + c) + B + b;$$

where $C + c = 3\cdot 1416(D + d)$, and $B + b = \cdot 7854(D^2 + d^2)$.

EXAMPLE.—Find the whole surface of a frustum of a right cone, the slant side being $=20$ feet, and the diameters of the ends $=2$ and 4 feet.

$$C+c=3\cdot1416(D+d)=3\cdot1416\times6=18\cdot8496,$$
$$B+b=\cdot7854(D^2+d^2)=\cdot7854\times20=15\cdot708,$$
and $$s=\tfrac{1}{2}p(C+c)+B+b=\tfrac{1}{2}\times20\times18\cdot8496+15\cdot708$$
$$=204\cdot204 \text{ square feet.}$$

The rule is easily derived from that in Prob. X. For let C, c be the circumferences of the two ends, p the slant side AE (fig. to Art. 382) of the frustum, and p', p'' the slant sides of the two cones VAB, VEG, and s', s'' their convex surfaces, and s that of the frustum; then (Art. 383)

$$s'=\tfrac{1}{2}p'C,\ s''=\tfrac{1}{2}p''c; \text{ therefore } s=s'-s''=\tfrac{1}{2}(p'C-p''c),$$
or $$s=\tfrac{1}{2}\{pC+p''C-p''c\}=\tfrac{1}{2}\{pC+p''(C-c)\} \quad \ldots \quad \ldots \quad [1].$$

But $C:c=p':p''$; and hence $C-c:c=p'-p''$, or $p:p''$; therefore $p''=\frac{pc}{C-c}$, and substituting this in [1] for p'', it becomes

$$s=\tfrac{1}{2}p(C+c)=\text{ the curve surface.}$$

Exercises

1. What is the convex surface of a frustum of a right cone whose slant side is $=10$ feet, and the circumferences of its two ends $=5$ and 15 feet? . . . . $=100$ square feet.

2. Find the convex surface of a frustum of a right cone whose slant side is $=39$ feet, and the circumferences of its two ends $=15$ feet 9 inches and 22 feet 6 inches. . $=745\cdot875$ square feet.

3. What is the surface of a frustum of a right cone, its length being $=31$ feet, and the diameters of its two ends $=12$ and 20 feet? $=1985\cdot4912$ square feet.

4. If a segment whose slant side is $=6$ feet is cut off from the upper part of a cone whose slant side is $=30$ feet, and the circumference of its base $=10$ feet, what is the convex surface of the frustum? . . . . $=144$ square feet.

388. Problem XV.—To find the solidity of a wedge.

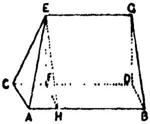
RULE.—To twice the length of the base add the length of the edge, and find the continued product of this sum, the breadth of the base and the height of the wedge, and take one-sixth of this product.

Let l, e, b, and h denote respectively the length of the base AB, the edge EG, the breadth of the base BD, and the height; then $$v=\tfrac{1}{6}(e+2l)bh.$$

EXAMPLE.—Find the solidity of a wedge, the length of which is $=5$ feet 4 inches, its base $=9$ inches broad, the edge of the wedge $=3$ feet 6 inches, and its height $=2$ feet 4 inches.

$$v=\tfrac{1}{6}(e+2l)bh=\tfrac{1}{6}(3\tfrac{1}{2}+10\tfrac{2}{3})\tfrac{3}{4}\times2\tfrac{1}{3}=\tfrac{1}{6}\times14\tfrac{1}{6}\times\tfrac{3}{4}\times\tfrac{7}{3}=4\cdot132 \text{ cubic feet.}$$

Let ADE be a wedge. Through E let a plane EFH pass parallel to the end GDB. Then EDH is a prism, and is equal to three times the pyramid, whose base is the triangle DBH, and vertex G (Solid Geom. II. 17), or it is



$$=\tfrac{1}{2}DH\,.\,\tfrac{1}{3}h\times 3=\tfrac{1}{2}DH\,.\,h=\tfrac{1}{2}HB\,.\,bh=\tfrac{1}{2}ebh.$$

Also, the pyramid ECAF $=$ AF . $\frac{1}{3}h=\frac{1}{3}$AH . $bh=\frac{1}{3}(l-e)bh$;

hence $$V=\{\tfrac{1}{2}e+\tfrac{1}{3}(l-e)\}bh=\tfrac{1}{6}(e+2l)bh.$$

Were the edge longer than the base, the formula would be the same; for the expression $\frac{1}{2}e+\frac{1}{3}(l-e)$ would then become $\frac{1}{2}e-\frac{1}{3}(e-l)=\frac{1}{6}(3e-2e+2l)=\frac{1}{6}(e+2l)$, as before.

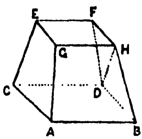
Exercises

1. Find the contents of a wedge whose base is $=16$ inches long and $2\frac{1}{4}$ broad, its height being $=7$ inches, and its edge $=10\frac{1}{2}$ inches.
$=111{\cdot}5625$ cubic inches.

2. The length and breadth of the base of a wedge are $=5$ feet 10 inches and $2\frac{1}{2}$ feet, the length of the edge is $=9$ feet 2 inches, and the height $=34{\cdot}29016$ inches; what is its solidity?
$=24{\cdot}8048$ cubic feet.

389. Problem XVI.—To find the solidity of a prismoid.

Rule.—To the sum of the areas of the two ends add four times the area of the middle section parallel to the ends; multiply this sum by the height, and take one-sixth of the product.



Let L $=$ the length of the base AB,
B $=$ " breadth of the base BD,
l $=$ " length of the top GH,
b $=$ " breadth of the top GH,
M $=$ " length of middle section,
m $=$ " breadth of middle section,
h $=$ " height of the prismoid;

then $M=\frac{1}{2}(L+l)$, and $m=\frac{1}{2}(B+b)$,

and $V=\frac{1}{6}(BL+bl+4Mm)h.$

Example.—Find the solidity of a prismoid, the length and breadth of its base being $=10$ and 8, those of the top $=6$ and 5, and the height $=40$ feet.

Here $M=\frac{1}{2}(L+l)=\frac{1}{2}(10+6)=\frac{1}{2}\times 16=8,$
$m=\frac{1}{2}(B+b)=\frac{1}{2}(8+5)=\frac{1}{2}\times 13=6{\cdot}5,$

and $V=\frac{1}{6}(BL+bl+4Mm)h$
$=\frac{1}{6}(10\times 8+6\times 5+4\times 8\times 6{\cdot}5)40=2120$ cubic feet.

The prismoid ADG is evidently equal to two wedges ADG and GFC; the base and edge of the former being AD and GH, and those of the latter FG and CD; and their height that of the prismoid.

Let V′ and V″ be the volumes of these wedges;

then $\quad V' = \frac{1}{6}(l + 2L)Bh$, $V'' = \frac{1}{6}(L + 2l)bh$, and $V = V' + V''$;

also, $\quad 4Mm = (B + b)(L + l) = BL + bl + Bl + bL$;

hence $\quad V = \frac{1}{6}(2BL + Bl + 2bl + bL)h = \frac{1}{6}(BL + bl + 4Mm)h$.

EXERCISES

1. What is the solidity of a log of wood of the form of a rectangular prismoid, the length and breadth of one end being 2 feet 4 inches and 2 feet, and those of the other end 1 foot and 8 inches, and the height or perpendicular length 61 feet?

$144{\cdot}\dot{5}9\dot{2}$ cubic feet.

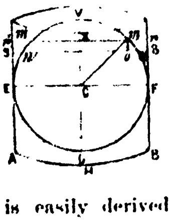
2. Find the capacity of a trough of the form of a prismoid, its bottom being 48 inches long, 40 inches broad, and its top 5 feet long and 4 feet broad, and its depth 3 feet. . $49\frac{2}{3}$ cubic feet.

3. What is the volume of a prismoid, the length and breadth of its greater end being 24 and 16 inches, those of its top 16 and 12 inches, and its length 120 inches? . . $19{\cdot}\dot{6}2\dot{9}$ cubic feet.

390. Problem XVII. To find the solidity of a sphere.

RULE I.—Find the solidity of the circumscribing cylinder that is, a cylinder whose diameter and height are equal to the diameter of the sphere and two-thirds of it will be the volume of the sphere.

RULE II.—Multiply the cube of the diameter of the sphere by ·5236, or more accurately by ·5235988, and the product will be its solidity.



Let d = the diameter of the sphere;

then $\quad V = {\cdot}5236d^3$.

EXAMPLE.—Find the solidity of a sphere whose diameter is 2 feet 8 inches.

$$V = {\cdot}5236d^3 = {\cdot}5236 \times (2\tfrac{2}{3})^3 = {\cdot}5236 \times (\tfrac{8}{3})^3$$
$$= {\cdot}5236 \times \tfrac{512}{27} = 9{\cdot}929 \text{ cubic feet.}$$

The first rule is derived from a theorem discovered by Archimedes. The second rule is easily derived from the first. For (Art. 378) the volume of a cylinder is $v = bh = {\cdot}7854d^2h$; and when $h = d$, which is the case for the cylinder

ABV circumscribing the sphere, then $v = \cdot 7854d^2d = \cdot 7854d^3$, and two-thirds of this is the volume of the sphere, or $V = \cdot 5236d^3$.

Note.—A sphere may also be considered as composed of an indefinite number of minute pyramids, whose bases are in the surface, and vertices in the centre of the sphere, and the sum of their solidities would be equal to the surface of the sphere, multiplied by one-third of its radius. But (Art. 391) the surface $= 4 \times \cdot 7854d^2$; hence the volume $= 4 \times \cdot 7854d^2 \times \frac{1}{6}d = \cdot 5236d^3$.

The volumes of spheres are proportional to the cubes of the radii of the spheres (Eucl. XII. 18).

Exercises

1. How many cubic inches are contained in a sphere = 25 inches in diameter? . . . . = 8181·25 cubic inches.
2. Find the solidity of a sphere, the diameter of which is = $8\frac{1}{2}$ inches. . . . . = 321·5558 cubic inches.
3. What is the solidity of a sphere whose diameter is = 5 inches? = 65·45 cubic inches.
4. How many cubic feet of gas can a balloon of a spherical form contain, its diameter being = 50 feet? . . = 65450 cubic feet.
5. Find the solidity of a sphere whose diameter is = 6 feet 2 inches. . . . . = 122·7866 cubic feet.
6. The diameter of a globe is = 4 feet 2 inches; what is its volume? . . . . = 37·876 cubic feet.

391. Problem XVIII.—To find the surface of a sphere.

I. The surface of a sphere is equal to four times the area of a great circle of the sphere; or,

II. The surface of a sphere is equal to the product of the square of its diameter by 3·1416; or,

III. The surface of a sphere is equal to the product of its circumference by its diameter; or,

IV. The surface of a sphere is equal to the convex surface of the circumscribing cylinder.

The surface and the volume of a sphere are two-thirds of the surface and the volume of the circumscribed cylinder.*

$$s = cd, \text{ or } s = 3\cdot 1416d^2 = \pi d^2 = 4\pi r^2.$$

* *Archimedes' Sphere and Cylinder*, I. 34, Cor. (Heiberg's edition).

Example.—How many square feet of sheet copper are contained in a hollow copper globe = 25 inches in diameter?

$$s = 3{\cdot}1416d^2 = 3{\cdot}1416 \times 25^2 = 3{\cdot}1416 \times 625$$
$$= 1963{\cdot}5 \text{ square inches} = 13{\cdot}6347 \text{ square feet.}$$

Let $mnm'n'$ and $rsr's'$ be two corresponding zones of the sphere GEVF and its circumscribing cylinder ABV (last fig.). The area of the cylindric zone is equal to the circumference of the cylinder $\times rs = 2\pi \,.\, Xr \,.\, rs$. Also, the surface of the spherical zone is, when its breadth is exceedingly small, equal very nearly to the surface of a frustum of a cone, and equal to the middle circumference of the zone $\times mn$ (Art. 387), or nearly $= 2\pi \,.\, Xm \,.\, mn$.

But, from similar triangles, $mn : mo = Cm : Xm$; hence $Xm \,.\, mn = Cm \,.\, mo = Xr \,.\, rs$; therefore, $2\pi \,.\, Xm \,.\, mn = 2\pi \,.\, Xr \,.\, rs$. That is, the surfaces of the spherical and cylindric zones are equal. The same can be similarly proved of the surfaces of all the other corresponding small zones of these two solids. Hence the whole surface of the sphere is equal to the convex surface of the circumscribing cylinder

$$= cd = 3{\cdot}1416dd = 3{\cdot}1416d^2 = \pi d^2 = 4\pi r^2.$$

This proposition can only be proved with rigorous accuracy and conciseness by means of the calculus.

Exercises

1. How many square inches of gold-leaf will gild a globe = 1 foot in diameter? . . . . 452·39 square inches.

2. What is the surface of a sphere whose diameter is 2 feet 9 inches? . . . . 23·75835 square feet.

3. Find the surface of a globe whose diameter is 51 inches. 56·74515 square feet.

4. Find the surface of a ball whose diameter is 5 inches. 78·54 square inches.

5. What is the surface of a sphere whose diameter is 2 feet 8 inches? . . . . 22·34 square feet.

6. What is the surface of a globe whose diameter is 9 inches? 1·767 square feet.

392. Problem XIX.—**To find the surface of any spherical segment or zone.**

Rule I.—Multiply the circumference of the sphere by the height of the segment or zone, and the product will be the area; or,

Rule II.—Multiply the diameter of the sphere by 3·1416, and

the product by the height of the segment or zone; the product will be the area.

Or, $s = ch$, or $s = 3{\cdot}1416dh = \pi dh$.

EXAMPLE.—What is the surface of a spherical zone whose height is $=4$ feet, the diameter of the sphere being $=5$ feet?

$$s = 3{\cdot}1416dh = 3{\cdot}1416 \times 5 \times 4 = 62{\cdot}832 \text{ square feet.}$$

It was proved in Prob. XVIII. that the surfaces of any two corresponding zones of a sphere and its circumscribing cylinder are equal. Now, the surface of any zone of the cylinder is evidently equal to the circumference of the cylinder, or of the sphere, multiplied by the height of the zone; and hence the surface of the spherical zone is found in the same manner.

EXERCISES

1. Find the convex surface of a spherical zone whose height is $=4$ inches, the diameter of the sphere being $=1$ foot. $=150{\cdot}7968$ square inches.

2. Find the convex surface of a spherical zone, the height of which is $=5$ inches, and the diameter of the sphere $=25$ inches. $=392{\cdot}7$ square inches.

3. What is the convex surface of a spherical segment whose height is $=3$ feet 6 inches, the diameter of the sphere being $=10$ feet? . . . . $=109{\cdot}956$ square feet.

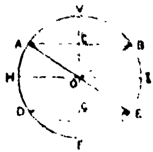
4. Find the number of square inches in the convex surface of a spherical segment whose height is $=2$ inches, the diameter of the sphere being $=6$ inches. . . . . $=37{\cdot}6992$ square inches.

5. What is the convex surface of a spherical segment whose height is $=9$ inches, the diameter of the sphere being $=3$ feet 6 inches? . . . . $=1187{\cdot}5248$ square inches.

393. **Problem XX.—To find the solidity of a spherical segment.**

RULE I.—To three times the square of the radius of the base of the segment add the square of its height; multiply the sum by the height, and the product by ·5236, and the result will be the solidity.

RULE II.—From three times the diameter of the sphere subtract twice the height of the segment; multiply the difference by the square of the height, and the product by ·5236; the result will be the solidity.



Let $ACBV$ be a spherical segment,

$h = VC$ its height,
$r = AC$ the radius of its base,
and d the diameter of the sphere;
then $V = \cdot 5236h(3r^2 + h^2)$,
or $V = \cdot 5236h^2(3d - 2h)$.

EXAMPLES. 1. The height of a spherical segment is 8 inches, and the radius of its base 14 inches; what is its solidity?

$$V = \cdot 5236h(3r^2 + h^2) = \cdot 5236 \times 8(3 \times 14^2 + 8^2)$$
$$= 4\cdot 1888 \times 652 = 2731\cdot 0976 \text{ cubic inches.}$$

2. The diameter of a sphere is 5 feet, and the height of a segment of it 2 feet; what is the solidity of the segment?

$$V = \cdot 5236h^2(3d - 2h) = \cdot 5236 \times 2^2(3 \times 5 - 2 \times 2)$$
$$= 2\cdot 0944 \times 11 = 23\cdot 0384 \text{ cubic feet.}$$

The spherical segment $ACBV$ is equal to the difference between the spherical sector $OAVB$ and the cone $OACB$. But the spherical sector is to the sphere as the surface of the segment to the surface of the sphere. Hence, if S, s be the surfaces of the sphere and segment, and V, V' the volumes of the sphere and sector,

$$S : s = V : V'; \text{ hence } V' = \frac{Vs}{S}.$$

But $S = 3\cdot 1416d^2$, $s = 3\cdot 1416dh$, $V = \cdot 5236d^3$;

hence $$V' = \frac{\cdot 5236d^3 \times 3\cdot 1416dh}{3\cdot 1416d^2} = \cdot 5236d^2h.$$

For the cone, volume $= V'' = \dfrac{3\cdot 1416}{3} r^2\left(\dfrac{d}{2} - h\right) = \cdot 5236r^2(d - 2h)$; hence volume of segment,

$$V = V' - V'' = \cdot 5236\{d^2h - r^2(d - 2h)\} \quad \ldots \quad \ldots \quad [1].$$

But $AC^2 = VC \cdot CE$, or $r^2 = h(d - h)$;

and hence $$d = \frac{r^2 + h^2}{h}.$$

Substituting this value of r^2 in [1], it becomes

$$V = \cdot 5236\{d^2h - h(d - h)(d - 2h)\} = \cdot 5236h^2(3d - 2h).$$

Substituting in this last expression the above value for d,

$$V = \cdot 5236h^2\left(\frac{3r^2 + 3h^2}{h} - 2h\right) = \cdot 5236h(3r^2 + h^2).$$

EXERCISES

1. Find the solidity of a spherical segment whose height is = 4 inches, and the radius of its base = 8 inches.

$= 435\cdot 6352$ cubic inches.

2. Find the volume of a spherical segment, the diameter of the base of which is $=20$, and its height $=9$ . . . $=1795\cdot4244$.

3. What is the solidity of a spherical segment, the radius of whose base is $=25$ inches, and its height $=6\cdot75$?

$=6787\cdot844$ cubic inches.

4. Find the solidity of a spherical segment, the height of which is $=2$ feet, the diameter of the sphere being $=10$ feet.

$=54\cdot4544$ cubic feet.

394. **Problem XXI.—To find the solidity of a spherical zone.**

RULE I.—Add together the squares of the radii of the two ends and one-third the square of the height; multiply the sum by the height, and this product by 1·5708, and the result will be the solidity.

RULE II.—For the middle zone, add together the square of the diameter of either end, and two-thirds of the square of the height, or find the difference between the square of the diameter of the sphere, and one-third of the square of the height of the zone; then multiply the sum or the difference by the height, and the product by ·7854, and the result will be the solidity.

Let R and r be the radii of the ends; then the first rule gives

$$V = 1\cdot5708h(R^2 + r^2 + \tfrac{1}{3}h^2).$$

And the rules for the middle zone give

$$V = \cdot7854h(D^2 + \tfrac{2}{3}h^2), \text{ or } V = \cdot7854h(d^2 - \tfrac{1}{3}h^2),$$

where d is the diameter of the sphere, and D that of either end of the zone.

EXAMPLES.—1. Find the solidity of a spherical zone, the diameters of its ends being $=4$ and 3 inches, and its height $=2$ inches.

$$V = 1\cdot5708h(R^2 + r^2 + \tfrac{1}{3}h^2) = 1\cdot5708 \times 2(4 + \tfrac{9}{4} + \tfrac{4}{3})$$
$$= 3\cdot1416 \times \tfrac{91}{12} = 23\cdot8238 \text{ cubic inches.}$$

2. Find the solidity of the middle zone of a sphere, the diameters of its ends being $=4$ feet, and its height $=6$ feet.

$$V = \cdot7854h(D^2 + \tfrac{2}{3}h^2) = \cdot7854 \times 6(4^2 + \tfrac{2}{3} \times 6^2)$$
$$= 4\cdot7124 \times 40 = 188\cdot496 \text{ cubic feet.}$$

Or, since $\frac{1}{4}d^2 = \frac{1}{4}D^2 + \frac{1}{4}h^2$ or $d^2 = D^2 + h^2 = 4^2 + 6^2 = 16 + 36 = 52$,

$$V = \cdot7854h(d^2 - \tfrac{1}{3}h^2) = \cdot7854 \times 6(52 - \tfrac{1}{3} \times 6^2)$$
$$= 4\cdot7124 \times 40 = 188\cdot496 \text{ cubic feet.}$$

The spherical zone ABED (fig. to Prob. XX.) is evidently equal

to the difference between the two segments VDE and VAB. Let r' = the radius of the sphere, and h' the height of the less segment; $h'+h$, the height of the greater, then h = the height of the zone;

also $\qquad 3\text{R}^2 = 6r'(h'+h) - 3(h'+h)^2,$

and $\qquad 3r^2 = 6r'h' - 3h'^2$; hence,

$$\text{V} = \frac{\pi}{6}\{6r'(h'+h) - 2(h'+h)^2\}(h'+h) - \frac{\pi}{6}h'(6r'h' - 2h'^2)$$

$$= \frac{\pi}{6}\{6r'h'^2 + 12r'h'h + 6r'h^2 - 2h'^3 - 6h'^2h - 6h'h^2 - 2h^3 - 6r'h'^2 + 2h'^3\}$$

$$= \frac{\pi h}{6}\{6r'(h'+h) - 3(h'+h)^2 + 6r'h' - 3h'^2 + h^2\}$$

$$= \frac{\pi h}{6}\{3\text{R}^2 + 3r^2 + h^2\} = \frac{\pi}{2}h(\text{R}^2 + r^2 + \tfrac{1}{3}h^2)$$

$$= 1{\cdot}5708h(\text{R}^2 + r^2 + \tfrac{1}{3}h^2).$$

But for the middle zone R = r, and if D be the diameter of the end, then $\text{R}^2 + r^2 = \tfrac{1}{4}\text{D}^2 + \tfrac{1}{4}\text{D}^2 = \tfrac{1}{2}\text{D}^2$; hence, for the middle zone,

$$\text{V} = \frac{\pi h}{2}(\tfrac{1}{2}\text{D}^2 + \tfrac{1}{3}h^2) = \frac{\pi h}{4}(\text{D}^2 + \tfrac{2}{3}h^2) = {\cdot}7854h(\text{D}^2 + \tfrac{2}{3}h^2)$$

$$= {\cdot}7854h(d^2 - \tfrac{1}{3}h^2), \text{ if } d = \text{the diameter of the sphere.}$$

Exercises

1. What is the volume of a spherical zone, the diameters of its ends being = 10 and 12 inches, and its height = 2 inches?

195·8264 cubic inches.

2. Find the solidity of the middle zone of a sphere whose diameter is = 40 inches, the diameter of its base being = 24, and its height = 32 inches. . . . . . 31633·8176 cubic inches.

3. What is the volume of a spherical zone whose height is = 15 inches, and the diameters of its ends = 20 and 30 inches?

9424·8 cubic inches.

4. The diameters of the ends of a spherical zone are = 8 and 12 inches, and its height = 10 inches; what is its solidity?

1340·416 cubic inches.

5. What is the volume of a middle zone of a sphere, its height being = 8 feet, and the diameters of its ends = 6 feet?

= 494·2784 cubic feet.

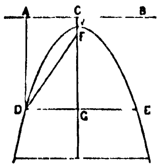
6. Find the volume of a spherical zone whose height is = 4 feet, and the end diameters = 6 feet. . . . = 146·608 cubic feet.

MENSURATION OF CONIC SECTIONS

395. The **conic sections** are the three curves—the parabola, the ellipse, and the hyperbola.

DEFINITIONS

396. A **parabola** is a curve such that any point in it is equidistant from a given point and a given straight line.



Thus, if the curve DVE is such that any point in it, as D, is equidistant from a given point F and a given line AB—that is, such that DF = DA—the curve is a parabola.

397. The given point is called the **focus** of the parabola, and the given line its **directrix.**

Thus, F is called the focus of the parabola, and AB is its directrix.

398. That part of a perpendicular to the directrix passing through the focus, which is contained within the curve, is called the **axis,** or **principal diameter**; and the extremity of the axis is called the **vertex** of the parabola.

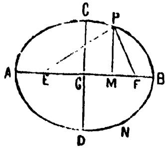
Thus, VG produced indefinitely is the axis, and V the vertex of the curve.

399. An **ordinate** is a perpendicular from any point in the curve on the axis, and when produced to meet the curve on the other side of the axis, it is a **double** ordinate; and the portion of the axis intercepted between the ordinate and the curve is called the **abscissa.**

Thus, DG is an ordinate to the axis, and GV is its abscissa; also DE is a double ordinate.

400. The **principal parameter** is four times the distance of the vertex from the directrix.

Thus, four times CV is the parameter. It is also equal to 4 VF, or to the double ordinate through the focus.



401. An **ellipse** is a curve such that the sum of the distances of any point in it from two given points is equal to a given line.

Thus, if any point, as P, in the curve ACBD has the sum of its distances from two given points, E and F—namely, PE + PF—equal to a given line, the curve is an ellipse.

402. The given points are called the **foci**; and the middle of the line joining them, the **centre.**

Thus, E and F are the foci, and G the centre.

403. The distance of the centre from either focus is called the **eccentricity**.

EG or GF is the eccentricity.

404. The **major axis** is a line passing through the foci, and terminated by the curve; and a line similarly terminated, passing through the centre, and perpendicular to the major axis, is named the **minor axis**. The former axis is also called the **transverse** diameter; and the latter axis, the **conjugate** diameter.

Thus, AB is the major, and CD the minor axis.

405. An **ordinate** to either axis is a line perpendicular to it from any point in the curve; and this line produced to meet the curve on the other side of the axis is called a **double** ordinate; also each of the segments into which the ordinate divides the axis is called an **abscissa**.

Thus, PM is an ordinate to the axis AB; and AM and MB abscissæ.

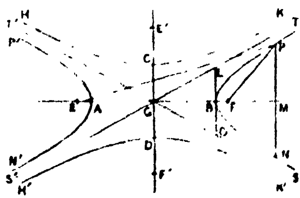
406. The **parameter** *of either axis is a third proportional to it and the other axis*.

Thus, the parameter of AB is a third proportional to AB and CD, and is the same with the double ordinate through the focus, called the focal ordinate.

407. A **hyperbola** is a curve such that the difference between the distances of any point in it from two given points is equal to a given line.

Thus, if any point, as P in the curve PBN, has the difference of its distances from the two given points E and F namely, PE, PF equal to a given line AB, the curve is an hyperbola.

If another curve, P'AN', similar to PBN, pass through A, these two branches are called **opposite** hyperbolas.



408. The two given points are called the **foci**; and the middle of the line joining them, the **centre**.

Thus, E and F are the foci, and G the centre.

409. The distance of the centre from either focus is called the **eccentricity**.

Thus, GE or GF is the eccentricity.

410. The **major axis** is that portion of the line joining the foci,

which is terminated by the opposite hyperbolas; it is also called the **transverse** diameter.

Thus, AB is the major axis.

411. A line passing through the centre perpendicular to the major axis, and having the distance of its extremities from those of this axis equal to the eccentricity, is called the **minor axis,** or **conjugate diameter.**

Thus, if the line CD is perpendicular to AB, and if the distances of C and D from A or B are equal to EG, CD is the minor axis.

412. An **ordinate** to the major axis is a line perpendicular to it from any point in the curve, and this line produced to meet the curve on the other side of the axis is called a **double** ordinate; and the segment of the axis between the ordinate and curve is called an **abscissa.**

Thus, PM is an ordinate, and PN a double ordinate to the axis AB; and BM an abscissa.

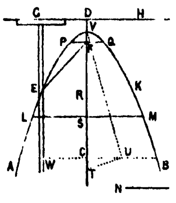
413. A third proportional to the major and minor axis is called the **parameter** of the former axis.

Thus, a third proportional to AB and CD is the parameter of AB, and is equal to the double ordinate through the focus.

414. Problem I.—Given the parameter of a parabola, to construct it.

Let the parameter of a parabola be equal to the line N, it is required to construct it.

Draw GH for the directrix, and DC perpendicular to it for the axis. Make DV and VF each one-fourth of N, then V will be the vertex, and F in the axis the focus of the parabola. Draw any line LM parallel to GH, and with the distance DS for a radius, and F as a centre, cut LM in L and M, and these are two points in the parabola. Draw any other parallel, as AB, and find the points A, B in a similar manner; and so on. Then a curve APVQB, passing through all these points, will be a parabola.



For the distance of L from GH—namely, SD—is equal to the distance of L from F; and the same holds for the other points.

When the length of the directrix is given in numbers, a line N must be taken from some convenient scale of equal parts of the required length, and the figure may then be constructed.

The curve may also be described by means of a bar GW, moved parallel to the axis with its extremity G on the directrix, and having a thread FEW with one end F fixed in the focus, and a pencil at E, held so as to keep the thread tight, will describe the curve.

EXERCISES

1. Construct a parabola having a parameter equal to the given line A. A———

2. Construct a parabola whose parameter is 200 on a scale of half an inch to the hundred.

415. Problem II.—Given an ordinate of a parabola and its abscissa, to find the parameter.

RULE.—Divide the square of the ordinate by the abscissa, and the quotient will be the parameter.

Let d = the ordinate BC (fig. to Prob. I.),
a = " abscissa CV, and
p = " parameter;

then $d^2 = pa$; hence $p = \frac{d^2}{a}$, and $a = \frac{d^2}{p}$.

EXAMPLE.—Given an ordinate of a parabola = 6 and its abscissa = 15, to find the parameter.

$$p = \frac{d^2}{a} = \frac{6^2}{15} = \frac{36}{15} = \frac{72}{30} = 2\cdot4.$$

EXERCISES

1. An ordinate of a parabola is = 20, and its abscissa = 36; find the parameter. . . . . $11\frac{1}{9}$.

2. Find the parameter of a parabola, an ordinate and abscissa being respectively = 12 and 25. . . . . 5·76.

3. What is the parameter of a parabola one of whose ordinates is = 16, and the corresponding abscissa = 18? . . . . $14\frac{2}{9}$.

4. Find the parameter of a parabola, one of its ordinates being = 25, and the corresponding abscissa = 20. . . . . 31·25.

416. Problem III.—To construct a parabola, any ordinate and its abscissa being given.

Find by Prob. II. the parameter, and then by Prob. I. construct the curve.

EXERCISE

Construct a parabola one of whose ordinates is = 120, and the corresponding abscissa = 225.

417. Problem IV.—Of two abscissæ and their ordinates, any three being given, to find the fourth.

The abscissæ are directly proportional to the squares of their ordinates by Prob. II.

Hence, if A, a are the abscissæ, and D, d their ordinates,

$$A : a = D^2 : d^2;\ \therefore d^2 = \frac{aD^2}{A},\ \text{and } D^2 = \frac{Ad^2}{a}.$$

Also $$D^2 : d^2 = A : a;\ \therefore a = \frac{Ad^2}{D^2},\ \text{and } A = \frac{aD^2}{d^2}.$$

EXAMPLE.—Given the abscissa VS = 10 (fig. to Prob. I.), and the abscissa VC = 12, also the ordinate SL = 9; required the ordinate AC.

$$VS : VC = SL^2 : AC^2;$$
$$10 : 12 = 9^2 : AC^2,$$

and $$AC^2 = \frac{12 \times 9^2}{10} = \frac{6 \times 81}{5} = \frac{486}{5} = 97 \cdot 2;$$

hence $$AC = \sqrt{97 \cdot 2} = 9 \cdot 859,$$

or $$d^2 = \frac{aD^2}{A} = \frac{12 \times 9^2}{10} = 97 \cdot 2,\ \text{and } d = 9 \cdot 859.$$

EXERCISES

1. Two abscissæ of a parabola are = 18 and 32, and the ordinate of the former is = 12; find the ordinate of the latter. . = 16.

2. Two abscissæ are = 3 and 6, the ordinate of the former is = 5; find that of the latter. . . . . = 7·07.

3. Two abscissæ are = 9 and 16, and the ordinate of the former = 6; find that of the latter. . . . . = 8.

4. Two ordinates are = 6 and 8, and the abscissa of the former = 9; find that of the latter. . . . . = 16.

5. Two ordinates are = 18 and 24, and the abscissa of the former = 18; find that of the latter. . . . . = 32.

418. Problem V.—To find the length of a parabolic curve cut off by a double ordinate.

RULE.—To the square of the ordinate add four-thirds of the square of the abscissa, and the square root of the sum, multiplied by two, will be the length of the curve nearly.

Let d = the ordinate, and a = the abscissa, and
l = " length of the curve;

then $$l^2 = 4\left(d^2 + \frac{4a^2}{3}\right),\ \text{or } l = 2\sqrt{\left(d^2 + \frac{4a^2}{3}\right)}.$$

Example.—The abscissa of a parabola is $=3$, and its ordinate $=9$; what is the length of the arc?

$$l^2=4\left(d^2+\frac{4a^2}{3}\right)=4\left(9^2+\frac{4\times 3^2}{3}\right)=4(81+12)=4\times 93,$$

and $$l=2\sqrt{93}=2\times 9\cdot 6436=19\cdot 2872.$$

Exercises

1. The abscissa of a parabolic arc is $=4$, and the ordinate is $=8$; what is its length? . . . . 18·47.

2. The abscissa and ordinate of a parabola are 10 and 8; what is the length of the curve? . . . . 28·09.

419. Problem VI.—To find the area of a parabola.

Rule.—Multiply the base by the height, and two-thirds of the product is the area.

Let $b=$ base or double ordinate, $a=$ height or abscissa; then $\mathcal{A}=\frac{2}{3}ba$.

Example.—What is the area of a parabola whose base is $=25$, and height $=18$?

$$\mathcal{A}=\tfrac{2}{3}ba=\tfrac{2}{3}\times 25\times 18=300.$$

It was proved, by the method of exhaustions, by Archimedes, and can be easily proved by the integral calculus, that the area of a parabola is two-thirds of the circumscribing rectangle, which has the same base and height as the curve, its upper side being a tangent through the vertex.

Exercises

1. Find the area of a parabola whose base or double ordinate is $=36$, and height or abscissa $=45$. . . . . . . 1080.

2. What is the area of a parabola whose base and height are 18 and 28 respectively? . . . . 336.

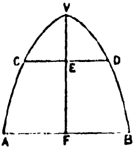
3. Find the area of a parabola whose base is $=30$, and height $=44$. . . . . 880.

4. Find the area of a parabola whose base is $=3\cdot 6$ feet, and height $=5\cdot 6$ feet. . . . . 13·44 feet.

420. Problem VII.—To find the area of a zone of a parabola.

Rule.—Divide the cubes of the two parallel sides by the difference of their squares; multiply the quotient by the height of the zone, and two-thirds of the product will be the area; or,

Divide the sum of the squares of the parallel ends, increased by their product, by the sum of the parallel ends; multiply this quotient by the altitude, and two-thirds of the product is the area.



Let D, d denote the two ordinates AB, CD, and h = the height EF;

then $$AR = \tfrac{2}{3}h\left(\frac{D^3 - d^3}{D^2 - d^2}\right);$$

or $$AR = \tfrac{2}{3}h\left(\frac{D^2 + Dd + d^2}{D + d}\right).$$

Example.—Find the area of a parabolic zone, the two terminating ordinates being = 18 and 30, and the altitude = 9.

$$AR = \tfrac{2}{3}h\frac{D^3 - d^3}{D^2 - d^2} = \tfrac{2}{3} \times 9 \times \frac{30^3 - 18^3}{30^2 - 18^2} = 6 \times \frac{900 + 540 + 324}{30 + 18}$$

$$= 6 \times \frac{1764}{48} = \frac{1764}{8} = 220 \cdot 5.$$

By means of the preceding problem, the above rule may be easily proved. Let A′, A″, and h', h'', denote respectively the areas and heights of the two parabolas VAB, VCD; then (last problem)

$$A' = \tfrac{2}{3}Dh', \text{ and } A'' = \tfrac{2}{3}dh'';$$

hence $$AR = A' - A'' = \tfrac{2}{3}(Dh' - dh'') = \tfrac{2}{3}\{Dh + h''(D - d)\}.$$

But by a property of the parabola,

$$D^2 : d^2 = h' : h''; \text{ hence } D^2 - d^2 : d^2 = h : h'', \text{ and } h'' = \frac{d^2h}{D^2 - d^2}.$$

Substitute this value of h'' in the above value of AR,

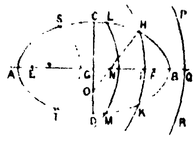
then $$AR = \tfrac{2}{3}h\left(D + \frac{d^2}{D + d}\right), \text{ or } AR = \tfrac{2}{3}h\left(\frac{D^3 - d^3}{D^2 - d^2}\right) = \tfrac{2}{3}h\left(\frac{D^2 + Dd + d^2}{D + d}\right).$$

Exercises

1. Find the area of a zone of a parabola whose parallel sides are = 5 and 3, and its height = 4. . . . . = $16\frac{1}{3}$.
2. Find the area of a parabolic zone whose parallel ends are = 6 and 10, and the height = 6. . . . . = 49.
3. Required the area of a zone of a parabola whose height is = 11, and its two ends = 10 and 12. . . . . = $121\frac{1}{3}$.
4. The ends of a parabolic zone are = 5 and 10, and its height = 6; what is its area? . . . . = $46\frac{2}{3}$.
5. The ends of a zone of a parabola are = 6 and 9, and their distance = 8; what is its area? . . . . = 60·8.
6. The parallel sides of a parabolic zone are = 10 and 15, and their distance = 15; required its area. . . . . . . = 190.

421. **Problem VIII.—To describe an ellipse, having given its major and minor axes.**

Let AB be the major axis. Draw a line CD, bisecting it perpendicularly, and make GC, GD each equal to half of the minor axis, then CD is this axis. From C as a centre, with half the major axis AG as a radius, cut AB in E and F, and these points are the foci.



Produce AB to Q till EQ = AB; then from E as a centre describe an arc PQ, and this arc is a species of directrix to the ellipse. With any radius EI, from E as a centre, describe an arc HK; and with the distance IQ as a radius, from F as a centre, cut HK in H and K, and these are points in the curve. Describe from E as a centre any other arc LM, and find as before the points L and M. Proceed in the same manner till a sufficient number of points are found, and the curve passing through them—namely, ADBC—is an ellipse.

The construction of the ellipse by this method is exactly similar to that of the parabola, PR being considered the directrix, and the concentric arcs HK, LM, &c. as parallels to the arc PR.

Cor. 1.—When the major axis and the eccentricity or the foci are given, the ellipse can be constructed nearly in the same manner as in the problem.

Cor. 2.—When the minor axis and the eccentricity are given, the ellipse may be constructed thus:

Draw AB, bisecting CD perpendicularly, and lay off GE, GF each equal to the eccentricity, then EC is equal to half the major axis. Hence, make GA, GB each = EC, and AB is the major axis; and the ellipse can now be constructed as before.

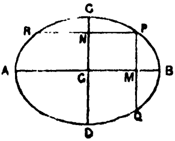
The ellipse may also be constructed by means of *elliptic compasses*, which consist of two brass bars AB, CD, with grooves and a third bar OH = AG, half the major axis, a part of it, NH, being = CG, half the minor axis, with two pins at O and N; and OH being moved, so that the pins at N and O move respectively in the grooves of AB and CD, the extremity H will move in the curve of the ellipse.

Or, if a thread EPF (fig. to Art. 401), equal in length to AB, have its extremities fixed in the two foci, and be drawn tight by means of a pencil moving in the angle P, the pencil will describe an ellipse.

422. Problem IX.—When the two axes and an abscissa are given, to find the ordinate.

RULE.—As the square of the major axis is to that of the minor, so is the rectangle under the two abscissæ to the square of the ordinate.

The halves of the two axes may be taken instead of the axes themselves in this rule.



Let a = the major axis $=AB$,
b = the minor " $=CD$,
h = one abscissa $=AM$;
then $a-h$ = the other abscissa $=MB$,
d = the ordinate $=PM$,
and $a^2 : b^2 = (a-h)h : d^2$,
or $d^2 = \frac{b^2}{a^2}(a-h)h$.

EXAMPLE.—The axes are $=30$ and 10, and one abscissa is $=24$; find the ordinate.

$$a=30,\ b=10,\ h=24;\ \text{hence } a-h=30-24=6;$$

hence $$d^2 = \frac{10^2}{30^2} \times 6 \times 24 = \frac{144}{9} = 16;\ \therefore\ d=\sqrt{16}=4.$$

EXERCISES

1. The major and minor axes of an ellipse are $=60$ and 20, and one abscissa is $=12$; find the ordinate. . . . . . . $=8$.

2. The axes are $=45$ and 15, and one abscissa is $=9$; what is the ordinate? . . . . $=6$.

3. The axes are $=52{\cdot}5$ and $17{\cdot}5$, and the abscissa $=42$; find the ordinate. . . . . $=7$.

4. The axes are $17{\cdot}5$ and $12{\cdot}5$, and an abscissa $=14$; find the ordinate. . . . . $=5$.

423. Problem X.—When the axes and an ordinate are given, to find the abscissæ.

RULE.—As the square of the minor axis is to the square of the major axis, so is the product of the sum and difference of the semi-axis minor and the ordinate to the distance of the ordinate from the centre.

This distance being added to the semi-axis major, and also subtracted from it, will give the greater and less abscissæ.

Let c = the distance MG (last fig.) from the centre, and a, b the semi-axes, and $h=AM$, and $d=PM$;

then $b^2 : a^2 = (b+d)(b-d) : c^2$, or $c^2 = \frac{a^2}{b^2}(b+d)(b-d)$,

and $h = a + c$, $2a - h = a - c$.

EXAMPLE.—The axes are $= 30$ and 10, and the ordinate $= 4$; what are the abscissæ?

$$c^2 = \frac{a^2}{b^2}(b+d)(b-d) = \frac{15^2}{5^2}(5+4)(5-4) = 9 \times 9 \times 1 = 81,$$

and $c = \sqrt{81} = 9$;

hence the greater abscissa $AM = a + c = 15 + 9 = 24$,

and the less abscissa $MB = a - c = 15 - 9 = 6$.

The rule depends on the same principle as the last; for $CG^2 : AG^2 = CN \,.\, ND : PN^2$, or MG^2.

EXERCISES

1. The axes are $= 45$ and 15, and the ordinate 6; what are the abscissæ? . . . . $= 36$ and 9.
2. The axes are $= 70$ and 50, and an ordinate $= 20$; find the abscissæ. . . . . 14 and 56.

424. Problem XI.—When the minor axis, an ordinate, and an abscissa are given, to find the major axis.

RULE.—Find the square root of the difference of the squares of the semi-axis minor and the ordinate, and, according as the less or greater abscissa is given, add this root to or subtract it from the semi-axis minor; then,

As the square of the ordinate is to the product of the abscissa and minor axis, so is the sum or difference found above to the major axis.

Or, if a, b are the semi-axes, and h the abscissa, $d^2 : 2bh = b \pm \sqrt{(b^2 - d^2)} : 2a$, and $2a = \frac{2bh}{d^2}\{b \pm \sqrt{(b^2 - d^2)}\}$.

EXAMPLE.—The minor axis is $= 10$, the smaller abscissa 6, and the ordinate $= 4$; find the major axis.

$$b^2 - d^2 = 5^2 - 4^2 = 25 - 16 = 9, \text{ and } \sqrt{(b^2 - d^2)} = 3;$$

and $d^2 : 2bh = b \pm \sqrt{(b^2 - d^2)} : 2a$.

Or, $4^2 : 10 \times 6 = 5 + 3 : 2a$, or $16 : 60 = 8 : 2a$, and $2a = 30$.

Or by the formula, $2a = \frac{2bh}{d^2}\{b + \sqrt{(b^2 - d^2)}\}$

$$= \frac{10 \times 6}{16}\{5 + \sqrt{(5^2 - 3^2)}\} = \frac{15}{4} \times (5+3) = 30.$$

The rule is derived from the same proposition as the last two.

Thus, $AG^2 : CG^2 = AM \,.\, MB : PM^2$,

or $a^2 : b^2 = (2a - h)h : d^2$; hence $a^2d^2 = b^2h(2a - h)$.

From this quadratic equation, the value of a, the unknown quantity, is easily found, and the result is the above value.

EXERCISES

1. The minor axis is $=15$, an ordinate $=6$, and the less abscissa $=9$; what is the major axis? . . . . $=45$.

2. The minor axis is $=50$, an ordinate $=20$, and the less abscissa $=14$; find the major axis. . . . . $=70$.

3. The minor axis is $=5$, the greater abscissa $=12$, and the ordinate $=2$; what is the major axis? . . . . $=15$.

425. Problem XII.—When the major axis, an ordinate, and one of the abscissæ are given, to find the minor axis.

RULE.—Find the other abscissa, then the product of the two abscissæ is to the square of the ordinate as the square of the major axis to that of the minor axis.

Or, $h(a - h) : d^2 = a^2 : b^2$, and $b^2 = \dfrac{a^2d^2}{h(a - h)}$.

When a and b are semi-axes, $b^2 = \dfrac{a^2d^2}{h(2a - h)}$.

EXAMPLE.—The major axis is $=15$, an ordinate $=2$, and an abscissa $=3$; what is the minor axis?

$$b^2 = \frac{a^2d^2}{h(a - h)} = \frac{15^2 \times 2^2}{3(15 - 3)} = \frac{75 \times 4}{12} = 25, \text{ and } b = \sqrt{25} = 5.$$

The rule is derived from the same theorem as that in Prob. IX.

426. If the abscissæ were segments of the minor axis, and the ordinate a perpendicular to it, then the major axis could be found by the analogous formula, $a^2 = \dfrac{b^2d^2}{h(b - h)}$.

EXERCISES

1. The major axis is $=70$, an ordinate $=20$, and one of the abscissæ $=14$; what is the minor axis? . . . . . $=50$.

2. The major axis is $=210$, an ordinate $=28$, and one of the abscissæ $=168$; what is the minor axis? . . . . . $=70$.

427. Problem XIII.—To find the length of the circumference of an ellipse when the axes are given.

RULE.—Multiply the square root of half the sum of the squares

of the two diameters by 3·1416, and the product will be the circumference nearly.

If l = the length of the curve, then $l = 3·1416\sqrt{\{\frac{1}{2}(a^2+b^2)\}}$.

EXAMPLE.—What is the length of the circumference of an ellipse whose axes are = 10 and 30?

$$l = 3·1416\sqrt{\{\tfrac{1}{2}(a^2+b^2)\}} = 3·1416\sqrt{\{\tfrac{1}{2}(30^2+10^2)\}} = 3·1416\sqrt{500}$$
$$= 3·1416 \times 22·3607 = 70·2484.$$

The rule is derived by means of the calculus; but it is only an approximation, though sufficiently accurate for practical purposes.

EXERCISES

1. The axes are = 10 and 12; what is the length of the curve of the ellipse? . . . . = 34·7001.
2. The axes are = 6 and 8; what is the length of the curve? = 22·214.
3. The axes are = 4 and 6; what is the length of the curve? = 16·019.

428. Problem XIV.—To find the area of an ellipse.

RULE.—Multiply the product of the two axes by ·7854, and the result will be the area.

Or, $\mathcal{A} = ·7854ab$.

EXAMPLE.—What is the area of an ellipse whose axes are = 15 and 20 feet?

$$\mathcal{A} = ·7854ab = ·7854 \times 20 \times 15 = 235·62 \text{ square feet.}$$

The rule can only be demonstrated rigorously by means of the integral calculus. The truth of it, however, will appear evident from the consideration that if a circle is described on the major axis, and an ordinate to this axis be produced to meet the circle, then if d' = the ordinate of the circle, $d'^2 = h(a-h)$ by Eucl. III. 35. But (Prob. IX.) $a^2 : b^2 = h(a-h) : d^2$; and hence

$$a^2 : b^2 = d'^2 : d^2, \text{ or } a : b = a' : d;$$

that is, each ordinate of the circle is to the corresponding one of the ellipse as $a : b$; hence, if A' = area of circle,

$$A' : \mathcal{A} = a : b, \text{ or } \mathcal{A} = \frac{bA'}{a} = \frac{b}{a} \times ·7854a^2 = ·7854ab.$$

EXERCISES

1. Find the area of an ellipse whose axes are = 5 and 10. = 39·27.
2. Find the area of an ellipse whose axes are = 5 and 7. = 27·489.

3. What is the area of an ellipse whose axes are $= 12$ and 16?

$= 150·7968$.

4. What is the area of an ellipse whose axes are $= 6$ and 7?

$= 32·9868$.

429. Problem XV.—To find the area of an elliptic segment.

RULE.—Find the area of the corresponding segment of the circle described upon that axis of the ellipse which is perpendicular to the base of the segment; then this axis is to the other axis as the circular segment to the elliptic segment; or,

Multiply the tabular area belonging to the corresponding circular segment by the product of the two axes of the ellipse, and the result will be the area.

Let $\mathcal{A}$ = the area of the elliptic segment, h = its height, and A' = the area of a segment of the same height of a circle described on the axis, of which the height is a part; then, when h is a part of the major axis a,

$$a : b = A' : \mathcal{A} \text{ and } \mathcal{A} = \frac{bA'}{a} = \text{segment PBQ (fig. to Prob. IX.)}.$$

When h is a part of the minor axis b,

$$b : a = A' : \mathcal{A}, \text{ and } \mathcal{A} = \frac{aA'}{b} = \text{segment RCP}.$$

EXAMPLE.—What is the area of an elliptic segment whose base is parallel to the minor axis, the height of it being $= 10$ feet, and the axes of the ellipse $= 35$ and 25?

Height of tabular circular segment $= \frac{10}{35} = \frac{2}{7} = ·2857$;
area of tabular circular segment $t = ·185154$;
then $\mathcal{A} = abt = 35 \times 25 \times ·185154 = 162·00975$.

The rule depends on the principle that an elliptic segment bears the same proportion to the corresponding circular segment that the whole ellipse does to the whole circle described on the axis of which the height is a part.

EXERCISES

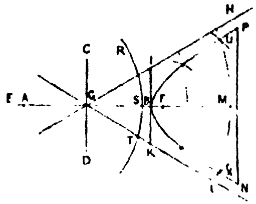
1. Find the area of an elliptic segment whose base is perpendicular to the major axis, its height being $= 6$, and the axes $= 30$ and 10 . . . . $= 33·5472$.

2. Find the area of an elliptic segment whose base is parallel to the major axis, its height being $= 2$, and the diameters $= 14$ and 10.

$= 15·65536$.

430. Problem XVI.—To describe an hyperbola, its two axes being given.

Make AB equal to the major axis; bisect it perpendicularly by CD, and make CG and GD each equal to half the minor axis. The distance CA or DB being laid off from G to E and F, these two points will be the foci of the hyperbola.



From E as a centre, with a radius =AB, describe an arc RST, and it will be a species of directrix. From E as a centre, describe any arc, as UMX; and with the distance MS of U from the directrix, and with F as a centre, cut UMX in U and X, and these are points in the curve. Find other two points in the same manner, and so on till a sufficient number are found; then the curve PBN passing through them all is an hyperbola. Another hyperbola similarly described, and passing through the point A, would be the opposite hyperbola.

If a tangent IK to the curve at its vertex B be drawn, such that BI and BK are each = half the minor axis CG, and straight lines GH, GL be drawn from the centre through its extremities I and K, they are called *asymptotes*, and possess the singular property of continually approaching to the curve without ever meeting it.

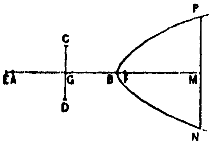
Cor. 1.—When the major axis AB and the eccentricity EG or GF are given, the minor axis CD can be found thus:—Bisect AB perpendicularly by CD, and then from A as a centre, with EG as a radius, cut CD in the points C and D, and they will be the extremities of this axis. The curve can then be described as in the above problem.

Cor. 2.—When the minor axis CD and the eccentricity EG are given, the major axis can be found thus: Bisect CD perpendicularly by EF, with EG as a radius and C as a centre, cut EF in A and B, then AB is the major axis. The curve can then be described as above.

431. Problem XVII.—The axes of an hyperbola and an abscissa being given, to find the ordinate.

Rule.—As the square of the major axis is to that of the minor, so is the product of the two abscissæ to the square of the ordinate.

Let the axes AB, CD be denoted by a and b, the abscissa BM by h, and the ordinate PM by d; then $AM = a + h$, and



$$a^2 : b^2 = (a+h)h : d^2, \text{ or } d^2 = \frac{b_2}{a^2}(a+h)h.$$

EXAMPLE.—The major axis of an hyperbola is $= 15$, the minor axis 9, and the less abscissa $= 5$; what is the ordinate?

$$BM = h = 9,\ AM = a + h = 15 + 9 = 24;$$

hence
$$d^2 = \frac{b^2}{a^2}(a+h)h = \frac{9^2}{15^2}(15+5)5 = 36,$$

and
$$d = \sqrt{36} = 6.$$

EXERCISES

1. The major and minor axes are $= 48$ and 42, and the less abscissa $= 16$; what is the ordinate? . . . . $= 28$.

2. The major axis is $= 25$, the minor $= 15$, and the less abscissa $= 8\frac{1}{3}$; what is the ordinate? . . . . $= 10$.

3. The major and minor axes are $= 15$ and $7\frac{1}{2}$, and the less abscissa $= 5$; what is the ordinate? . . . . $= 5$.

432. Problem XVIII.—The two axes and an ordinate being given, to find the abscissæ.

RULE.—As the square of the minor axis is to that of the major axis, so is the sum of the squares of the semi-axis minor and the ordinate to the square of the distance between the ordinate and the centre.

The sum of this distance and the semi-axis major will give the greater abscissa, and their difference the less.

Let $c =$ this distance $=$ GM (fig. to Prob. XVII.), and a, b half the axes;

then
$$4b^2 : 4a^2 = b^2 + d^2 : c^2, \text{ or } c^2 = \frac{a^2}{b^2}(b^2 + d^2),$$

and
$$AM = 2a + h = a + c,\ BM = h = c - a.$$

EXAMPLE.—The major and minor axes are $= 30$ and 18, and the ordinate $= 12$; what are the abscissæ?

$$c^2 = \frac{a^2}{b^2}(b^2 + d^2) = \frac{15^2}{9^2}(9^2 + 12^2) = \frac{225}{9^2} \times 225 = 25^2;\ \therefore c = 25.$$

Hence $a + c = 15 + 25 = 40$, $c - a = 25 - 15 = 10$; and the two abscissæ are $= 10$ and 40.

EXERCISES

1. The major and minor axes are = 24 and 21, and the ordinate = 14; what are the abscissæ? . . . . . . = 32 and 8.

2. The major and minor axes are = 55 and 33, and an ordinate is = 22; required the abscissæ. . . . . = $73\frac{1}{3}$ and $18\frac{1}{3}$.

3. The major and minor axes are = 60 and 45, and an ordinate = 30; what are the abscissæ? . . . . = 80 and 20.

433. Problem XIX.—The major axis, an ordinate, and the two abscissæ being given, to find the minor axis.

RULE.—The product of the abscissæ is to the square of the ordinate as the square of the major axis is to that of the minor axis.

$$(a+h)h : d^2 = a^2 : b^2, \text{ or } b^2 = \frac{a^2d^2}{(a+h)h}, \text{ or } b = \frac{ad}{\sqrt{(a+h)h}},$$

where a and b are the axes.

EXAMPLE.—The major axis is = 30, the ordinate 12, and the two abscissæ = 40 and 10; what is the minor axis?

$$b = \frac{ad}{\sqrt{(a+h)h}} = \frac{30 \times 12}{\sqrt{40 \times 10}} = \frac{360}{\sqrt{400}} = 360 \times \frac{1}{20} = 18.$$

The rule depends on the same principle as that of Prob. XVII.

EXERCISES

1. The major axis is = 15, an ordinate = 6, and the two abscissæ = 20 and 5; what is the minor axis? . . . . . . = 9.

2. The major axis is 36, and ordinate = 21, and the abscissæ = 12 and 48; find the minor axis. . . . . . . 31·5.

434. Problem XX.—The minor axis, ordinate, and the two abscissæ being given, to find the major axis.

RULE.—Find the square root of the sum of the squares of the semi-axis minor and the ordinate; and, according as the less or greater abscissa is given, find the sum or difference of this root and the semi-axis minor; then,

As the square of the ordinate is to the product of the abscissa and minor axis, so is the sum or difference found above to the major axis.

Let a, b be the semi-axes;

then $d^2 : 2bh = b \pm \sqrt{(b^2+d^2)} : 2a$,

and $$2a = \frac{2bh}{d^2}\{b \pm \sqrt{(b^2+d^2)}\}.$$

EXAMPLE.—The minor axis is $=18$, the ordinate $=12$, and the less abscissa $=10$; what is the major axis?

$$2a=\frac{2bh}{d^2}\{b+\sqrt{(b^2+d^2)}\}=\frac{18\times 10}{12^2}\{9+\sqrt{(9^2+12^2)}\}=\tfrac{5}{4}\times(9+15)$$
$$=\tfrac{5}{4}\times 24=30.$$

The rule is derived from the same theorem as that in last problem. When a, b are the semi-axes, the proportion in the last problem becomes $a^2:b^2=h(2a+h):d^2$; hence $a^2d^2=b^2h(2a+h)$. From this quadratic equation, the value of a, the unknown quantity, is easily found, and the result is the value given above.

EXERCISES

1. The minor axis is $=45$, the less abscissa $=30$, and the ordinate $=30$; required the major axis. . . . . $=90$.
2. The minor axis is $=15$, an ordinate $=10$, and the less abscissa $=8\frac{1}{3}$; what is the major axis? . . . . . . $=25$.

435. Problem XXI.—To find the length of an arc of an hyperbola, reckoning from the vertex of the curve.

RULE.—To 15 times the major axis add 21 times the less abscissa, and multiply the sum by the square of the minor axis; add this product to 19 times the product of the square of the major axis by the abscissa, and add it also to 9 times the same product; divide the former sum by the latter, multiply the quotient by the ordinate, and the product will be the length of the arc.

Let $l=$ the length of the arc,

then $$l=\frac{(15a+21h)b^2+19a^2h}{(15a+21h)b^2+9a^2h}d=\left(1+\frac{10a^2h}{(15a+21h)b^2+9a^2h}\right)d.$$

EXAMPLE.—The major and minor axes of an hyperbola are $=15$ and 9, an ordinate at a point in it is $=6$, and the abscissa $=5$; what is the length of the arc to this point from the vertex?

$$l=\frac{(15a+21h)b^2+19a^2h}{(15a+21h)b^2+9a^2h}d=\frac{(15\times 15+21\times 5)9^2+19\times 15^2\times 5}{(15\times 15+21\times 5)9^2+9\times 15^2\times 5}\times 6.$$

Expunge 15, which is a common factor to the terms of this fraction,

$$l=\frac{(15+7)81+1425}{(15+7)81+675}\times 6=\frac{3207}{2457}\times 6=7\cdot 8315.$$

The rule can be demonstrated by means of the integral calculus.

EXERCISES

1. The major and minor axes of an hyperbola are $=30$ and 18, the

ordinate = 12, and the smaller abscissa = 10; what is the length of the arc? . . . . 15·663.

2. The major and minor axes are = 105 and 63, a double ordinate = 84, and the less abscissa = 35; what is the length of the whole arc? . . . . = 109·641.

436. Problem XXII.—To find the area of an hyperbola, the axes and abscissa being given.

RULE.—To 7 times the major axis add 5 times the abscissa; multiply the sum by 7 times the abscissa, and multiply the square root of this product by 3.

To this last product add 4 times the square root of the product of the major axis and abscissa.

Multiply this sum by 16 times the product of the minor axis and abscissa; divide this product by 300 times the major axis, and the quotient will be nearly the required area.

Or, $A = 16bh\{3\sqrt{7h(7a+5h)} + 4\sqrt{ah}\} \div 300a$.

EXAMPLE.—The major and minor axes of an hyperbola are = 10 and 6, and the abscissa = 5; what is its area?

$$A = 16bh\{3\sqrt{7h(7a+5h)} + 4\sqrt{ah}\} \div 300a$$
$$= 16 \times 6 \times 5\{3\sqrt{7 \times 5(70+25)} + 4\sqrt{10 \cdot 5}\} \div 300 \times 10$$
$$= 16 \times 3(3\sqrt{3325} + 4\sqrt{50}) \div 300 = 16 \times 201{\cdot}273 \div 100 = 32{\cdot}20368.$$

EXERCISES

1. In an hyperbola the major and minor axes are = 15 and 9, and the abscissa = 5; what is the area? . . . . . = 37·92.

2. The major and minor axes are = 20 and 12, and the abscissa $6\frac{2}{3}$; find the area. . . . . 67·414.

THE SOLIDS OF REVOLUTION OF THE CONIC SECTIONS

437. The **solids of revolution** generated by the conic sections are the paraboloid, the spheroid or ellipsoid, and the hyperboloid.

438. A paraboloid is a solid generated by the revolution of a parabola about its axis, which remains fixed.

The paraboloid is also called the **parabolic conoid,** as it is like a cone.

439. A **frustum of a paraboloid** is a portion of it contained by two parallel planes perpendicular to its axis.

440. A **spheroid** is a solid generated by the revolution of an ellipse about one of its axes, which remains fixed.

The spheroid is said to be **oblate** or **prolate** according as the minor or major axis is fixed. The fixed axis is called the **polar** axis, and the revolving one the **equatorial** axis.

441. A **segment of a spheroid** is a portion cut off by a plane perpendicular to one of its axes.

When the plane is perpendicular to the fixed axis, the segment may be said to be **circular,** as its base is a circle; and when the plane is parallel to the fixed axis, the segment may be said to be **elliptical,** as its base is an ellipse.

442. The **middle zone** or **frustum of a spheroid** is a portion of it contained by two parallel planes at equal distances from the centre, and perpendicular to one of the axes.

The frustum may be said to be **circular** or **elliptic** according as its ends are perpendicular or parallel to the fixed axis.

443. An **hyperboloid** is a solid generated by the revolution of one of the opposite hyperbolas about its axis remaining fixed.

This hyperboloid is also called a **hyperbolic conoid.**

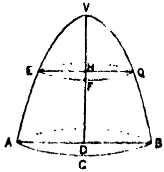
444. A **frustum of a hyperboloid** is a portion of it contained between two parallel planes perpendicular to the axis.

445. Problem I.—To find the solidity of a paraboloid.

RULE.—Multiply the area of the base by the height, and half the product will be the solidity; or,

Multiply the square of the diameter of the base by the

height, and this product by ·7854, and half the result will be the solidity.



Let ABV be the paraboloid,
b = the area of the base,
h = the height;
then $V = \frac{1}{2}bh$.
Or, if d = the diameter of the base,
$b = \cdot 7854d^2$,
and $V = \frac{1}{2} \times \cdot 7854d^2h = \cdot 3927d^2h$.

EXAMPLE.—What is the solidity of a paraboloid the diameter of whose base is = 10, and its height = 15?

$$V = \cdot 3927d^2h = \cdot 3927 \times 10^2 \times 15 = 589 \cdot 05.$$

Take any ordinate HG at a distance VH = h' from the vertex, and another at the same distance from the base at D, then the abscissa of the latter is $h - h'$; and if d' denote the ordinate HG, and d'' the other ordinate, then (Art. 415) if p = the parameter, $ph' = d'^2$, and $p(h - h') = d''^2$; hence $d'^2 + d''^2 = ph = \frac{1}{4}d^2$, and therefore $\pi d'^2 + \pi d''^2 = \frac{1}{4}\pi d^2$; that is (Art. 273), the circular sections of the paraboloid perpendicular to the axis, of which d', d'' are the radii, are equal to the base ACB; and the same can be proved of every two sections of the paraboloid that are equidistant from the vertex and base. Therefore, if a cylinder were described on the base ACB, having a height = the half of VD, any horizontal section of it would be = the corresponding section of the paraboloid, at the same distance from the base, together with the section equidistant from the vertex. Hence the whole paraboloid is equal to a cylinder on the same base, and having half the altitude, which proves the rule.

EXERCISES

1. What is the volume of a paraboloid the height of which is = 10, and the diameter of its base = 20? . . . . = 1570·8.

2. Find the solidity of a paraboloid whose altitude is = 21, and the diameter of its base = 12. . . . . . . = 1187·5248.

3. What is the solidity of a paraboloid whose height is = 15, and the diameter of its base = 20? . . . . = 2356·2.

446. Problem II.—To find the solidity of a frustum of a paraboloid.

RULE.—Multiply the sum of the areas of the two ends by the height, and half the product will be the solidity; or,

Multiply the sum of the squares of the diameters of the two ends

by ·7854, and this product by the height, and half the last product will be the solidity.

Let D and d be the diameters of the ends, and h the height of the frustum;

then $$V = \tfrac{1}{2} \times \cdot 7854(D^2 + d^2)h = \cdot 3927(D^2 + d^2)h.$$

EXAMPLE.—Find the solidity of a frustum of a paraboloid, the diameters of its ends being $=15$ and 12, and its height $=9$.

$$V = \cdot 3927(D^2 + d^2)h = \cdot 3927(15^2 + 12^2)9$$
$$= 3 \cdot 5343 \times 369 = 1304 \cdot 1567.$$

Let h', h'', and V', V'', be the heights and solidities of the two paraboloids ABV, EGV (fig. to Prob. I.);

then $V' = \frac{\pi}{8} D^2 h'$, $V'' = \frac{\pi}{8} d^2 h''$, and $V = V' - V'' = \frac{\pi}{8}(D^2 h' - d^2 h'')$.

But $D^2 : d^2 = h' : h''$; hence $D^2 - d^2 : d^2 = (h' - h'')$, or $h : h''$;

and therefore $h'' = \dfrac{d^2 h}{D^2 - d^2}$. Also, $h' = h + h'' = \dfrac{D^2 h}{D^2 - d^2}$.

Substituting these values of h' and h'' in the above value of V, it becomes

$$\frac{\pi}{8}(D^2 + d^2)h = \cdot 3927(D^2 + d^2)h.$$

EXERCISES

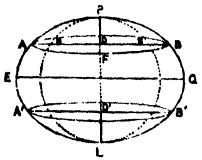
1. What is the solidity of a frustum of a paraboloid, the diameters of its ends being $=30$ and 24, and its height $=9$?

$= 5216 \cdot 6268$.

2. Find the solidity of a frustum of a paraboloid, the diameters of its ends being $=29$ and 15, and its height $=18$. . $= 7535 \cdot 1276$.

447. Problem III.—To find the solidity of a spheroid.

RULE.—Multiply the square of the equatorial axis by the polar axis, and this product by ·5236, and the result will be the solidity.



Let PELQ be an oblate spheroid, the minor axis PL being the fixed axis, or that of the spheroid, and EQ the major axis being the revolving axis.

Let the major axis $=a$, and the minor $=b$;
then $V = \cdot 5236 a^2 b$ for an oblate spheroid,
and $V = \cdot 5236 a b^2$ for a prolate spheroid.

EXAMPLE.—What is the solidity of the oblate spheroid whose polar axis is $=30$, and equatorial axis $=50$?

$$V = \cdot 5236 a^2 b = \cdot 5236 \times 50^2 \times 30 = 39270.$$

If a sphere PMLN be described on the axis PL, and if a section AB of the spheroid be taken perpendicular to its axis by a plane passing through D, and a section of the sphere by the same plane be taken, the section of the spheroid—namely, the circle AFB—would be to that of the sphere, whose diameter is MN, as the squares of their radii; that is, as the square of the ordinate AD to the square of the corresponding ordinate MD of the circle PMLN. But the squares of these ordinates are as $a^2 : b^2$; hence any section of the spheroid perpendicular to the axis is to the corresponding section of the sphere as $a^2 : b^2$. And, therefore, if V' = the solidity of the sphere,

$$V : V' = a^2 : b^2, \text{ and } V = \frac{a^2V'}{b^2} = \frac{a^2}{b^2} \times \cdot 5236b^3 = \cdot 5236a^2b.$$

The rule for the prolate spheroid may be similarly proved.

EXERCISES

1. Find the solidity of an oblate spheroid whose polar axis is = 15, and equatorial axis = 25. . . . . . . = 4908·75.

2. The axes of an oblate spheroid are = 12 and 20; what is its solidity? . . . . = 2513·28.

3. Find the solidity of the prolate spheroid whose polar axis is = 7, and equatorial axis = 5. . . . . = 91·63.

4. What is the solidity of the prolate spheroid whose axes are = 18 and 14? . . . . = 1847·2608.

448. Problem IV.—To find the solidity of a segment of a spheroid whose base is perpendicular to one of the axes.

1. When the segment is circular.

RULE.—Find the difference between three times the polar axis and twice the height of the segment, and multiply it by the square of the height, and the product by ·5236; then

The square of the polar axis is to that of the equatorial axis as the last product to the solidity of the segment.

When the segment is a portion of an oblate spheroid,

$$b^2 : a^2 = \cdot 5236(3b - 2h)h^2 : V;$$

hence
$$V = \cdot 5236(3b - 2h)\frac{a^2h^2}{b^2}.$$

When the segment is a portion of a prolate spheroid, it is similarly shown that $V = \cdot 5236(3a - 2h)\dfrac{b^2h^2}{a^2}$.

2. When the segment is elliptical.

RULE.—Find the difference between three times the equatorial

axis and twice the height of the segment, and multiply it by the square of the height, and this product by ·5236; then

The equatorial axis is to the polar axis as the last product to the solidity of the segment.

When the segment is a portion of an oblate spheroid,

$$a : b = \cdot 5236(3a - 2h)h^2 : V;$$

hence $$V = \cdot 5236(3a - 2h)\frac{bh^2}{a}.$$

When the segment is a portion of a prolate spheroid, it is similarly shown that $V = \cdot 5236(3b - 2h)\frac{ah^2}{b}$.

EXAMPLES.—1. The axes of an oblate spheroid are $=50$ and 30, and the height of a circular segment of it is $=6$; what is its volume?

$$V = \cdot 5236(3b - 2h)\frac{a^2h^2}{b^2} = \cdot 5236(90 - 12)\frac{50^2 \times 6^2}{30^2}$$
$$= \cdot 5236 \times 78 \times 100 = 4084 \cdot 08.$$

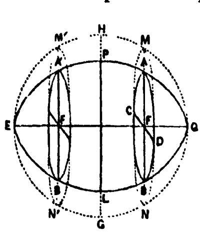
2. What is the solidity of an elliptical segment of a prolate spheroid, its height being $=12$, and the axes $=100$ and 60?

$$V = \cdot 5236(3b - 2h)\frac{ah^2}{b} = \cdot 5236(180 - 24)\frac{100 \times 12^2}{60}$$
$$= \cdot 5236 \times 156 \times 240 = 19603 \cdot 584.$$

The first rule is easily derived thus:—Let APB be a circular segment of an oblate spheroid (fig. to Prob. III.). Then it was shown in last problem that the corresponding sections of the spheroid and sphere, such as those whose diameters are AB and MN, were to one another as $a^2 : b^2$. Hence, if $V' =$ the volume of the spherical segment MPN, $V' = \cdot 5236(3b - 2h)h^2$ (by Art. 393), and

$$b^2 : a^2 = V' : V, \text{ and } V = \frac{a^2V'}{b^2} = \cdot 5236(3b - 2h)\frac{a^2h^2}{b^2}.$$

When the spheroid is prolate, $a^2 : b^2 = V' : V$.



Let the segment be elliptical, as AQB, the spheroid PELQ being oblate. Then, if HEGQ be a sphere described on the axis EQ, and ACBD, MCND be two corresponding sections of the spheroid and sphere, it is evident that CD is a diameter of each of these sections, and equal to MN. Also the diameter MN : AB = HG : PL = EQ : PL $= a : b$.

Now, MN and AB are any corresponding chords of the sphere and spheroid parallel to PL; hence any

other two corresponding chords in the plane of the section MCND, parallel to AB, have the same proportion. Hence ACBD is an ellipse. Therefore the area of the elliptic section ACBD is to that of the sphere MCND as b to a. Hence, V′ being the volume of the spherical segment,

$$a : b = V' : V, \text{ and } V = \frac{bV'}{a} = \cdot 5236(3a - 2h)\frac{bh^2}{a}.$$

When the spheroid is prolate, the rule for the elliptic segment may be proved in the same manner, by describing a sphere on the equatorial axis.

EXERCISES

1. Find the solidity of a circular segment of a prolate spheroid, the axes being = 40 and 24, and the height = 4. . . = 337·7848.

2. The axes of an oblate spheroid are = 25 and 15, and the height of a circular segment of it is = 3; what is the solidity of the segment? . . . . = 510·51.

3. What is the solidity of an elliptic segment of an oblate spheroid whose height is = 10, the axes of the spheroid being = 100 and 60? . . . . = 8796·48.

4. Find the solidity of an elliptic segment of a prolate spheroid whose axes are = 25 and 15, the height of the segment being = 3. = 306·306.

449. **Problem V.—To find the solidity of the middle frustum of a spheroid.**

1. When the frustum is circular.

RULE.—To twice the square of the middle diameter add the square of the end diameter; multiply this sum by the length of the frustum, and then this product by ·2618.

When the frustum is a portion of an oblate spheroid,

$$V = \cdot 2618(2a^2 + d^2)l.$$

When the frustum is a portion of a prolate spheroid,

$$V = \cdot 2618(2b^2 + d^2)l.$$

2. When the frustum is elliptical.

RULE.—To double the product of the axes of the middle section add the product of the axes of one end, multiply this sum by the length of the frustum, and by ·2618.

Let d and e be the greater and less axes of one end, then, whether the frustum is a portion of an oblate or prolate spheroid,

$$V = \cdot 2618(2ab + de)l.$$

EXAMPLES.—1. What is the solidity of a circular middle frustum

of an oblate spheroid, the middle diameter being $=25$, the end diameters $=20$, and the length $=9$?

$$V = \cdot 2618(2a^2 + d^2)l = \cdot 2618(2 \times 25^2 + 20^2)9 = 2\cdot 3562 \times 1650 = 3887\cdot 73.$$

2. Find the solidity of an elliptic middle frustum of a spheroid, the axes of the middle section being $=50$ and 30, those of the ends $=30$ and 18, and the length $=40$.

$$V = \cdot 2618(2ab + de)l = \cdot 2618(2 \times 50 \times 30 + 30 \times 18)40$$
$$= 10\cdot 472 \times 3540 = 37070\cdot 88.$$

The rules are easily proved by means of those in Problems III. and IV.

Let ABA′B′ (fig. to Art. 447) be a circular middle frustum of the oblate spheroid PELQ; then the volume of the middle zone of the sphere described on PL is $V' = \cdot 7854(b^2 - \frac{1}{3}l^2)l$ by Art. 394, where $l = DD'$. But $PL^2 : EQ^2 = LD \,.\, DP : AD^2$, or $b^2 : a^2 = \frac{1}{2}(b+l) \,.\, \frac{1}{2}(b-l) : \frac{1}{4}d^2$, or $b^2 : a^2 = b^2 - l^2 : d^2$; hence, $b^2 - l^2 = \dfrac{b^2 d^2}{a^2}$.

Hence $\quad b^2 - \frac{1}{3}l^2 = \frac{2}{3}b^2 + \frac{1}{3}(b^2 - l^2) = \frac{1}{3}\left(2b^2 + \dfrac{b^2 d^2}{a^2}\right) = \frac{1}{3}(2a^2 + d^2)\dfrac{b^2}{a^2}$,

and $$V' = \cdot 2618(2a^2 + d^2)\frac{b^2}{a^2}l.$$

But any section, as MN, of the sphere perpendicular to the axis PL, is to the corresponding section AB of the spheroid as $b^2 : a^2$, or $b^2 : a^2 = V' : V$;

therefore, $$V = \frac{a^2 V'}{b^2} = \cdot 2618(2a^2 + d^2)l.$$

Again, let ABA′B′ (fig. to Art. 448) be an elliptic middle frustum of an oblate spheroid PELQ; then the volume of the middle frustum MNM′N′ of the sphere described on EQ is

$$V' = \cdot 7854(a^2 - \tfrac{1}{3}l^2)l, \text{ where } l = FF'.$$

But $\quad EQ^2 : PL^2 = EF \,.\, FQ : AF^2$,

or $\quad a^2 : b^2 = \frac{1}{2}(a+l)\frac{1}{2}(a-l) : \frac{1}{4}c^2$,

or $\quad a^2 : b^2 = a^2 - l^2 : c^2$;

therefore, $$a^2 - l^2 = \frac{a^2 c^2}{b^2}.$$

Hence $$a^2 - \tfrac{1}{3}l^2 = \tfrac{2}{3}a^2 + \tfrac{1}{3}(a^2 - l^2) = \tfrac{2}{3}a^2 + \frac{a^2c^2}{3b^2} = \tfrac{1}{3}\left(2a^2 + \frac{a^2c^2}{b^2}\right).$$

But $EQ : PL = CD : AB$, or $a : b = d : c$; hence $d^2 = \dfrac{a^2c^2}{b^2}$, and $V' = \cdot 7854 \times \frac{1}{3}(2a^2 + d^2)l$. But any section of the spherical frustum, as MN, is to the corresponding one of the spheroidal frustum AB as $a : b$, by Prob. IV.;

hence $$a : b = V' : V, \text{ and } V = \frac{bV'}{a} = \cdot2618(2a^2 + d^2)\frac{lb}{a}$$
$$= \cdot2618(2ab + de)l, \text{ for } d = \frac{ae}{b}.$$

Exercises

1. Find the solidity of a circular middle frustum of a spheroid, the middle diameter being = 100, those of the ends = 80, and the length = 36. . . . . $= 248814 \cdot 72$.

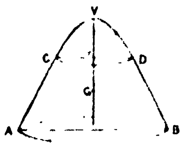
2. What is the solidity of a circular middle frustum of a spheroid, its middle diameter being = 30, its end diameters = 18, and its length = 40? . . . . 22242·528.

3. Find the solidity of an elliptic middle frustum of an oblate spheroid, the axes of the middle section being 25 and 15, those of each end 15 and 9, and the height = 20. . . 4633·86.

4. What is the solidity of an elliptic middle frustum of an oblate spheroid, the axes of the middle section being 100 and 60, those of each end = 60 and 36, and the length = 80? . . 296567·04.

450. Problem VI.—To find the solidity of an hyperboloid.

Rule.—To the square of the radius of the base add the square of the middle diameter between the base and the vertex; multiply this sum by the altitude, and then by ·5236.



Let r = the radius of the base AE, d = the middle diameter, and h = the height EV;
then $$V = \cdot5236(r^2 + d^2)h.$$

When the diameter of the base, the height, and the axes of the generating hyperbola are given, but not the middle diameter, it may be found by Art. 431; thus, if $h' = \frac{1}{2}h$,

$$a^2 : b^2 = (a + h')h' : \tfrac{1}{4}d^2;\ \text{hence, } d^2 = \frac{4b^2(a + h')h'}{a^2}.$$

Example.—Find the solidity of an hyperboloid, the altitude of which is = 25, the radius of the base = 26, and the middle diameter = 34.

$$V = \cdot5236(r^2 + d^2)h = \cdot5236(26^2 + 34^2)25$$
$$= 13 \cdot 09 \times 1832 = 23980 \cdot 88.$$

Let the figure in Art. 430 be supposed to revolve about its axis GM, and the cone, generated by the asymptotes GH, GL, is called the *asymptotic cone*. Let V denote the volume of the conoid BPN; V′ that of the frustum of the cone of the same height with

the conoid, the diameter of its upper end being IK; and V″ that of the cylinder of the same height with these two solids, and whose radius is equal to half the minor axis CG or IB; then it can be shown in the following manner that $V = V' - V''$.

Let r and r' denote the radii of the bases of the conoid and conic frustum, and h their height, d the double ordinate at the middle point between the base and vertex, and a, b the semi-axes; then $(r'+r)(r'-r) = b^2$, or $r^2 = r'^2 - b^2$; hence the section of the conoid is equal to the difference between the corresponding section of the cone and of the cylinder whose volume is V″; and as this can be shown to be the case for every section, therefore $V = V' - V''$.

Now, $\quad V' = \frac{1}{3}\pi(r'^2 + b^2 + br')h, \; V'' = \pi b^2 h,$

therefore, $\quad V = \frac{1}{3}\pi(r'^2 + br' - 2b^2)h.$

But since $GB = a$, and $BI = b$, from similar triangles,

$$a : b = a + h : r', \text{ and } r' = \frac{b}{a}(a+h);$$

hence $$V = \frac{1}{3}\frac{\pi}{a^2}(6ab^2h + 2b^2h^2)h.$$

Now, $$a^2 : b^2 = \left(2a + \frac{h}{2}\right)\frac{h}{2} : \frac{1}{4}d^2, \text{ and } d^2 = \frac{b^2}{a^2}(4a+h)h.$$

Also, $a^2 : b^2 = (2a+h)h : r^2$, and $r^2 = \frac{b^2}{a^2}(2a+h)h$, and if d^2 and r^2 be substituted in the preceding expression for V, by eliminating a and b, it becomes $\cdot 5236(r^2 + d^2)h$.

EXERCISES

1. Find the solidity of an hyperboloid whose altitude is $= 50$, the radius of its base $= 52$, and the middle diameter $= 68$. $= 191847 \cdot 04$.

2. What is the solidity of an hyperboloid whose altitude is $= 20$, the radius of its base $= 24$, and the middle diameter $= 31 \cdot 749$? $= 16587 \cdot 6375$.

451. Problem VII.—To find the solidity of a frustum of an hyperboloid.

RULE.—Add together the squares of the radii of the two ends, and of the middle diameter between them, multiply the sum by the altitude, and this product by ·5236.

Let R and r be the radii of the two ends AE, CF (fig. to Prob. VI.), d the middle diameter through G, and h the height EF;

then $$V = \cdot 5236(R^2 + r^2 + d^2)h.$$

EXAMPLE.—Find the solidity of a frustum of an hyperboloid,

the diameters of its ends being $=6$ and 10, the middle diameter $=8\frac{1}{2}$, and the height $=12$.

$$V = \cdot 5236(R^2 + r^2 + d^2)h = \cdot 5236\{5^2 + 3^2 + (8\tfrac{1}{2})^2\}12$$
$$= \cdot 5236 \times 425 \times 3 = \cdot 5236 \times 1275 = 667 \cdot 59.$$

Let the axes of the generating hyperbola be a, b, and h', the height VF, then $a^2 : b^2 = (a + h')h' : r^2$; and hence $r^2 = \frac{b^2}{a^2}(a + h')h'$. And similar values can be found for R^2 and $(\frac{1}{2}d)^2$, in terms of a, b, h', and h. From the three equations thus formed, if the values of the unknown quantities a, b, and h' be found in terms of R, r, d, and h, the solidities of the two hyperboloids VAB, VCD can then be found in terms of the same given quantities, and the difference of these solidities would give that of the frustum, and hence the formula for finding it.

EXERCISES

1. What is the solidity of a frustum of an hyperboloid, the diameters at its two ends and at its middle being $=12$, 20, and 17, and its height $=18$? . . . . 4005·54.

2. Find the solidity of a frustum of an hyperboloid, the diameters at its ends and middle being $=3$, 5, and $4 \cdot 25$, and its height $=8$.
111·265.

REGULAR SOLIDS

There are only five **Regular Solids**, or, as they are sometimes called, **Platonic Bodies**, and it can be proved that no more can exist.

DEFINITIONS

The regular solids are the five following :—

452. The **tetrahedron** is a regular triangular pyramid whose sides are equilateral triangles.

453. The **hexahedron** is a cube.

454. The **octahedron** is contained by eight equilateral triangles.

455. The **dodecahedron** is contained by twelve regular pentagons.

456. The **icosahedron** is contained by twenty equilateral triangles.

Each side of a regular solid, except the tetrahedron, has an opposite face parallel to it, and the edges of these faces are also respectively parallel.

457. Problem I.—To find the solidity of a regular tetrahedron.

RULE.—Multiply the cube of one of its edges by the square root of 2, or 1·414214, and take one-twelfth of the product.

Let e = one of the edges;

then $V = \frac{1}{12}e^3\sqrt{2} = \frac{1}{12}e^3 \times 1{\cdot}414214 = {\cdot}117851e^3$;

also, the surface $s = e^2\sqrt{3}$ (by Art. 257), or $s = 1{\cdot}732e^2$.

EXAMPLE.—What is the solidity and surface of a tetrahedron whose edge is = 15?

$v = \frac{1}{12}e^3\sqrt{2}$, or $v = {\cdot}117851e^3 = {\cdot}117851 \times 15^3 = {\cdot}117851 \times 3375 = 397{\cdot}747$.

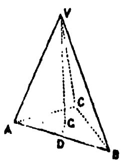
The surface may be found by the rules formerly given for the areas of regular polygons.

Thus, the surface of the four sides of this pyramid, as they are equilateral triangles, is (A' being the tabular area),

$$= 4e^2A' = 4 \times 15^2 \times {\cdot}433 = 900 \times {\cdot}433 = 389{\cdot}7;$$

or $$s = e^2\sqrt{3} = 15^2\sqrt{3} = 225 \times 1{\cdot}732 = 389{\cdot}7.$$

Let VABC be a regular tetrahedron. Draw VG perpendicular to the base; join AG, and draw DG perpendicular to AB.



It can be easily shown that AG bisects the angle CAB, and that DG bisects AB. Hence angle GAD = 30°, and therefore AG = 2DG, as may be easily proved; and hence $AG^2 = 4DG^2$.

Now, $AG^2 = AD^2 + DG^2$,

or $4AG^2 = 4AD^2 + 4DG^2$;

that is, $3AG^2 = AB^2 = AV^2 = AG^2 + GV^2$;

hence $2AG^2 = GV^2$, and $GV = AG\sqrt{2}$.

Now, the base $b = ABC = 6ADG = 3AD \,.\, DG$;

and hence V or $\frac{1}{3}bh = AD \,.\, DG \,.\, GV = \frac{1}{2}AB \,.\, \frac{1}{2}AG \,.\, AG\sqrt{2}$;

or $V = \frac{1}{4}AB \,.\, AG^2\sqrt{2} = \frac{1}{4}AB \,.\, \frac{1}{3}AB^2\sqrt{2} = \frac{1}{12}e^3\sqrt{2}$.

EXERCISES

1. Find the solidity of a tetrahedron whose edge is = 8. = 60·3397.
2. Find the solidity of a tetrahedron whose side is = 3. = 3·1819.
3. What is the volume and surface of a tetrahedron whose edge is = 6? . . . . = 25·4558 and 62·352.

The rules for finding the volume and surface of a cube were given in Articles 375 and 377.

458. Problem II.—To find the solidity of an octahedron.

RULE.—Multiply the cube of one of the edges by the square root of 2, and take one-third of the product.

$$V = \tfrac{1}{3}e^3\sqrt{2}, \text{ or } v = \cdot 471405e^3,$$

and $$s = 8e^2A', \text{ or } s = 2e^2\sqrt{3}$$

by Articles 269 and 257.

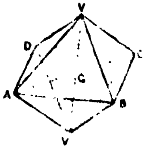
EXAMPLE.—Find the volume and surface of an octahedron whose side is $= 6$.

$$V = \tfrac{1}{3}e^3\sqrt{2} = \cdot 471405e^3 = \cdot 471405 \times 6^3 = 101\cdot 823,$$

and $$s = 8e^2A' = 8 \times 6^2 \times \cdot 433 = 124\cdot 704;$$

or $$s = 2e^2\sqrt{3} = 2 \times 6^2 \times 1\cdot 732 = 124\cdot 704.$$

Let AVCV′ be a regular octahedron. Draw VG perpendicular to the plane ADC; join AC.



It is easily proved that $AG = GC$,

and $$4AG^2 = AC^2 = 2AB^2;$$

hence $$AG^2 = \tfrac{1}{2}AB^2, \text{ or } AG = \tfrac{1}{2}AB\sqrt{2}.$$

Also, $$VG^2 = AV^2 - AG^2 = AB^2 \quad AG^2 - AG^2;$$

or $$VG = AG.$$

Now, area of square $AC = b = AB^2 = e^2 = 2AG^2$.

Volume of $VABCD = \tfrac{1}{3}bh = \tfrac{1}{3} \times 2AG^2 . AG = \tfrac{1}{3}AB^2 . \tfrac{1}{2}AB\sqrt{2} = \tfrac{1}{6}e^3\sqrt{2}$; hence the whole solid $V = \tfrac{1}{3}e^3\sqrt{2}$.

Also the surface is $s = 2e^2\sqrt{3}$ (by Art. 257), or it is $8e^2A'$ (by Art. 269).

EXERCISES

1. Find the solidity of an octahedron whose edge is $= 16$.

$= 1930\cdot 87$.

2. What are the volume and surface of an octahedron whose edge is $= 3$? . . . . $= 12\cdot 7279$ and $31\cdot 1769$.

459. Problem III.—To find the solidity of a dodecahedron.

RULE.—To 47 add 21 times the square root of 5; divide this sum by 40; find the square root of the quotient, and multiply it by five times the cube of the edge; or multiply the cube of the edge by 7·6631.

Or, $$V = 5e^3\sqrt{\frac{47 + 21\sqrt{5}}{40}} = 5e^3 \times 1\cdot 53262 = 7\cdot 6631e^3.$$

Also the surface $$s = 15e^2\sqrt{\frac{5 + 2\sqrt{5}}{5}} = 15e^2 \times 1\cdot 376382.$$

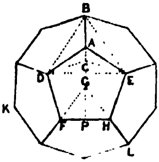
EXAMPLE.—What are the solidity and surface of a dodecahedron whose edge is $= 2$?

$$V = e^3 \times 7\cdot 6631 = 8 \times 7\cdot 6631 = 61\cdot 3048,$$

and $$s = 15e^2 \times 1\cdot 376382 = 15 \times 4 \times 1\cdot 376382 = 82\cdot 58292.$$

Let ABKL be a regular dodecahedron.

Draw BD, DE, EB on three contiguous sides, and AC perpendicular to the plane DBE, and draw DC, CE. Then, in the isosceles triangle ADE, angle A $=108°$; hence angle $E=36°$; and hence by trigonometry DE can be found, the side AD being given. Hence the sides of the equilateral triangle DBE are known; and C is evidently its centre, also angle $CDE=CED=\frac{1}{2}BED$ $=\frac{1}{2}\times 60°=30°$; hence $C=120°$; therefore, DE being known, CD can be found. Hence, in the right-angled triangle ACD, $AC^2=AD^2-CD^2$, and AC can thus be found.



Now AC, if produced, would evidently pass through the centre of the polyhedron, or of its circumscribing sphere; AC is the versed sine of an arc of it passing through AD; hence (as in Art. 276), if $D=$ the diameter of the sphere, $D.AC=AD^2$, therefore $D=\frac{AD^2}{AC}=\frac{e^2}{p}$, if $p=AC$.

Again, ADH being a regular pentagon, if G be its centre, and GP be perpendicular to FH, then angle $GFP=54°$, and $FP=\frac{1}{2}e$, is known; hence FG can be found.

Now, the lines joining the centre of the polyhedron, and the points G and F, are evidently the radii of the inscribed and circumscribing spheres, and with FG form a right-angled triangle. Hence, if R and r are their radii, and $r'=FG$, $R^2=r^2+r'^2$; and hence $r^2=R^2-r'^2$; and $R=\frac{1}{2}D$, and r' being known, r can thus be found.

But every regular polyhedron is composed of regular pyramids, whose altitudes are the radius of the inscribed sphere, and base one of the sides of the solid, and their number is the number of its sides. Hence, if $n=$ the number of sides, $A=$ area of one side, then $V=\frac{1}{3}nAr$.

By actually calculating the values of the preceding quantities, the result would be the rule $7{\cdot}6631e^3$.

The first expression in the rule given above would be found by using, instead of the natural sines of the various angles, the following values—namely, $\sin 36°=\frac{1}{4}\sqrt{(10-2\sqrt{5})}$, $\sin 108°=\frac{1}{4}\sqrt{(10+2\sqrt{5})}$, $\sin 30°=\frac{1}{2}$, $\sin 120°=\frac{1}{2}\sqrt{3}$, $\sin 72°=\frac{1}{4}\sqrt{(10+2\sqrt{5})}$, and $\sin 54°=\frac{1}{4}(1+\sqrt{5})$.

Exercises

1. Find the solidity of a dodecahedron the side of which is $=6$.

$=1655{\cdot}2296$.

2. What are the solidity and surface of a dodecahedron whose side is $=4$? . . . . $=490 \cdot 4384$ and $330 \cdot 33168$.

460. Problem IV.—To find the solidity of an icosahedron.

RULE.—To 7 add three times the square root of 5, divide this sum by 2, find the square root of the quotient, and multiply it by the cube of the edge, then take five-sixths of this product.

Or, $V = \frac{5}{6}e^3\sqrt{\frac{7+3\sqrt{5}}{2}}$, or $V = \frac{5}{6}e^3 \times 2 \cdot 61803 = 2 \cdot 18169e^3$,

and $s = 20e^2A'$, or $s = 5e^2\sqrt{3}$.

EXAMPLE.—What are the solidity and surface of an icosahedron whose edge is $=2$?

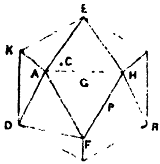
$$V = \tfrac{5}{6}e^3\sqrt{\frac{7+3\sqrt{5}}{2}} = \tfrac{5}{6} \times 2^3\sqrt{\frac{7+3 \times 2 \cdot 23606}{2}}$$

$$= \tfrac{20}{3}\sqrt{6 \cdot 85409} = \tfrac{20}{3} \times 2 \cdot 61803 = 17 \cdot 4535,$$

and $s = 5e^2\sqrt{3} = 5 \times 2^2 \times 1 \cdot 73205 = 34 \cdot 641$;

or $s = 20e^2A' = 20 \times 2^2 \times \cdot 433 = 34 \cdot 64$.

Let fall from A a perpendicular AC upon the plane of the regular pentagon DFHEK; then C will be the centre of the pentagon, and CD may be found as FG in the preceding figure. Then $AC^2 = AD^2 - CD^2$; and AC is thus found.



Let G be the centre of one of the triangular sides AFH, and find FG as CD was found in the preceding figure.

Then D, R, r, p, and r' denoting the same quantities as in the preceding problem, $D \cdot AC = AD^2$, and $D = \frac{AD^2}{AC} = \frac{e^2}{p}$. Also, $R = \frac{1}{2}D$, and $r^2 = R^2 - r'^2$.

As r can thus be found, then $V = \frac{1}{3}nrA'$, where $n = 20$. If the value of r be calculated, and substituted in this expression, the result will be the preceding formula.

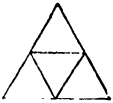
EXERCISES

1. Find the solidity of an icosahedron whose edge is $=6$. $=471 \cdot 24504$.

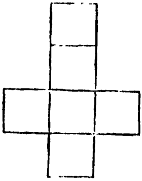
2. What are the volume and surface of an icosahedron whose edge is $=5$? . . . . $272 \cdot 711$ and $216 \cdot 506$.

461. The five regular solids may be easily made by cutting a piece of pasteboard into the following figures. The pasteboard should be cut nearly half-through along all the lines of the

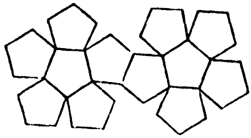
figure, and it will then be easily folded up into the form of the solid.



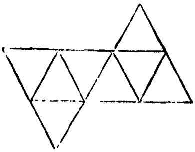
Tetrahedron



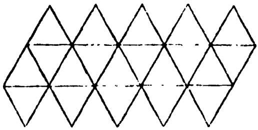
Cube



Dodecahedron



Octahedron



Icosahedron

The solidities and surfaces of regular solids may also be found by means of a Table containing the surfaces and solidities of regular solids, whose edges are = 1. The Table may be calculated by means of the preceding rules.

No. of Sides	Names	Surfaces	Solidities
4	Tetrahedron	1·7320508	0·1178513
6	Hexahedron	6·	1·
8	Octahedron	3·4641016	0·4714045
12	Dodecahedron	20·6457288	7·6631189
20	Icosahedron	8·6602540	2·1816950

The rules for finding the solidities and surfaces by means of this Table are:—

For the solidity of a regular solid, multiply the tabular solidity of the corresponding solid by the cube of the edge.

Or, $V = e^3 V'$, if V' = the tabular solidity for edge = 1.

For the surface of a regular solid, multiply the tabular surface of the corresponding solid by the square of the edge.

Or, $s = e^2 s'$, if s' = the tabular surface for edge = 1.

Example.—What are the solidity and surface of an icosahedron whose edge is = 2?

$$V = e^3 V' = 2^3 \times 2{\cdot}181695 = 17{\cdot}45356,$$

and $$s = e^2 s' = 2^2 \times 8{\cdot}660254 = 34{\cdot}641016.$$

The student may perform the preceding exercises in regular solids by means of this rule.

CYLINDRIC RINGS

462. A **cylindric ring** is a solid formed by the revolution of a circle about an axis in its own plane, the centre of revolution being without the circle.

463. The circle described by the centre of the generating circle is the **axis** of the ring.

The centre of the axis is the **centre** of the ring.

464. A **cross section** of a cylindric ring is one perpendicular to the axis. This section is equal to the generating circle.

465. The **interior diameter** of the ring is a line passing through its centre in the plane of its axis, and limited by its interior surface; and an **external diameter** is one terminated by its exterior surface.

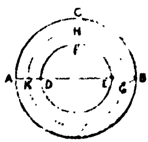
466. Problem I.—To find the solidity of a cylindric ring.

RULE.—Multiply the area of a cross section by the axis of the ring; or,

Multiply the square of the thickness by the diameter of the axis, and this product by 2·4674.

The diameter of a cross section of the ring is equal to half the difference of the interior and exterior diameters; and the diameter of the axis is half the sum of these diameters.

Let d' and d'' be the exterior and interior diameters AB and DE of the ring, and d that of the axis GHK, and t the thickness of the ring EB;



then $d = \frac{1}{2}(d' + d'')$, and $t = \frac{1}{2}(d' - d'')$.

Also, if $\mathcal{A}$ = the area of a cross section,

and c = the length of the axis;

then $\mathcal{A} = {\cdot}7854t^2$, and $c = 3{\cdot}1416d$,

and $V = \mathcal{A}c = 2{\cdot}4674dt^2$.

EXAMPLE.—Find the solidity of a cylindric ring whose inner diameter is $=12$, and its exterior diameter $=16$.

$$t=\tfrac{1}{2}(16-12)=2, \text{ and } d=\tfrac{1}{2}(16+12)=14;$$

hence $\quad V=2{\cdot}4674dt^2=2{\cdot}4674\times 14\times 2^2=138{\cdot}1744.$

By what is sometimes called the theorem of Guldinus (it is in reality due to Pappus), it appears that the solidity of the ring is equal to the area of the generating circle, multiplied by the line described by its centre of gravity or centre—that is, by the axis of the ring—or $={\cdot}7854t^2\times 3{\cdot}1416d=2{\cdot}4674dt^2$, as above.

EXERCISES

1. Find the solidity of a cylindric ring whose interior and exterior diameters are $=16$ and 24. . . . . . $=789{\cdot}568$.

2. Find the solidity of a cylindric ring, its diameters being $=8$ and 14 inches. . . . . $=244{\cdot}2726$.

3. The interior diameter of a cylindric ring is $=26$ inches, and its thickness $=8$ inches; what is its solidity? . . $=5369{\cdot}0624$.

467. Problem II.—To find the surface of a cylindric ring.

RULE.—Multiply the circumference of a cross section of the ring by the axis.

Or, $\quad s=3{\cdot}1416t\times 3{\cdot}1416d=9{\cdot}8696dt,$

where d and t are found, as in the last problem.

EXAMPLE.—What is the surface of a cylindric ring whose thickness is $=1$ inch, and inner diameter $=9$?

$$s=9{\cdot}8696dt=9{\cdot}8696\times 10\times 1=98{\cdot}696.$$

The rule for the area of the surface can also be proved by the theorem of Pappus; for by it the surface is equal to the product of the circumference of the generating circle by the line described by its centre of gravity or centre—that is, by the axis of the ring

EXERCISES

1. Find the surface of a cylindric ring whose diameters are $=36$ and 52. . . . . $=3474{\cdot}0992$.

2. What is the surface of a cylindric ring whose thickness is $=6$ inches, and inner diameter $=24$ inches? . . $=1776{\cdot}528$.

SPINDLES

468. A **spindle** is a solid generated by the revolution of an arc of a curve, cut off by a double ordinate, about that ordinate as an axis.

469. The spindle is said to be **circular, parabolic elliptic,** or **hyperbolic** according as the generating arc is a portion of a circle, a parabola, an ellipse, or hyperbola.

I. THE CIRCULAR SPINDLE

The **central distance** of a circular spindle is the distance between the centre of the circle and the centre of the spindle.

470. **Problem I.—To find the solidity of a circular spindle.**

RULE.—From one-twelfth of the cube of the length of the spindle subtract the product of the central distance and the area of the generating segment, and multiply this remainder by 6·2832.

The length of the spindle and half its middle diameter are the chord and height of the generating circular segment: hence the radius of the circle can be calculated by Art. 276, and the area of the segment by Art. 285.

From the radius of the circle subtract half of the middle diameter, and the remainder is the central distance.

Let CKDL be a circular spindle, and $l = KL$, the length of the spindle; $d = CD$, the middle diameter; $c = SM$, the central distance; $\mathcal{A} = KCL$, the area of the generating segment; A' = the tabular segment (Art. 286), and h' its height.

Then $$V = 6{\cdot}2832(\tfrac{1}{12}l^3 - \mathcal{A}c),$$

and $h' = \frac{1}{2}d \div 2r$; hence A' is known, and $\mathcal{A} = 4r^2A'$.

EXAMPLE.—Find the solidity of a circular spindle whose length is = 40 inches, and middle diameter = 30 inches.

By Art. 276, if r = radius of circle = SC, then

$$2r = \frac{(\frac{1}{2}l)^2 + (\frac{1}{2}d)^2}{\frac{1}{2}d} = \frac{20^2 + 15^2}{15} = 41\tfrac{2}{3}, \text{ and } r = 20\tfrac{5}{6};$$

hence $$c = r - \tfrac{1}{2}d = 20\tfrac{5}{6} - 15 = 5\tfrac{5}{6},$$

and $h' = (\frac{1}{2}d) \div 2r = 15 \div 41\frac{2}{3} = 15 \times \frac{3}{125} = \frac{9}{25} = {\cdot}36$ = height of tabular segment. to which corresponds the tabular area

$$A' = {\cdot}254551.$$

Therefore, $\text{Æ} = \cdot254551 \times (41\tfrac{2}{3})^2 = 441\cdot9288$,

and $V = 6\cdot2832(\tfrac{1}{12}l^3 - \text{Æ}c) = 6\cdot2832(\tfrac{1}{3} \times 16000 - 441\cdot9288 \times \tfrac{35}{6})$

$= 6\cdot2832 \times 2755\cdot415 = 17312\cdot8235$.

Exercises

1. The length of a circular spindle is $= 8$, and its middle diameter $= 6$; what is its solidity? . . . . . $= 138\cdot503$.

2. The length of a circular spindle is $= 24$, and its middle diameter $= 18$; find its solidity. . . . . . $= 3739\cdot5696$.

471. **Problem II.—To find the solidity of the middle frustum of a circular spindle.**

Rule.—From three times the square of the length of the spindle subtract the square of the length of the frustum, multiply the difference by the length of the frustum, and take one-twenty-fourth of this product; from the last result subtract the product of the area of the generating surface by the central distance, and multiply this remainder by $6\cdot2832$.

Let EGHF be a middle frustum of the spindle CKDL (last fig.): and let $CD = D$, $EG = d$, $AB = l$, $KL = L$, $h = CI$, $A =$ area of segment CEF, of which CEI is the half, $g =$ area of generating surface CEABF, and c, r, and h', as in last problem;

then $h = \tfrac{1}{2}(D - d)$, $2r = \dfrac{l^2}{4h} + h$, $h' = \dfrac{h}{2r}$; hence $A' =$ tabular area is known, and $\text{Æ} = 4r^2A'$.

Also, $2EM = \tfrac{1}{2}dl$; and hence $g = \text{Æ} + \tfrac{1}{2}dl$.

Then $c = r - \tfrac{1}{2}D$, and $\tfrac{1}{4}L^2 = r^2 - c^2$,

and $V = 6\cdot2832\{\tfrac{1}{24}(3L^2 - l^2)l - cg\}$.

Example.—Find the solidity of a middle frustum of a circular spindle, the middle and one of the end diameters being $= 16$ and 12, and the length of the frustum $= 20$.

$$h = \tfrac{1}{2}(D - d) = \tfrac{1}{2}(16 - 12) = \tfrac{1}{2} \times 4 = 2,$$

$$2r = \frac{l^2}{4h} + h = \frac{400}{8} + 2 = 52, \text{ and } r = 26.$$

$$h' = \frac{h}{2r} = \frac{2}{52} = \cdot038462; \text{ and hence } A' = \cdot009940.$$

$$\text{Æ} = 4r^2A' = 2704 \times \cdot009940 = 26\cdot87776.$$

$$g = a + \tfrac{1}{2}dl = 26\cdot87776 + 6 \times 20 = 146\cdot87776.$$

$$c = r - \tfrac{1}{2}D = 26 - 8 = 18,$$

$$\tfrac{1}{4}L^2 = r^2 - c^2 = 26^2 - 18^2 = 676 - 324 = 352,$$

and $$V=\{\tfrac{1}{24}(3L^2-l^2)l-cg\}6\cdot2832$$
$$=\{\tfrac{1}{24}(4224-400)20-2643\cdot8\}\times6\cdot2832$$
$$=(3186\cdot\dot{6}-2643\cdot8)6\cdot2832=542\cdot86\dot{6}\times6\cdot2832=3410\cdot93984.$$

Exercises

1. Find the solidity of a middle frustum of a circular spindle, the middle diameter being $=18$, an end diameter $=8$, and the length of the frustum $=20$. . . . . $=3657\cdot142$.

2. What is the solidity of a middle frustum of a circular spindle, its middle diameter being $=32$, an end diameter $=24$, and the length of the frustum $=40$? . . . . $=27287\cdot54$.

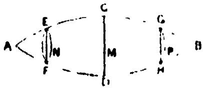
II. THE PARABOLIC SPINDLE

472. Problem III.—To find the solidity of a parabolic spindle.

Rule.—Multiply the square of the middle diameter by the length of the spindle, and this product by $\cdot41888$; or,

Take eight-fifteenths of the circumscribing cylinder.

Let CADB be a parabolic spindle, CD $=d$, $l=$ AB;



then $$V=\cdot41888d^2l,$$
or $$V=\tfrac{8}{15}\times\cdot7854d^2l.$$

Example.—Find the solidity of a parabolic spindle whose length is $=40$, and middle diameter $=16$.

$$V=\cdot41888d^2l=\cdot41888\times16^2\times40=4289\cdot33.$$

Exercises

1. The length of a parabolic spindle is $=30$, and its middle diameter $=17$; what is its solidity? . . . . . $=3631\cdot6896$.

2. Find the solidity of a parabolic spindle whose length is $=18$, and middle diameter $=6$ feet. . . . . . . $=271\cdot434$.

3. What is the solidity of a parabolic spindle whose length is $=50$, and middle diameter $=10$? . . . . . $=2094\cdot4$.

473. Problem IV.—To find the solidity of the middle frustum of a parabolic spindle.

Rule.—Add together 8 times the square of the middle diameter, 3 times the square of an end diameter, and 4 times the product of these diameters; multiply this sum by the length of the frustum, and then this product by $\cdot05236$.

Let $D = CD$, the middle diameter (last fig.),
$d = EF$, an end diameter,
$l = NP$, the length of the frustum;
then $V = \cdot05236(8D^2 + 3d^2 + 4Dd)l$.

EXAMPLE.—Find the solidity of a middle frustum of a parabolic spindle, its middle and an end diameter being $= 20$ and 16, and its length $= 20$ feet.

$$V = \cdot05236(8 \times 20^2 + 3 \times 16^2 + 4 \times 20 \times 16)20 = 1\cdot0472(3200 + 768 + 1280)$$
$$= 1\cdot0472 \times 5248 = 5495\cdot7056.$$

EXERCISES

1. What is the solidity of the middle frustum of a parabolic spindle, its middle and an end diameter being $= 16$ and 12, and its length $= 30$? . . . . $= 5101\cdot9584$.

2. Find the solidity of a middle frustum of a parabolic spindle, an end and its middle diameters being $= 10$ and 18, and its length $= 40$. . . . . $= 7564\cdot9728$.

III. THE ELLIPTIC SPINDLE

474. The **central distance** of an elliptic spindle is the distance from the centre of the generating ellipse to the centre of the spindle.

475. A diameter at one-fourth the length from the end of a spindle or a frustum is called the **quarter diameter.**

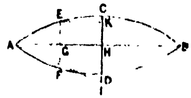
476. Problem V.—To find the solidity of an elliptic spindle.

RULE.—Divide 3 times the area of the generating segment by the length of the spindle, and from the quotient subtract the middle diameter; multiply this remainder by 4 times the central distance, and subtract this product from the square of the middle diameter; multiply this remainder by the length of the spindle, and the product by $\cdot5236$ for the solidity.

The central distance is found thus:—From 3 times the square of the middle diameter take 4 times the square of the quarter diameter, and from 4 times the latter diameter take 3 times the former; divide the former difference by the latter, and one-fourth of the quotient will be the central distance.

To the central distance add half the middle diameter, and the sum will be the semi-axis minor; and the major axis can then

be found by Art. 426, and the area of the generating segment by Art. 429.



Let l = AB, the length of the spindle; D = CD, the middle diameter; d = EF, the quarter diameter; c = IH, the central distance; b = IC, the semi-axis minor.

Then, if a = semi-axis major, $h = \frac{1}{2}D$, s = area of segment ACB, s' that of the corresponding circular segment, also h' and A′ the height and area of the tabular segment,

$$c = \tfrac{1}{4} \cdot \frac{3D^2 - 4d^2}{4d - 3D},\ b = c + \tfrac{1}{2}D,\ a = \frac{bl}{\sqrt{(4b - D)D}},$$

$$h' = \frac{D}{4b},\ s' = 4b^2A',\ s = \frac{as'}{b},\ \text{or } s = 4abA',$$

and
$$V = \cdot 5236\left\{D^2 - 4\left(\frac{3s}{l} - D\right)c\right\}l.$$

EXAMPLE.—Find the solidity of an elliptic spindle whose length is = 20, its middle diameter = 6, and its quarter diameter = 4·7477.

$$c = \tfrac{1}{4} \cdot \frac{3D^2 - 4d^2}{4d - 3D} = \tfrac{1}{4} \cdot \frac{108 - 90.1624}{18.9908 - 18} = 4\cdot 5,$$

and
$$b = c + \tfrac{1}{2}D = 7\cdot 5,$$

$$a = \frac{bl}{\sqrt{(4b - D)D}} = \frac{7\cdot 5 \times 20}{\sqrt{(30 - 6)6}} = \frac{150}{12} = 12\cdot 5,$$

$$h' = \frac{D}{4b} = \frac{6}{30} = \cdot 2, \text{ and } A' = \cdot 111824,$$

$$s = 4abA' = 4 \times 12\cdot 5 \times 7\cdot 5 \times \cdot 111824 = 41\cdot 934.$$

Then
$$V = \cdot 5236\left\{D^2 - 4\left(\frac{3s}{l} - D\right)c\right\}l$$

$$= \cdot 5236\left\{36 - 4\left(\frac{3 \times 41\cdot 934}{20} - 6\right)4\cdot 5\right\}20$$

$$= \cdot 5236 \times (36 - 18 \times \cdot 2901)20 = \cdot 5236 \times 30\cdot 7782 \times 20 = 322\cdot 3093.$$

Let h = CH = $\frac{1}{2}D$, h' = CK = $\frac{1}{2}(D - d)$; then EK = $\frac{1}{4}l$, and by Art. 423, $b^2 : a^2 = (2b - h)h : (\frac{1}{2}l)^2$, $b^2 : a^2 = (2b - h')h' : (\frac{1}{4}l)^2$. Hence $(2b - h)h = 4(2b - h')h'$, from which $2b = \dfrac{4h'^2 - h^2}{4h' - h}$. Substituting the above values of h' and h, it appears that $b - \frac{1}{2}D = c = \frac{1}{4}\dfrac{3D^2 - 4d^2}{4d - 3D}$; and hence $b = c + \frac{1}{2}D$ is known. Then (Art. 426) $a = \dfrac{2bl}{\sqrt{(2b - h)h}} = \dfrac{bl}{\sqrt{(4b - D)D}}$. The value of s is found by Art. 429 to be $= 4abA'$. The aid of the calculus is required to prove the expression for the solidity.

Exercise

Find the solidity of an elliptic spindle whose length is $=40$, the middle diameter $=12$, and the quarter diameter $=9{\cdot}49547$.

$=2578{\cdot}4748$.

477. Problem VI.—To find the solidity of the middle frustum of an elliptic spindle.

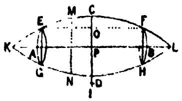
Rule.—Find the area of the elliptic segment whose chord is the length of the frustum; divide 3 times this area by the length of the frustum, from the quotient subtract the difference of the middle and an end diameter, and multiply this remainder by 8 times the central distance.

Find the sum of the square of an end diameter and twice the square of the middle diameter; from this sum subtract the product last found; multiply this difference by the length, and this product by $\cdot 2618$ for the solidity.

The central distance is found thus:—From the sum of 3 times the square of the middle diameter and the square of an end diameter take 4 times the square of the quarter diameter; and from 4 times the last diameter take the sum of an end diameter and three times the middle diameter; divide the former difference by the latter, and one-fourth of the quotient will be the central distance.

To the central distance add half the middle diameter, and the sum will be the semi-axis minor; the major axis can then be found by Art. 426, and the area of the above elliptic segment by Art. 429.

Let $l=AB$, the length of the frustum; $D=CD$, the middle diameter; $d = EG$, an end diameter; $q = MN$, the quarter diameter; $c = IP$, the central distance; $h=CO$, the height of segment ECF; and $s =$ the area of segment ECF.



Then
$$c=\tfrac{1}{4}\cdot\frac{3D^2+d^2-4q^2}{4q-(3D+d)},\ b=c+\tfrac{1}{2}D,\ h=\tfrac{1}{2}(D-d),$$
$$2a=\frac{bl}{\sqrt{(2b-h)h}},\ h'=\frac{h}{2b},\ s=4abA',$$
and
$$V=\cdot 2618\{2D^2+d^2-8\left(\frac{3s}{l}-D+d\right)c\}l.$$

Example.—Find the solidity of a middle frustum of an elliptic spindle whose length is $=14$, its middle diameter $=12$, an end diameter $=10{\cdot}8$, and a quarter diameter $=11{\cdot}7045$.

$$c = \tfrac{1}{4} \cdot \frac{3D^2 + d^2 - 4q^2}{4q - 3D - d} = \tfrac{1}{4} \cdot \frac{432 + 116{\cdot}64 - 547{\cdot}9813}{46{\cdot}818 - 36 - 10{\cdot}8} = \frac{{\cdot}6587}{{\cdot}072} = 9{\cdot}1,$$

and $$b = c + \tfrac{1}{2}D = 9{\cdot}1 + 6 = 15{\cdot}1, \text{ and } h = \tfrac{1}{2}(D - d) = {\cdot}6;$$

$$2a = \frac{bl}{\sqrt{(2b - h)h}} = \frac{15{\cdot}1 \times 14}{\sqrt{(30{\cdot}2 - {\cdot}6){\cdot}6}} = \frac{211{\cdot}4}{\sqrt{17{\cdot}76}} = \frac{211{\cdot}4}{4{\cdot}214} = 50{\cdot}16,$$

and $$a = 25{\cdot}08;$$

$$h' = \frac{h}{2b} = \frac{{\cdot}6}{30{\cdot}2} = {\cdot}01987, \ A' = {\cdot}003712,$$

$$s = 4abA' = 4 \times 25{\cdot}08 \times 15{\cdot}1 \times {\cdot}003712 = 5{\cdot}623,$$

and $$V = {\cdot}2618\left\{2D^2 + d^2 - 8\left(\frac{3s}{l} - D + d\right)c\right\}l$$

$$= {\cdot}2618\left\{288 + 116{\cdot}64 - 8\left(\frac{3 \times 5{\cdot}623}{14} - 1{\cdot}2\right)9{\cdot}1\right\}14$$

$$= {\cdot}2618(404{\cdot}64 - {\cdot}357)14 = {\cdot}2618 \times 404{\cdot}283 \times 14 = 1481{\cdot}778.$$

Exercise

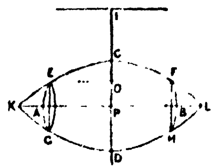
The length of a middle frustum of an elliptic spindle is 20, its middle and an end diameter = 16 and 12, and a quarter diameter = 15·07878; what is its solidity? . . . . . 3427·4856.

IV. THE HYPERBOLIC SPINDLE

The formulæ for the solidities of an hyperbolic spindle, and for the middle frustum of this spindle, are the same as for the elliptic spindle and its middle frustum, with a slight change in the signs.

1. For the hyperbolic spindle.

The notation remaining as in Prob. V.,



$$IP = c = \tfrac{1}{4} \cdot \frac{4d^2 - 3D^2}{4d - 3D},$$

$$a = c - \tfrac{1}{2}D = IC,$$

$$b = \frac{al}{\sqrt{(4a + D)D}}.$$

Then $s = KCL$ is found by Art. 436,

and then $$V = {\cdot}5236\left\{D^2 + 4\left(\frac{3s}{l} - D\right)c\right\}l.$$

2. For the middle frustum of an hyperbolic spindle.

The notation remaining as in Prob. VI.,

$$c = \tfrac{1}{4}\frac{4q^2 - 3D^2 - d^2}{4q - 3D - d}, \ a = c - \tfrac{1}{2}D, \ h = \tfrac{1}{2}(D - d),$$

$$2b = \frac{al}{\sqrt{(2a + h)h}}; \text{ then } s = AECFB = ECF + EB,$$

and then $$V = {\cdot}2618\left\{2D^2 + d^2 + 8\left(\frac{3s}{l} - D + d\right)c\right\}l.$$

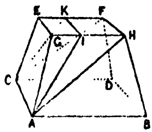
UNGULAS

Ungulas are portions cut off from pyramids, prismoids, cylinders, and cones, by plane sections not parallel to the base.

I. PYRAMIDAL AND PRISMOIDAL UNGULAS

478. **Problem I.—To find the solidity of the two ungulas into which a frustum of a rectangular pyramid or a prismoid is cut by a plane inclined to its base.**

CASE 1.—When the section passes through two opposite edges of its ends.



Let ABDGF be the frustum or prismoid, and ACFH the section. The solid is thus divided into two wedges BCH, EHC; hence,

Find the solidities of the two wedges into which the frustum is divided, by Art. 388.

For both wedges, $V=\frac{1}{6}(e+2l)bh$.

EXAMPLE.—Find the solidities of the two wedges, BCH, EHC, into which the frustum of a rectangular pyramid AF is divided, the length and breadth of its base $=30$ and 20 inches, and of its top $=24$ and 16 inches, and its height $=36$ inches.

For the wedge BCH,

$$V=\frac{1}{6}(e+2l)bh=\frac{1}{6}(16+2\times 20)30\times 36=10080.$$

For the wedge EHC,

$$V=\frac{1}{6}(e+2l)bh=\frac{1}{6}(20+2\times 16)24\times 36=7488.$$

EXERCISE

Find the solidities of the two wedges into which a frustum of a rectangular pyramid is divided by a plane passing through two of the shorter opposite edges of its ends, the length and breadth of its base being $=45$ and 30, those of its top $=36$ and 24, and its height $=40$. . . . . $=25200$ and 18720.

CASE 2.—When the section passes through an edge of one end and cuts off a part of the other end.

Let the section be ACKI (last fig.). The frustum is thus divided into a wedge EIC, and a rectangular prismoid ADHK, the volumes of which can be found by Articles 388 and 389.

For the wedge $V = \frac{1}{6}(c + 2l)bh$,
the prismoid $V = \frac{1}{6}(BL + bl + 4Mm)h$,
where $M = \frac{1}{2}(L + l)$, and $m = \frac{1}{2}(B + b)$.

EXAMPLE.—Find the solidities of the wedge EIC and the prismoid ADI, the dimensions of the frustum being the same as in the former example, and the distance of I from G = 10 inches.

$$EIC = V = \tfrac{1}{6}(c + 2l)bh = \tfrac{1}{6}(20 + 32)10 \times 36 = 3120,$$
$$ADI = V = \tfrac{1}{6}(BL + bl + 4Mm)h$$
$$= \tfrac{1}{6}(30 \times 20 + 14 \times 16 + 4 \times 22 \times 18)36 = 2408 \times 6 = 14448.$$

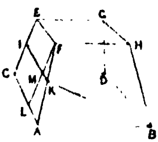
EXERCISE

Find the solidities of the wedge and prismoid into which a frustum of a rectangular pyramid is cut by a plane passing through one end of its base, and cutting off a portion of the top = 15 inches distant from its corresponding end; the dimensions of the frustum being the same as in the exercise of the last case.

= 7800 and 36120.

CASE 3.—When the section passes through an edge of one of the ends and cuts off a part from the opposite side.

Let BI be the section. The frustum is thus cut into a wedge ADK and an irregular polyhedron DKG, the volume of which is found by deducting that of the wedge from that of the frustum.



Let H = height of frustum, and
h = „ „ wedge; then
for the wedge $v = \frac{1}{6}(c + 2l)bh$, and when AG is a pyramidal frustum, its volume V' is found by Art. 384, or when it is a prismoid, V' is found by Art. 389, and then the polyhedron $= V' - v$.

Draw FL parallel to EC, then $AL = AC - FE$, and $IK = IM + MK = EF + MK$.

Or, if $AL = D$, $MK = d$, and $h' = H - h$, then $H : h' = D : d$, and $d = \dfrac{Dh'}{H}$; hence $c = IK = EF + d$.

EXAMPLE.—Let the section DK cut the side AE in a line IK at a perpendicular distance of 27 inches from the base, to find the volumes of the wedge and polyhedron, the dimensions of the frustum being the same as in the example of the first case.

Here $D = 20 - 16 = 4$, $d = \dfrac{Dh'}{H} = \dfrac{4 \times 9}{36} = 1$,

and $e = EF + d = 16 + 1 = 17$.

Then $v = \frac{1}{6}(e + 2l)bh = \frac{1}{6}(17 + 40)30 \times 27 = 7695$,

and $V' = \frac{1}{3}\left(\frac{BE - be}{E - e}\right)H = \frac{1}{3} \cdot \frac{600 \times 20 - 384 \times 16}{20 - 16} \times 36 = 17568$.

Hence $V = V' - v = 9873$.

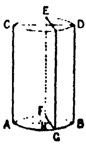
Exercise

Find the solidities of the wedge and polyhedron into which the frustum AG is divided, the height of the wedge being $= 30$, and the dimensions of the frustum the same as in the exercise of the first case. . . . . $= 19237{\cdot}5$ and $24682{\cdot}5$.

II. CYLINDRIC UNGULAS

470. **Problem II.—To find the solidity of an ungula of a cylinder cut off by a section perpendicular to the base.**

Multiply the area of the circular segment, which is the base of the ungula, by the height of the cylinder.



Let FGBDE be the ungula, and let

$d =$ AB, the diameter of the cylinder,
$h =$ BH, " height of the base,
$c =$ FG, " chord of the base,
$l =$ BD, " length of the ungula,
$\mathcal{A} =$ FBG, " area of the base.

Then $h' = \frac{h}{d} =$ height of tabular segment; let its area $= A'$;

then $\mathcal{A} = d^2A'$, and $V = l\mathcal{A}$.

Example.—The length of a cylindric ungula is $= 10$ feet, the diameter of the cylinder $= 18$ inches, and the section $= 6$ inches distant from the axis; what is the solidity of the two ungulas?

For the ungula EBFG, $h' = \frac{h}{d} = \frac{3}{18} = \frac{1}{6} = {\cdot}1\dot{6}$, and $A' = {\cdot}086042$.

$A = d^2A' = 1{\cdot}5^2 \times {\cdot}086042 = {\cdot}19359$,

and $V = lA = 10 \times {\cdot}1935945 = 1{\cdot}935945$.

For the ungula EAFG,

cylinder $AD = {\cdot}7854d^2l = {\cdot}7854 \times 1{\cdot}5^2 \times 10 = 17{\cdot}6715$,

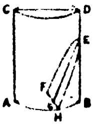
ungula $= 17{\cdot}6715 - 1{\cdot}9359 = 15{\cdot}7356 - 1{\cdot}935945 = 15{\cdot}735555$.

Exercise

What is the solidity of the two cylindric ungulas cut off by a plane parallel to the axis of the cylinder, at a distance of 2 feet

from it, the diameter and length of the cylinder being $= 6$ and 20 feet respectively? . . . . . . $= 61 \cdot 95$ and $503 \cdot 53776$.

480. **Problem III.—To find the solidity of a cylindric ungula cut off by a section inclined to the base.**



Let AD be the cylinder, and EFHB the ungula.

$l = \mathrm{BE}$, the length of the ungula,

$h = \mathrm{GB}$, " height of the base,

$c = \mathrm{FH}$, " chord of the base,

$2r$ or $d = \mathrm{AB}$, " diameter of the cylinder,

$\mathcal{A} =$ area of segment FBH.

Then $\qquad h' = \dfrac{h}{d}$, and $\mathcal{A} = d^2\mathrm{A}'$, where $\mathrm{A}' =$ tabular area;

and $\qquad \mathrm{V} = \frac{1}{2}\{\frac{1}{6}c^3 - \mathcal{A}(d - 2h)\}\dfrac{l}{h}$.

EXAMPLE.—Find the solidity of a cylindric ungula, the diameter of the cylinder being $= 25$, the length of the ungula $= 60$, and the height of its base $= 5$.

By Art. 256, $\frac{1}{4}c^2 = \mathrm{AG}\,.\,\mathrm{GB} = (d - h)h = 20 \times 5 = 100$; hence $c = 20$,

$$h' = \frac{h}{d} = \frac{5}{25} = \cdot 2\text{; hence } \mathrm{A}' = \cdot 111824,$$

and $\qquad \mathcal{A} = d^2\mathrm{A}' = 25^2 \times \cdot 111824 = 69 \cdot 89,$

and $\qquad \mathrm{V} = \frac{1}{2}\{\frac{1}{6}c^3 - \mathcal{A}(d - 2h)\}\dfrac{l}{h}$

$$= \tfrac{1}{2}\{\tfrac{1}{6} \times 20^3 - 69 \cdot 89(25 - 10)\}\tfrac{60}{5} = \tfrac{1}{2}(1333 \cdot \dot{3} - 1048 \cdot 35)12 = 1709 \cdot 9.$$

EXERCISES

1. A cylindric vessel ACDB, 10 inches diameter, containing some fluid, is inclined till the horizontal surface of the fluid EFH meets the bottom in FH, leaving AG 8 inches of the diameter dry, and meets the side at E 24 inches from the bottom; how many cubic inches of fluid is contained in it? . . $= 109 \cdot 4334$.

2. Suppose that the vessel stated in last example is inclined till the surface of the fluid bisects the base, and that the surface rises to the same height on the side as before; find the quantity of fluid. . . . . $= 400$ cubic inches.

In this example, $2h = d$, and $c = d$, and the above formula becomes $V = \frac{1}{2} \times \frac{1}{6}d^3 \times \dfrac{2l}{d} = \frac{1}{6}d^2l$.

3. Suppose that the fluid in the same vessel leaves only 2 inches of the bottom diameter dry, and that it rises to the same height as before; what is the quantity of fluid? $= 734 \cdot 218$ cubic inches.

III. CONIC UNGULAS

481. Conic ungulas are elliptic, parabolic, or hyperbolic.

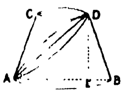
An **elliptic** conic ungula is a portion of a cone cut off by a plane which, produced if necessary, would cut the opposite slant sides of the cone, and would form an elliptic section.

A **parabolic** conic ungula is a portion of a cone cut off by a plane parallel to the slant side of the cone, and which forms a parabolic section.

A **hyperbolic** conic ungula is a portion of a cone cut off by a plane which neither cuts the opposite slant sides nor is parallel to the slant side, and which forms an hyperbolic section.

482. **Problem IV.—To find the solidity of the elliptic ungulas of a conic frustum made by a section passing diagonally through opposite edges of the ends.**

Let ACDB be a conic frustum, and ADB, ACD two ungulas into which it is divided by the section AD.



Let $D = AB$, the diameter of the greater end,
$d = CD$, the diameter of the smaller end,
$l = DE$, the perpendicular length,
$V =$ solidity of the greater ungula,
$v =$ solidity of the less ungula;

then $V = \cdot2618\left(\frac{D^2 - d\sqrt{Dd}}{D-d}\right)Dl$, and $v = \cdot2618\left(\frac{D\sqrt{Dd} - d^2}{D-d}\right)dl$.

EXAMPLE.—Find the solidity of the greater ungula ADB of a conic frustum ACB, the diameters of the ends being $= 15$ and $9\cdot6$ inches, and the height of the frustum being $= 20$ inches.

$$V = \cdot2618\left(\frac{D^2 - d\sqrt{Dd}}{D-d}\right)Dl = \cdot2618 \times \frac{225 - 9\cdot6\sqrt{15 \times 9\cdot6}}{15 - 9\cdot6} \times 15 \times 20$$

$$= \frac{\cdot2618 \times 109\cdot8 \times 300}{5\cdot4} = 1596\cdot98.$$

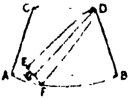
EXERCISES

1. A vessel of the form of a conic frustum is inclined till the surface of a quantity of fluid contained in it just covers the bottom and reaches the edge of its mouth; how many cubic inches of fluid does it contain, the diameters of the mouth and bottom being $= 38\cdot4$ and 60, and the depth of the vessel $= 40$ inches? . . $= 51103\cdot36$.

2. A vessel of the same dimensions as that of the preceding example, the bottom of which is the narrower end, contains a

quantity of fluid similarly disposed; how many cubic inches of fluid are there? . . . . $= 26164·92$.

483. Problem V.—To find the solidity of the elliptic ungulas of a conic frustum made by a section cutting off a part of the base.



Let $D = AB$, the diameter of the greater end of frustum,

$d = CD$, the diameter of the smaller end of frustum,

l = perpendicular height of frustum,

$h = BG$, height of base,

A' = tabular area of segment for which $h' = \dfrac{h}{D}$,

A'_1 = tabular area of segment for which the height is

$$h'' = \frac{h - D + d}{d},$$

V = volume of ungula DEFB,

v = " " complemental ungula EAFCD,

V' = " " the frustum AFBDC.

Then $$V = \tfrac{1}{3}\left(D^3A' - d^3A'_1\frac{h}{h - D + d}\sqrt{\frac{h}{h - D + d}}\right)\frac{l}{D - d}.$$

Or, if $\dfrac{h}{h - D + d} = q$, $V = \tfrac{1}{3}(D^3A' - d^3A'_1 q\sqrt{q})\dfrac{l}{D - d}$

and $v = V' - V$, where $V' = ·2618(D^2 + d^2 + Dd)l$.

EXAMPLE.—Find the volume of the ungula DEFB of a conic frustum ABCD, the diameters of its ends being 15 and 9·6, the perpendicular length = 20, and the height of the base of the ungula BG = 10 inches.

Here $$h' = \frac{h}{D} = \frac{10}{15} = ·\dot{6}, \text{ and } A' = ·556226.$$

Also, $$h'' = \frac{h - D + d}{d} = \frac{10 - 15 + 9·6}{9·6} = \frac{4·6}{9·6} = ·4791\dot{6},$$

and $$A'_1 = ·371872.$$

$$q = \frac{10}{10 - 15 + 9·6} = \frac{10}{4·6} = 2·1739,$$

and $$V = \tfrac{1}{3}(D^3A' - d^3A'_1 q\sqrt{q})\frac{l}{D - d}$$

$$= \tfrac{1}{3}(15^3 \times ·556226 - 9·6^3 \times ·371872 \times 2·1739\sqrt{2·7139}) \times \frac{20}{15 - 9·6}$$

$$= \tfrac{1}{3}(1877·2628 - 1054·537)\frac{20}{5·4} = 822·7258 \times \tfrac{100}{81} = 1015·701.$$

Exercises

1. A vessel in the form of a conic frustum, whose bottom and top diameters are $=30$ and $19\cdot2$ inches, contains a quantity of fluid, which, when the vessel is inclined, just reaches the lip, and leaves 10 inches of the bottom diameter dry; how many cubic inches of fluid are there, supposing the depth of the vessel to be $=20$ inches? $=4062\cdot787$.

2. If a vessel of the form of a conic frustum, equal in dimensions to that of the last example, but close at both ends, be so inclined that a quantity of fluid in it just covers the smaller end and 10 inches of the diameter of the greater, what is the quantity of fluid contained in it? . . . . . $=5595\cdot748$ cubic inches.

484. **Problem VI.—To find the solidity of the parabolic ungulas of a conic frustum.**

Let D, d, V, v, V′, l, and h have the same meaning as in the last problem, and A = the area of the base EBF of the ungula (last fig.); then $$h = D - d \text{ (Art. 481)},$$
and $$V = \tfrac{1}{3}\left\{\frac{AD}{D-d} - \tfrac{4}{3}d\sqrt{(D-d)d}\right\}l,$$
also $$v = V' - V, \text{ where } V' = \cdot2618(D^2 + d^2 + Dd)l.$$

Example.—Find the solidity of the parabolic ungula DEFB (last fig.) of a conic frustum, the diameters of the ends being $=15$ and $9\cdot6$ inches, and its height $=20$ inches, and the upper edge of the ungula terminating in the edge of the upper end of the frustum.

Here $$h = D - d = 15 - 9\cdot6 = 5\cdot4,$$
and $$h' = \frac{h}{D} = \frac{5\cdot4}{15} = \cdot36, \text{ and } A' = \cdot254551\,;$$
hence $$A = d^2A' = 15^2 \times \cdot254551 = 57\cdot273975,$$
and $$V = \tfrac{1}{3}\left\{\frac{A\,.\,D}{D-d} - \tfrac{4}{3}d\sqrt{(D-d)d}\right\}l$$
$$= \tfrac{1}{3}\left(\frac{57\cdot273975 \times 15}{5\cdot4} - 12\cdot8\sqrt{5\cdot4 \times 9\cdot6}\right)20$$
$$= \tfrac{1}{3}(159\cdot0943 - 92\cdot16)20 = 446\cdot229.$$

Exercises

1. Let a vessel in the form of a conic frustum, the diameters of its bottom and top being $=30$ and $19\cdot2$ inches, be inclined so that its upper slant side shall be parallel to the horizon; to find the quantity of fluid it is capable of containing in this position, the depth of the vessel being $=20$ inches. . $=1784\cdot9166$ cubic inches.

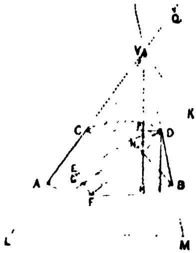
2. Find the ungula which is complementary to that in the preceding exercise. . . . . . . $=7873\cdot6184$ cubic inches.

The preceding rules for finding the volumes of conic ungulas may be proved thus :—

Let VAB be a cone, and EFBD an ungula of the conic frustum ABDC.

Produce GD to meet VA produced in L, draw LM parallel to AB, VK perpendicular to GD produced, and VH, DI perpendicular to AB, and BN to GD, and join VE, VF.

The ungula EFDB = conic solid EFBV − conic solid EFDV.



Let $L'=VH$, $D=AB$, $h=GB$;

then $D-h=AG$;

and let $l=DI$, $d=CD$, $A=$ segment EFB,

$l'=VP$, $d'=LM$, $A_1=$ segment EDF,

$l''=VK$, $h'=GD$, $a=LD$,

then $a-h'=GL$.

Also, let $b=$ minor axis of the ellipse of which EFD is a segment, and V, V', v', the volumes of the ungula EFBD, and the solids VEFB, VEFD.

By the similar triangles, ABV, CDV,

$$AB:CD=VH:VP, \text{ or } D:d=L':l' \quad \ldots \quad \ldots \quad [1].$$

Hence $$D-d:D=l:L'; \text{ therefore } L'=\frac{Dl}{D-d} \quad \ldots \quad \ldots \quad [2];$$

and by [1], $$l'=\frac{dL'}{D}=\frac{dl}{D-d} \quad \ldots \quad \ldots \quad \ldots \quad [3].$$

Also, from the similar triangles GBN, GDI, and BDN, VDK, $GB:BN=GD:DI$, and $BN:VK=BD:DV=DI:VP$.

Hence $$GB:GD=VK:VP, \text{ or } h:h'=l'':l'.$$

Therefore, $$l''=\frac{hl'}{h'}=\frac{hdl}{h'(D-d)} \quad \ldots \quad \ldots \quad \ldots \quad [4].$$

Now, $$V'=\tfrac{1}{3}AL',\ v'=\tfrac{1}{3}A_1l'', \text{ and } V=V'-v'.$$

Or, $$V=\tfrac{1}{3}(AL'-A_1l'')=\tfrac{1}{3}\cdot\frac{l}{D-d}\left(AD-A_1d\frac{h}{h'}\right) \quad \ldots \quad [5].$$

This is an expression for the volume of an ungula of any cone or pyramid.

1. When angle VAB exceeds DGB, the section EDF is a segment of an ellipse, of which DL is the major axis $=a$, and the minor axis $b=\sqrt{CD\,.\,LM}=\sqrt{dd'}$. For if G were the middle of LD, then EF would be the minor axis, and $CD=2AG$, $LM=2GB$, and $EF^2=4AG\,.\,GB=CD\,.\,LM$ or $b^2=dd'$.

From similar triangles LAG, LCD, $LD:LG=CD:AG$.

Or, $a : a - h' = d : D - h$; hence $a : h' = d : h - (D - d)$;

hence $$a = \frac{dh'}{h - (D - d)}, \text{ and } h' = \frac{a}{d}(h - D + d).$$

Also, $$h' : a = h : d';$$

hence $$d' = \frac{ah}{h'} = \frac{dh}{h - (D - d)}; \text{ hence } b = d\sqrt{\frac{h}{h - (D - d)}}.$$

Let A'_1 = a circular segment, height $= h'$, and diameter $= a$, then $a : b = A'_1 : A_1$; hence $A_1 = \frac{b}{a}A'_1 = \frac{\sqrt{h\{h - (D - d)\}}}{h'}A'_1$.

Let A''_1 = a segment similar to A'_1, but of a circle AEB, so that its height is $= \frac{Dh'}{a} = \frac{D}{d}(h - D + d)$;

then $$A''_1 : A'_1 = D^2 : a^2; \text{ hence, } A'_1 = \frac{a^2 A''_1}{D^2},$$

and $$A_1 d\frac{h}{h'} = \frac{d^3}{D^2}\left(\frac{h}{h - D + d}\right)^{\frac{3}{2}} A''_1.$$

Therefore, $$V = \frac{1}{3}\frac{l}{D - d}\left\{AD - \frac{d^3}{D^2}\left(\frac{h}{h - D + d}\right)^{\frac{3}{2}} A''_1\right\}.$$

And if A_2 and A_3 denote the areas of segments of a circle whose diameter $= 1$, similar to A and A''_1, then $A = A_2 D^2$, and $A''_1 = A_3 D^2$;

hence $$V = \frac{1}{3} \cdot \frac{l}{D - d}\left\{A_2 D^3 - A_3 d^3 \frac{h}{h - D + d}\sqrt{\frac{h}{h - D + d}}\right\}.$$

2. When the plane GD passes through A, then EFD or A becomes a whole ellipse, and EFB becomes the circle AFB. Also, $A = \cdot 7854 D^2$, $A_1 = \cdot 7854 ab$, $h = D$, $h' = a$, $b^2 = dd' = dD$.

Hence by [5], $$V = \cdot 2618 \frac{Dl}{D - d}(D^2 - d\sqrt{Dd});$$

and this being subtracted from the volume of the frustum $= \cdot 2618\left(\frac{D^3 - d^3}{D - d}\right)l$, gives for the complementary ungula

$$v = \cdot 2618 \frac{dl}{D - d}(D\sqrt{Dd} - d^2).$$

3. When DG is parallel to AC, the section EFD is a parabola; and hence its area $A_1 = \frac{2}{3}EF \cdot GD$. But in this case $AG = CD = d$, $h = D - d$, and $EF^2 = 4AG \cdot GB = 4dh = 4d(D - d)$; hence $A_1 = \frac{2}{3}h' \times 2\sqrt{(D - d)d}$; and substituting this value of A_1 in [5],

$$v = \frac{1}{3} \cdot \frac{l}{D - d}\{A \cdot D - \frac{4}{3}d(D - d)\sqrt{(D - d)d}\}$$

$$= \frac{1}{3}\left\{\frac{A \cdot D}{D - d} - \frac{4}{3}d\sqrt{(D - d)d}\right\}l.$$

4. When the angle DGB is greater than VAB, the section EDF becomes an hyperbola, and GD produced would then meet AV produced in some point, as Q above V. And its major axis would be $DQ = a = \frac{dh'}{D - d - h}$, and the minor axis would be $b = d\sqrt{\frac{h}{D - d - h}}$; for in this case $h - (D - d)$ becomes $(D - d) - h$. The expression for the area of the now hyperbolic segment EDF being found, and substituted in [5] for A_1, the resulting expression would be the volume of the hyperbolic ungula.

IRREGULAR SOLIDS

485. Problem.—To find the solidity of an irregular solid.

RULE I.—When the solid is of an oblong form, find the areas of several equidistant sections perpendicular to some line that measures the length of the solid, and proceed with these areas exactly as with equidistant ordinates (Art. 292), and the result will be the cubic contents.

Or, $$V = \tfrac{1}{3}(A + 4B + 2C)D.$$

RULE II.—Divide the solid, by parallel sections, into portions nearly equal to frustums of conic solids, find the area of a middle section of each portion, and multiply it by the length of that portion, and the product will be nearly its solidity, and the sum of the solidities of all the portions will be nearly the solidity of the whole.

RULE III.—When the solid is not great, and is very irregular and insoluble in water, immerse it in water in some vessel of a regular form containing a sufficient quantity of water to cover the solid, then take out the body, and measure the capacity of that portion of the vessel which is contained between the two positions of the surface of the water before and after the body was removed.

EXAMPLE.—Find the solidity of an oblong solid whose length is $= 100$ feet, and the areas of five equidistant sections $= 50$, 55, 70, 80, and 80 square feet.

Here $A = 50 + 80 = 130$, $4B = 4 \times 135 = 540$, and $2C = 140$;

hence $V = \frac{1}{3}(130 + 540 + 140) \times 25 = 6750$ cubic feet.

EXERCISES

1. Find the quantity of excavation of a portion of a canal, the areas of five equidistant vertical sections being $= 200$, 240, 360, 300.

and 280 square feet, and the common distance of the sections =25 feet. . . . . =28000 cubic feet.

2. What is the solidity of an oak-tree of irregular form, the lengths of four portions of it being respectively=8, 5, 6, and 7 feet, and the areas of their middle sections=10, 8, 7, and 5 square feet? =197 cubic feet.

3. The surface of a portion of excavated earth is nearly of a rectangular form, its length is=60 feet, its mean breadth=40 feet, and the mean depth of the excavation=8 feet; required the number of cubic yards of excavation. . . . . =711·$\dot{1}$.

ADDITIONAL EXERCISES IN MENSURATION

1. What is the difference between the superficial contents of a floor=28 feet long and 20 broad, and that of two others of only half its dimensions? . . . . =280 feet.

2. It is required to cut off a piece of a yard and a half from a plank=26 inches broad; what must be the length of the piece? =6·23 feet.

3. The area of an equilateral triangle is=720; required its side. =40·784.

4. What must be the length of the radius of a circle which contains an acre? . . . . =117·752 feet.

5. A circular fish-pond is to be dug in a garden; what must be the length of the cord with which its circumference is to be described, so that it shall just occupy half an acre? . . . . =83·263 feet.

6. What length of a plank=10 inches broad will make 4½ square feet? . . . . =5·4 feet.

7. A log of wood is=15 inches broad and 11 thick; what length of it will make 10 cubic feet? . . . . =8 feet $8\frac{8}{11}$ inches.

8. A round cistern is=26·3 inches in diameter; what must be the diameter of another to contain twice as much, the depth being the same? . . . . =37·19 inches.

9. What will be the expense of painting a conical church-spire, at 8d. per yard, the circumference of the base being=64 feet, and its slant height=118 feet? . . . . . . =£13, 19s. 8⅜d.

10. What would be the expense of gilding a spherical ball of 6 feet diameter, at 3½d. the square inch? . . . =£237, 10s. 1·19d.

11. How many 3-inch cubes can be cut out of a cubic foot? =64.

12. The numbers expressing the surface and solidity of a sphere are the same; what is its diameter? . . . . =6.

13. To what height above the earth's surface must a person ascend to see one-third of its surface? =A height equal to its diameter.

14. A cylindric vessel=3 feet deep is wanted that will contain twice as much as another=28 inches deep and=46 inches diameter; what must be the diameter of the former? . =57·372 inches.

15. A cubic foot of brass is to be drawn into a cylindric wire $=\frac{1}{40}$ of an inch in diameter; what will be the length of the wire? =97784·6 yards.

16. A rectangular bowling-green, 300 feet long and 200 broad, is to be raised one foot higher by means of earth dug from a ditch to be made around it; what must be the depth of the ditch, its breadth being=8 feet? . . . . $=7\frac{23}{86}$ feet.

17. A frustum of a square pyramid is=18 feet long, and the sides of its ends are=1 and 3 feet, and it is to be divided into three equal portions; what must be the length of each? =3·269, 4·559, and 10·172.

18. A cone=40 inches high is to be cut into three equal parts by planes parallel to its base; what must be their lengths? =5·057, 7·209, and 27·734 nearly.

19. The same number expresses the solidity and convex surface of a cylinder; what is its diameter? . . . . =4.

20. The base and head diameters of a tub are=20 and 10 inches respectively; what ought to be its depth in order that it may contain 9163 cubic inches? . . . . =50 inches.

21. A circle=60 inches in diameter is to be divided into three equal portions by means of two concentric circles; what must be their diameters? . . . . =34·641 and 48·9898.

22. A square inscribed within a circle contains 16 square yards; what is the area of the circumscribed square? =32 square yards.

23. The side of the cubic altar of Apollo at Delphi was=1 cubit; what must be the side of the new cubic altar, which was to be twice the size of the former? . . . . . =1·259921 cubits.

24. A pot of the form of a conic frustum is 5·7 inches deep, and its top and bottom diameters are=3·7 and 4·23 inches; supposing it at first to be filled with liquid, and that a quantity of it is poured out till the remaining liquid just covers the bottom, what is the excess of the remaining quantity above that poured out? =7·0534 inches.

25. A conical glass, whose depth is=6 inches, and the diameter of its mouth=5 inches, is filled with water, and a sphere 4 inches in diameter, of greater specific gravity than water, is put into it; how much water will run over? . . =26·2722 cubic inches.

26. If a sphere and cone are the same as in the last exercise, and

the cone only one-fifth full of water, what portion of the vertical diameter of the sphere is immersed? . =·546 inch very nearly.

27. A cone equal to that in the former exercise being one-fifth full of water, what is the diameter of a sphere which, when placed in it, would just be covered with the water? . =2·446 inches.

28. A coppersmith proposes to make a flat-bottomed kettle, of the form of a conic frustum, to contain 13·8827 gallons; the depth of the kettle to be=12 inches, and the diameters of the top and bottom to be in the ratio of 5 to 3; what are the diameters?

=25 and 15 inches.

29. A piece of marble, of the form of a frustum of a cone, has its end diameters=1½ and 4 feet, and its slant side is=8 feet; what will it cost at 12s. the cubic foot? . . . =£30, 1s. 11⅝d.

30. The price of a ball, at 1d. the cubic inch, is as great as the gilding of it at 3d. the square inch; what is its diameter? =18 inches.

31. A garden=500 feet long and 400 broad is surrounded by a terrace-walk, the surface of which is one-eighth of that of the garden; what is the breadth of the walk? . . =13·4848 feet.

32. The paving of a square court at 6d. a yard cost as much as the enclosing of it at 5s. a yard; what was its side? =40 yards.

33. A reservoir is supplied from a pipe 6 inches in diameter; how many pipes of 3 inches diameter would discharge the same quantity, supposing the velocity the same? . . . . . . =4 pipes.

34. A pipe of 4 inches diameter is sufficient to supply a town with water; what must be the diameter of a pipe which, with the same velocity, will supply it when its population is increased by a half? . . . . =4·899 inches.

35. The ditch of a fortification is=1000 feet long, 9 deep, 20 broad at bottom, and 22 at top; how many cubic yards of excavation are there in it? . . . . =7000.

36. When the pressure of the atmosphere is=15 lb. on the square inch, what would be the pressure on the surface of a man's body, supposing it to be=16 square feet? . . =34560 lb.

37. A silver cup, of the form of a conic frustum, whose top and bottom diameters are=3 and 4 inches, and depth=6 inches, being filled with liquor, a person drank out of it till he could just see the middle of the bottom; how much did he drink?

=42·8567 cubic inches.

38. The sanctuary of Butis, in Egypt, was formed of one stone, in the form of a cube of 60 feet, open at top, and hollowed so that it was everywhere=6 feet thick; required its weight, at the rate of 157½ lb. avoirdupois the cubic foot. . . . =6439½ tons.

THE COMMON SLIDING-RULE

486. The **common** or **carpenter's sliding-rule** consists of two pieces, each a foot long, connected by a folding-joint. It is used for computing the quantity of timber and the work of artificers.

When the rule is opened out, one side or face of it is divided into inches and eighths of an inch, with other scales of parts of an inch; and one-half of the other side contains several tables of practical use. But the part of it used for performing arithmetical operations is one face of one of the pieces, in the middle of which is a narrow slip of brass, which slides in a groove.

487. On each of the two parts into which the stock is divided by the slider is a scale, and there are also two scales on the slider. The scales on the stock are named A and D, and those on the slider B and C, the scales A and B being contiguous, as are also C and D. The scales A, B, C are exactly equal, and are just a scale of logarithmic numbers like that in Article 150. The numbers on the scale D are the square roots of those opposite to them on the scale C.

488. Suppose the slider in its place, with 1 on its extremity, coinciding with 1 on the contiguous scales, and let the number d on D be opposite to the number c on C, then $d^2 = c$; and were these numbers on scales of the same standard, then would $2\text{L}d = \text{L}c$; but $\text{L}d$ on the scale D is $= \text{L}c$ on the scale C; and hence—

489. The logarithms of the numbers on the scale D are double the logarithms of the same numbers on the scale C.

In finding any number on any of the scales which is the result of some operation, as of multiplication, division, &c., it is necessary to know previously how many places of figures the number will contain; but this is generally easily known.

490. Problem I.—To find the product of two numbers.

RULE.—Set 1 on B to one of the numbers on A, then opposite to the other number on B is the product on A.

EXAMPLE.—Multiply 24 by 25.

Set 1 on B opposite to 24 on A, then opposite to 25 on B is 600 on A.

For if a, b, and p are the two numbers and their product,

then $$p = ab, \text{ and } \frac{b}{1} = \frac{p}{a}; \therefore \text{L}b - \text{L}1 = \text{L}p - \text{L}a;$$

that is, the extent from 1 to b on the line B is = that from a to p on the line A; and, therefore, if 1 on b is opposite to a on A, then b on B will be opposite to p on A.

491. Problem II.—To divide one number by another.

RULE.—Set 1 on B opposite to the divisor on A, then opposite to the dividend on A is the quotient on B.

EXAMPLE.—Divide 800 by 32.

Set 1 on B opposite to 32 on A, then opposite to 800 on A is 25 on B, which is the quotient.

Let a, b, and q be the divisor, dividend, and quotient,

then $$q = \frac{b}{a}, \text{ or } \frac{b}{a} = \frac{q}{1}; \therefore \text{L}b - \text{L}a = \text{L}q - \text{L}1;$$

that is, the distance from a to b on A is = that from 1 to q on B.

492. Problem III.—To perform proportion.

RULE.—Set the first term on B to the second on A, then opposite to the third on B is the fourth term on A.

EXAMPLE.—Find a fourth proportional to 20, 28, and 25.

Set 20 on B opposite to 28 on A, then opposite to 25 on B is 35 on A, which is the fourth term required.

Let a, b, c, and d be four terms of a proportion,

then $a : b = c : d$, or $a : c = b : d$; $\therefore \text{L}a - \text{L}c = \text{L}b - \text{L}d$;

that is, the distance between the numbers a and c on one logarithmic line is = the distance between b and d on the same or on an equal line, as in Art. 152.

493. Problem IV.—To find the square of a number.

RULE.—Set 1 on D to 1 on C, then opposite to the given number on D is its square on C.

EXAMPLE.—Find the square of 15.

Set 1 on D to 1 on C, then opposite to 15 on D is 225 on C.

The reason of the rule is evident from Art. 487.

The square of a number can also be found by Prob. I. Art. 490. For it is just the product of the number by itself. Thus, $15^2 = 15 \times 15$; and, by rule in Art. 490, this product is 225.

494. **Problem V.—To find the square root of a given number.**

RULE.—Set 1 on C to 1 on D, then opposite to the given number on C is its square root on D.

EXAMPLE.—Find the square root of 256.

Set 1 on C to 1 on D, then opposite to 256 on C is 16 on D.

Since the numbers on C are the squares of those opposite to them on D, therefore, according as 1 on D is reckoned 1, 10, 100,... the 1 on C must be reckoned 1, 100, 10,000...

495. **Problem VI.—To find a mean proportional between two numbers.**

RULE.—Set one of the numbers on C to the same on D, then opposite to the other number on C is the mean proportional on D.

EXAMPLE.—Find a mean proportional between 9 and 16.

Set 16 on C to 16 on D, and opposite to 9 on C will be 12 on D.

The rule is proved thus:—Since $a : x = x : b$; therefore,

$$\frac{a}{b} = \frac{a}{x} \cdot \frac{x}{b} = \frac{a}{x} \cdot \frac{a}{x} = \frac{a^2}{x^2}; \quad \therefore \ \mathrm{L}a - \mathrm{L}b = 2\mathrm{L}a - 2\mathrm{L}x.$$

Therefore the distance between the numbers a and b on the line C will be equal to the distance between a and x on the line D (Art. 489).

MEASUREMENT OF TIMBER

496. The measurement of timber is merely a particular application of the principles of the mensuration of surfaces and solids; but as approximate rules are sometimes adopted on account of their practical utility in measuring timber, it is necessary to treat this subject separately.

497. **Problem I.—To find the superficial content of a board or plank.**

RULE.—Multiply the length by the breadth, and the product is the area.

When the board tapers gradually, take half the sum of the two extreme breadths, or the breadth at the middle, for the mean breadth, and multiply it by the length.

Let b = the breadth in inches,

l = " length in feet,

and $Æ$ = " superficial content in feet; then $Æ=\frac{1}{12}bl$.

By the Sliding-rule.—Set the breadth in inches on B to 12 on A, and opposite to the length in feet on A will be the content on B in feet and decimal parts of a foot.

EXAMPLE.—How many square feet are contained in the surface of a plank = 10 feet 6 inches long and 8 inches broad?

$$Æ=bl=\tfrac{8}{12}\times10\tfrac{1}{2}=\tfrac{2}{3}\times\tfrac{21}{2}=7 \text{ square feet.}$$

Or, set 8 on B to 12 on A, and opposite to 10·5 on A is 7 on B.

The first rule depends on Art. 247.

The reason of the method by the sliding-rule is this:—The area or surface $Æ=bl$, b and l being the breadth and length in feet. When b is given in inches, then $Æ=\frac{b}{12}\cdot l$, or $bl=12Æ$, which is convertible into the proportion $12 : b = l : Æ$, and 12, b and l being given, $Æ$ is found by Prob. III.

EXERCISES

1. Find the area of a board = 18 inches broad and 16 feet 3 inches long. . . . . . . = 24 square feet 54 square inches.
2. What is the price of a plank, the length of which is = 12 feet 6 inches, and breadth = 1 foot 10 inches, at $1\frac{1}{2}$d. per square foot? = 2s. $10\frac{3}{8}$d.
3. Find the price of a plank, the length of which is = 17 feet, and breadth = 1 foot 3 inches, at $2\frac{1}{2}$d. a square foot. . = 4s. $5\frac{1}{8}$d.
4. What is the superficial content of a board = 29 feet long and 22 inches broad? . . . . = $53\frac{1}{6}$ square feet.
5. The length of each of five oaken planks is = $17\frac{1}{2}$ feet, two of them have the mean breadth of $13\frac{1}{2}$ inches, one is = $14\frac{1}{2}$ inches at the middle, and the remaining two are each = 18 inches at the broader end, and = $11\frac{1}{4}$ inches at the other end; what is their price at 3d. per square foot? . . . . = £1, 5s. $9\frac{1}{4}$d.

498. Problem II.—To find the cubic content of squared timber of uniform breadth and thickness.

RULE.—Find the continued product of the length, breadth, and thickness, and the result is the content. (See Art. 374.)

Let b, t, l, and V be the breadth, thickness, length, and volume or solidity; then $V=btl$.

By the Sliding-rule.—Find the mean proportional between the breadth and thickness in inches (Art. 495), then set the length on C to 12 on D, and opposite to the mean proportional on D is the content on C in feet.

When the timber is square, the mean proportional is = the side of the square. When the mean proportional is in feet, 1 on D is to be used instead of 12.

EXAMPLE.—Find the solidity of a squared log of timber, of the invariable breadth and thickness of 32 and 20 inches, its length being = 40 feet 6 inches.

$$V=btl=\tfrac{32}{12}\times\tfrac{20}{12}\times 40\tfrac{1}{2}=180 \text{ cubic feet.}$$

Or, find (Art. 495) the mean proportional between 32 and 20, which is 25·3; then set 40·5 on C to 12 on D, and opposite to 25·3 on D is 180 on C, the content.

The method by the sliding-rule is derived thus:—Let m = the mean proportional between b and t, then $m^2=bt$, and as m is in inches,

$$V=b\,.\,\frac{m}{12}\,.\,\frac{m}{12}=b\left(\frac{m}{12}\right)^2;\ \text{hence}\ \frac{V}{b}=\left(\frac{m}{12}\right)^2,$$

and $$LV-Lb=2Lm-2L12;$$

that is, the distance between 12 and m on D is equal to that between b and V on C.

EXERCISES

1. Find the solidity of a log of wood = 30 inches broad, 18 thick, and 16 feet long. . . . . = 60 cubic feet.

2. What is the content of a log the end of which is = 30 inches by 20, and length = 20 feet? . . . . . . = $83\tfrac{1}{3}$ cubic feet.

3. What is the content of a square log of wood, the side being = 14 inches, and the length = 12 feet? . . . . = $16\tfrac{1}{3}$ cubic feet.

4. The side of a square block of sandstone is = 3 feet, and its length = 6 feet; what is its content? . . . . = 54 cubic feet.

5. Find the cubic content of a log of wood = 20 feet 3 inches long, its ends being = 32 by 20 inches. . . . . . = 90 cubic feet.

6. The side of a square log of wood is = 2 feet, and its length = 24 feet 1 inch; what is its content? . . . . = $96\tfrac{1}{3}$ cubic feet.

499. Problem III.—To find the content of squared tapering timber.

RULE.—Find the mean breadth and thickness, and multiply their product by the length.

As in last problem, $V = btl$.

By the Sliding-rule.—The method is the same as that of last problem, using the mean breadth and thickness.

Example.—The breadth of a tapering plank of wood at the two ends is $=18$ and 12 inches, and its thickness at the ends $=14$ and ·10 inches, and its length $=19$ feet 10 inches; what is its solidity?

Here $\qquad b=\frac{1}{2}(18+12)=15$, and $t=\frac{1}{2}(14+10)=12$,

and $\qquad V=btl=\frac{15}{12}\cdot\frac{12}{12}\times 19\frac{5}{6}=24\frac{19}{24}$ cubic feet.

The above rule, though generally used, is correct only in one case—namely, when two of the sides are parallel and the other two converge; for the solid is then a prism, having one of the parallel sides for its base. In other cases this rule gives the content a little less than the real solidity, and the error is greater the more the difference between the breadth and thickness. But the true solidity can always be found by considering the log a prismoid, and calculating its content by the rule in Art. 389.

The preceding example, calculated thus, gives for the content 25·067 cubic feet instead of $24\frac{19}{24}$.

500. When the breadth is irregular, it may be measured at several places, and the sum of these breadths, divided by their number, may be taken for the mean breadth. In the same way the mean thickness may be found.

Exercises

1. Find the content of a squared tapering log of wood, the breadth and thickness at one end being $=34$ and 20 inches, and those at the other end $=26$ and 16 inches, and the length $=32$ feet. $=120$ cubic feet.

2. Find the cubic content of a log, the breadth and thickness at one end being $=33$ and 22 inches, and those at the other end $=27$ and 18 inches, and the length $=40$ feet. . . $=166\frac{2}{3}$ cubic feet.

3. The breadth and thickness of one end of a piece of timber are $=21$ and 15 inches, and those at the other end are $=18$ and 12 inches, and the length is $=41$ feet; what is its solidity? $=74·95$ cubic feet.

4. The breadth and thickness at the greater end of a piece of timber are $=1·78$ and 1·23 feet, and at the smaller end $=1·04$ and 0·91 feet; what is its content, its length being $=27·36$ feet? $=41·278$ cubic feet.

501. **Problem IV.—To find the content of round or unsquared timber.**

RULE I.—Find the quarter girt—that is, one-fourth of the mean circumference—and multiply its square by the length.

By the Sliding-rule.—Set the length on C to 12 on D, and opposite to the quarter girt in inches on D is the content on C.

RULE II.—Find one-fifth of the girt, and multiply its square by twice the length.

By the Sliding-rule.—Set twice the length on C to 12 on D, and opposite to one-fifth of the girt on D is the content on C.

Let l and c denote the length and mean circumference of a piece of round timber, and V its volume; then,

by Rule I., $$V = \left(\frac{c}{4}\right)^2 l = \tfrac{1}{16}c^2 l = \cdot0625c^2 l;$$

by Rule II., $$V = 2\left(\frac{c}{5}\right)^2 l = \cdot08c^2 l.$$

EXAMPLE.—The mean circumference of a piece of unsquared timber is $=6$ feet 8 inches, and its length $=16$ feet 4 inches; what is its content?

By Rule I., $$V = \left(\frac{c}{4}\right)^2 l = (\tfrac{1}{4}\cdot\tfrac{20}{3})^2 \cdot 16\tfrac{1}{3} = (\tfrac{5}{3})^2 \cdot \tfrac{49}{3} = 45\cdot37 \text{ cubic feet.}$$

By Rule II., $$V = 2\left(\frac{c}{5}\right)^2 l = (\tfrac{1}{5}\cdot\tfrac{20}{3})^2 \cdot 32\tfrac{2}{3} = (\tfrac{4}{3})^2 \cdot \tfrac{98}{3} = 58\cdot074 \text{ cubic feet.}$$

Note.—When the piece of timber is of a cylindric form, its volume, by Art. 378, is $V = bh = \cdot07958c^2 l$, if $l = h$. Therefore the first rule in this case gives the content too small by more than one-fifth part of the true solidity; and the second gives it too much by about the 191st part.

When the tree tapers uniformly, it is then a frustum of a cone, and its true volume can be found by the rule in Art. 386. In this case the first rule gives a result still further from the truth, for a conic frustum exceeds a cylinder of the same length, whose circumference is the mean girt of the frustum.

The first rule is generally followed in practice, and the deficiency in the content given by it is intended to be a compensation to the purchaser for the loss of timber caused by squaring it.

The following formula commends itself :—

$G = \frac{1}{4}$ girt of tree at middle in feet,
$g = \frac{1}{4}$ " " one end "
$h = \frac{1}{4}$ " " other end "
L = length of log in feet,
c = cubic contents of log in feet,

$$c = L\left(\frac{G+g+h}{3}\right)^2.$$

Allowance is to be made for bark by deducting from each $\frac{1}{4}$ girt. The allowance varies from half an inch in trees with thin bark to 2 inches for trees with thick bark.

MEASURES OF TIMBER

100 superficial feet of planking	= 1 square.
120 deals	= 1 hundred.
50 cubic feet of squared timber	= 1 load.
40 feet of unhewn timber .	= 1 load.
600 superficial feet of inch planking	= 1 load.
Boards 7 inches wide . .	= battens.
" 9 " . .	= deals.
" 12 " . .	= planks.

To cut the best beam from a log.

Divide the diameter, *ab*, into 3 equal parts, *af*, *fe*, and *eb*, and

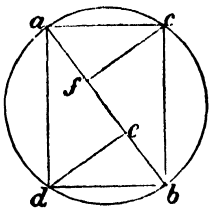


Fig. 1.

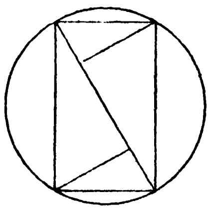


Fig. 2.

from *e* and *f* draw the lines *ed*, *fc* at right angles to *ab* ; join *ac*, *ad*, *bc*, and *bd*, then *acbd* is the cross section of the strongest beam (fig. 1).

To cut the stiffest beam, divide the diameter into 4 instead of 3 parts (fig. 2).

In the following exercises the first answer is the result by the first rule, and the other is that by the second rule:—

EXERCISES

1. The mean girt of a tree is = 8 feet, and its length = 24 feet; required its content. . = 96 cubic feet, or 122·88 cubic feet.

2. What is the content of a piece of round timber, the girt at the thicker end being = 16 feet, and at the smaller = 12 feet, and its length = 19 feet? . . = $232\frac{7}{9}$ cubic feet, or 297·92 cubic feet.

3. Find the content of a tree whose mean girt is = 3·15 feet, and length = 14 feet 6 inches. = 8·992 cubic feet, or 11·51 cubic feet.

4. The girts of a piece of round timber at five different places are = 9·43, 7·92, 6·15, 4·74, and 3·16 feet, and its length is = 17 feet 3 inches; what is its content?

= 42·5195 cubic feet, or 54·425 cubic feet.

RELATIONS OF WEIGHT AND VOLUME OF BODIES

502. The **relations of the weights and volumes of bodies** are determined by means of their specific gravities.

503. The **specific gravity,** or specific density, of any solid or liquid is the ratio which its density bears to that of distilled water at its maximum density point (4° C., or 39° F.).

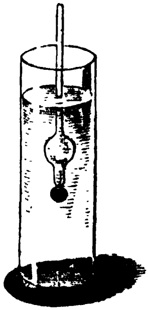
Tables of specific gravities are formed for reference, of which a specimen is appended. Thus, the specific gravity of mercury is 13·6, by which we mean that any given bulk of mercury will weigh 13·6 times as much as an equal bulk of water at the same temperature.

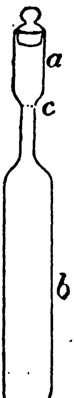
When a body is immersed in a liquid, it loses as much weight as the weight of the liquid displaced. This is the Principle of Archimedes, and is the foundation of the method adopted for finding the specific gravity of solids. See Prob. I., below.

The following instruments are commonly used for finding specific gravities, namely:—

(1) Nicholson's hydrometer; (2) Tweddel's hydrometer; and (3) the specific gravity bottle, or Pyknometer.

The second (Tweddel's) hydrometer is used for finding the specific gravity of liquids heavier than water, such as sulphuric acid. It acts by 'variable immersion'—that is, measures specific gravities by the depth to which it sinks. It is made of glass, with two globes, one for flotation, the other for balancing it in an upright position; and the stem is so graduated that the reading of the number of degrees multiplied by 5 and added to 1000 gives the specific gravity of the liquid as compared with water, whose specific gravity is for convenience taken to be 1000. Thus, 15 Tweddel represents the specific gravity of 1075; or, calling the specific gravity of water 1, it represents a specific gravity of 1·075.





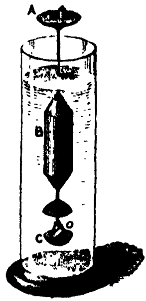
There are other hydrometers constructed on the principle of variable immersion, such as those for determining the density of alcohol, which is lighter than water, and those employed for determining the density of salt water in a boiler, where the graduations are so arranged as to indicate in a ready manner either the strength of the alcohol or the quantity of salt held in solution in the water.

The specific gravity bottle, in one form, is a bottle (*b*) with a fine stem (*c*), ending in a wide tube (*a*) having a glass stopper. The bottle is first weighed when empty, then when filled with water up to the mark (*c*), and finally, when filled with a given liquid up to the same mark.

We thus ascertain the weight (1) of a given volume of water, (2) of the same volume of a given liquid, and

ratio of the second to the first gives the specific gravity of the liquid.

Nicholson's Hydrometer and its Manipulation.—Nicholson's hydrometer consists of a hollow cylinder (B) which ensures flotation, having at its base a loaded pan (C) to keep it upright, and at the top a stem supporting a dish (A); upon the stem a standard point (m) is marked.



This instrument may be used for finding the specific gravity of a solid or a liquid. For example, let the solid be a piece of sulphur. Put the hydrometer in water, when it will require a given weight placed in (A) in order to sink the hydrometer to (m). Let this weight be 125 grs. Now place the sulphur in (A), and add, say, 55 grs. in order to sink the instrument again to (m). It follows that the weight of the sulphur is 70 grs.

Next place the sulphur in (C) as marked (o), and let 34·5 grs. be placed in the dish (A), in addition to the 55 grs., in order to sink the instrument to (o).

Then weight of sulphur = 70 grs.,
weight of water displaced by sulphur = 34·5 grs.

$$\therefore \text{ Specific gravity of sulphur} = \frac{70}{34 \cdot 5} = 2 \cdot 03.$$

In order to find the specific gravity of a liquid (B), let x be the weight which sinks the instrument to the point (o) in water, (y) the weight which sinks it to the same point in the liquid (B), and let W be the weight of the instrument.

Then weight of water displaced by instrument = $W + x$,
weight of liquid displaced by instrument = $W + y$.

$$\therefore \text{ Specific gravity of liquid (B)} = \frac{W + y}{W + x}.$$

EXAMPLE.—The standard weight of a Nicholson's hydrometer is 1250 grs.; a small substance is placed in the upper pan, and it is

found that 530 grs. are needed to sink the instrument to the standard point; but when the substance is put in the lower pan, 620 grs. are required. What is the specific gravity of the substance?

Standard weight	. . . . . .	= 1250 grs.
Weight required to sink the instrument to standard point	. . . . .	= 530 "
∴ Weight of body	. . . . . .	= 720 "

Extra weight required when body is placed in the lower pan $= 620 - 530 = 90$ grs.

∴ Specific gravity of the body $= \frac{720}{90} = 8$.

Definition of a Perfect Fluid.—A 'perfect' fluid is defined to be a substance which offers no resistance to a continuous change of shape. There are two kinds of fluids—those which are practically incompressible, termed liquids; and those which are easily compressed, called gases and vapours. We know of no substance which completely fulfils the above definition; but water, many other liquids, and all gases so nearly comply with it that for many purposes we may in practice consider them as perfect fluids.

Viscosity.—All known fluids, however, do offer some resistance to a change of shape, although they have no elasticity of form or power of recovery when the stress that has produced the change is removed; and the property in virtue of which they do so is called the viscosity of the fluid. The viscosity of a fluid is measured by the shearing stress required to deform it at the uniform rate of unit shear per unit time.

In many investigations it is necessary, for simplicity, to assume that we are dealing with a perfect fluid; it is therefore of the first importance that these definitions be clearly understood.

504. The following table contains the specific gravities of the most common solids and fluids, those for solids and liquids being referred to water as standard, those for gases to air as standard. Referred to water, the specific gravity of air at 0° C. and 30 inches barometric pressure is 0·001293.

TABLE OF SPECIFIC GRAVITIES

Metals	
Aluminium, sheet,	2·670
" cast,	2·560
Antimony, "	6·720
Bismuth, "	9·822
Copper bolts,	8·850
" wire,	8·900
Gold,	18·417
Iron, cast, average,	7·248
" wrought, average,	7·780
Lead, cast,	11·360
" sheet,	11·400
Mercury,	13·596
Platinum,	21·531
" sheet,	23·000
Silver,	10·474
Steel,	8·000
Tin, cast,	7·291
Zinc, "	7·000

Alloys	
Aluminium bronze, 90 to 95 per cent. copper,	7·680
Bell-metal (small bells),	8·050
Brass, cast,	8·400
" sheet,	8·440
" wire,	8·540
Gold (standard),	17·724
Gun-metal (10 copper, 1 tin),	8·464
Silver (standard),	10·312
Speculum (metal),	7·447
White-metal (Babbett),	7·310

Timber		
Acacia,	from	0·710
	to	0·790
Ash,	from	0·690
	to	0·760
Beech,	from	0·690
	to	0·696
Birch,	from	0·711
	to	0·730
Box,		1·280
Cedar, West Indian,		0·748
" American,		0·554
" Lebanon,		0·486
Chestnut,		0·606
Cork,		0·240
Deal, Christiania,		0·689
Ebony,		1·187
Elm, English,		0·553
" "		0·579
" Canadian,		0·725
Fir, spruce,		0·512
" male,		0·550
" female,		0·498
Hornbeam,		0·760
Ironwood,		1·150
Greenheart,		1·143
Larch,		0·543
"		0·556
Lignum-vitæ,		1·333
Lime,		0·564
Mahogany, Nassau,		0·668
" Honduras,		0·560
" Spanish,		0·852
Maple,		0·675
Oak, African,		0·988
" American, red,		0·850
" " white,		0·779
" English,		0·777
" "		0·934
Pine, red,	from	0·576
	to	0·657
" white,	from	0·432
	to	0·553
" yellow,		0·508

Pine, Dantzic, . . .		0·649
" Memel, .	from	0·550
	to	0·601
" Riga, .	from	0·466
	to	0·654
Satinwood, . . .		0·960
Teak, . .	from	0·740
	to	0·860

STONES, &c.

Agate, ...		2·590
Amethyst, common, .		2·750
Basalt, Scotch, . .		2·950
" Greenstone, .		2·900
" Welsh, . .		2·750
Chalk, . .	from	2·330
	to	2·620
Firestone, . . .		1·800
Granite, Aberdeen gray,		2·620
" " red,		2·620
" Cornish, . .		2·660
" Mount Sorrel, .		2·670
Limestone, compact, .		2·580
" Purbeck, .		2·600
" Blue Lias, .		2·467
" Lithographic,		2·600
Marble, Statuary, . .		2·718
" Italian, . .		2·726
" Brabant block, .		2·697
Oolite, Portland Stone, .		2·423
" Bath " .		1·978
Sandstone (Arbroath pavement),		2·477
" Bramley Fall,		2·500
" Caithness, .		2·638
" Craigleith, .		2·450
" Derby grit, .		2·150
" Red (Cheshire),		2·510
Slate, Anglesea, . .		2·870
" Cornwall, . .		2·510
" Welsh, . . .		2·880
Trap, ...		2·720

MISCELLANEOUS SUBSTANCES

Asphalt, ...		2·500
Brick, common,	from	1·600
	to	2·000
" London stock, .		1·840
" Red, . . .		2·160
" Welsh fire, . .		2·400
" Stourbridge fire, .		2·200
Cement, Portland, .	from	3·100
" " in powder,		3·155
" Roman, . .		1·600
Clay, ...		1·900
Coal, Anthracite, . .		1·530
" Cannel, . .		1·272
" Glasgow, . .		1·290
" Newcastle, . .		1·269
Coke, ...		0·744
Concrete, ordinary, .		1·900
" in cement, .		2·200
Earth, . .	from	1·520
	to	2·000
Glass, flint, . . .		3·078
" crown, . . .		2·520
" common green, .		2·528
" plate, . . .		2·760
Gutta-percha, . . .		0·966
Gypsum, ...		2·286
Ice, if from water purged from air, . . .		0·954
India-rubber, . . .		0·930
Ivory, ...		1·820
Lime, quick, . . .		0·843
Mortar, . .	from	1·380
	to	1·900
" average, . .		1·700
Pitch, ...		1·150
Plumbago, . . .		2·267
Snow, ...		0·083
Sand, quartz, . . .		2·750
" river, . . .		1·880
" pit (coarse), . .		1·610
" " (fine), . .		1·520

Sand, pit, Thames,	1·640
Shingle,	1·420
Tallow,	0·940
Tar,	1·016
Tile, common,	1·810
" "	1·850

Liquids, &c.

Water, distilled,	1·000
" sea,	1·027
Acetic acid,	1·060
Alcohol, absolute,	0·792
" of commerce,	0·800
" proof,	0·916
Chloroform,	1·490
Citric acid,	1·034
Ether,	0·716
Fluoric acid,	1·060
Hydrochloric acid,	1·200
Milk,	1·032
Nitric acid,	1·420
Oil, Linseed,	0·940
" Olive,	0·915
" Whale,	0·923
Petroleum, crude,	0·885
" refined,	0·910
Sulphuric acid,	1·843
Oil, Amber,	0·868
" Cinnamon,	1·043
Oil, Lavender,	0·894
" Turpentine,	0·864
" Sweet almonds,	0·932
" Codfish,	0·923
" Hempseed,	0·926
Porter, brown stout,	1·011
Proof spirit,	0·922
Strong ale,	1·050
Wine, Port,	0·997
" Champagne,	0·997
Brandy, French,	0·941
Rectified spirit,	0·838

Gases

Atmospheric air,	1·000
Ammoniacal gas,	0·590
Carbonic acid gas,	1·527
" oxide gas,	0·972
Carburetted hydrogen gas,	0·972
Chlorine gas,	2·500
Cyanogen gas,	1·805
Hydriodic Acid gas,	4·340
Hydrogen gas,	0·069
Iodine, vapour of,	8·716
Nitrous oxide gas,	1·527
Oxygen gas,	1·111
Prussic acid gas,	0·937
Steam of water at 212°,	0·623

505. The weight of a cubic foot of water is very nearly 1000 ounces, or 62½ lb. avoirdupois, and therefore, if the decimal point in the numbers in the preceding table for solids and fluids is removed three figures to the right, the numbers will denote very nearly the weight in ounces of a cubic foot of the different substances.

The weight of a cubic foot of water at the maximum density, and in a vacuum, is 999·278 ounces avoirdupois, and that of a cubic inch is 253 grains, or ·527 ounce troy, or ·5783 ounce avoirdupois.

The weight of a cubic foot of air at 0° C. and 30 inches barometric pressure is 0·08071 lb.

USEFUL MEMORANDA IN CONNECTION WITH WATER *

1 cubic foot of water = 62·425 lb. = ·557 cwt. = ·028 ton.
1 gallon " = 10 lb. = ·16 cubic foot.
1 cubic inch " = ·03612 lb.
1 " foot " = 6·24 gallons = say 6¼ gallons.
1 cwt. " = 1·8 cubic feet = 11·2 gallons.
1 ton " = 35·9 " = 224 gallons.
1 cubic foot of sea water = 64·11 lb.
Weight of sea water = 1·027 weight of fresh water.
1 cubic inch of ice at 32° F. = ·0334 lb.
1 " foot " " = 57·8 lb.
1 lb. " " = 29·94 cubic inches.
Snow, 1 cubic inch = ·003 lb.
" 1 " foot = 5·2 lb.
" 1 lb. = 332·3 cubic inches = ·1923 cubic foot.
Snowfall = ·433 lb. per inch depth per superficial foot.
Inches of rainfall × 2323200 = cubic feet per square mile.
" " × 14½ = millions of gallons " "

506. **Problem I.—To find the specific gravity of a body.**

CASE 1.—When the body is heavier than water.

RULE.—Find the weight of the body in air and also in water; then the difference of these weights is to the former weight as the specific gravity of water to that of the body.

When water is the standard, its specific gravity is 1, and the specific gravity of the body is the quotient, obtained by dividing the whole weight by the difference of the weights.

Let w, w' be the weights of the body in air and water, and s, s' the specific gravities of the body and of water;

then $w - w' : w = s' : s$, or $w - w' : w = 1 : s$, when $s' = 1$,

or $$s = \frac{ws'}{w - w'}, \text{ or when } s' = 1,\ s = \frac{w}{w - w'}.$$

EXAMPLE.—A piece of silver weighs 33·6 ounces in air, and 29·56 ounces in water; what is its specific gravity?

* For ordinary calculations the weight of a cubic foot of fresh water is assumed = to 62·5 lb. or 1000 ounces.

$$s=\frac{ws'}{w-w'}=\frac{33\cdot6s'}{33\cdot6-29\cdot56}=\frac{33\cdot6s'}{3\cdot04}=11\cdot05s'.$$

Or, when $s'=1,\ s=11\cdot05.$

507. The rule is founded on the hydrostatic principle that a body immersed in a fluid lighter than itself loses as much of its weight as that of an equal volume of the fluid. Hence, if w'' = the weight of a portion of water equal in volume to that of the immersed body, then $w''=w-w'$; and hence $w'':w=s':s$, or $w-w':w=s':s$.

The weight of a portion of air, equal in volume to that of the body, is here disregarded.

EXERCISES

1. A piece of limestone weighs in air 20 lb., and in water 13½ lb.; what is its specific gravity? . . . . . $=3\cdot077$.

2. A piece of steel was found to weigh 78·5 lb. in air, and 68·5 lb. in water; what was its specific gravity? . . $=7\cdot85$.

3. A bar of lead weighed 30 cwt. in air, and only 27 cwt. 1 quarter 11 lb. 5 ounces in water; required its specific gravity. . $=11\cdot325$.

CASE 2.—When the body is lighter than water.

RULE.—Find the weight of the body in air, and the weight in water of another body, which, when attached to the former, will make it sink; find also the weight in water of the compound mass; from the sum of the two former weights subtract the latter; then

The remainder is to the weight of the given body as the specific gravity of water to that of the given body.

Let w denote the weight of the given body,

w' the weight in water of the attached body,

W' " " " compound mass;

then, s and s' remaining as formerly,

$$w+\mathrm{w}'-\mathrm{W}':w=s':s,\ \text{or}=1:s,\ \text{if } s'=1,$$

and

$$s=\frac{ws'}{w+\mathrm{w}'-\mathrm{W}'},\ \text{or } s=\frac{w}{w+\mathrm{w}'-\mathrm{W}'}.$$

EXAMPLE.—A piece of ash weighs 60 lb. in air, and to it is affixed a piece of copper which weighs in water 40 lb., and the compound weighs also in water 25 lb.; what is the specific gravity of the ash?

$$s=\frac{ws'}{w+\mathrm{w}'-\mathrm{W}'}=\frac{60s'}{60+40-25}=\frac{60s'}{75}=\cdot8,\ \text{if } s'=1.$$

Since the attached body is weighed in water both times, its weight remains the same; hence the weight lost in water by the lighter body $=(w+\mathrm{w}'-\mathrm{W}')$; therefore, by the last case,

$$w+\mathrm{w}'-\mathrm{W}':w=s':s=1:s,\ \text{if } s'=1.$$

EXERCISES

1. If a piece of elm weighs 30 lb. in air, and a piece of copper, which weighs 32 lb. in water, be affixed to it, and the compound weigh 12 lb. in water, what is the specific gravity of the elm? . . . . $=\cdot 6$.

2. If a piece of cork weighs 25 lb. in air, and a piece of lead, weighing 91·17 lb. in water, be attached to it, and the compound mass weigh 12 lb. in water, what is the specific gravity of the cork? . . . . $=\cdot 24$.

3. If a piece of beech weigh 42·6 lb. in air, and a piece of iron, weighing 40·7 lb. in water, be attached to it, and the compound mass weigh 33·3 lb. in water, what is the specific gravity of the beech? . . . . $=\cdot 852$.

508. **Problem II.—To find the weight of a body when its cubic content and specific gravity are given.**

RULE.—Multiply the number of cubic feet in the volume by the specific gravity of the body, and this product by 1000, and the result is the weight in ounces; or,

$$w = 1000sv \text{ ounces} = 62\tfrac{1}{2}sv \text{ lb.}$$

For, by Art. 505, 1000 times the specific gravity is the weight of a cubic foot of the body in avoirdupois ounces; hence the rule is obvious.

EXAMPLE.—Find the weight of a bar of cast iron, its breadth and thickness being $=4$ and $2\frac{1}{2}$ inches, and length $=8$ feet.

$$V = btl = \tfrac{1}{3} \times \tfrac{5}{24} \times 8 = \tfrac{5}{9} \text{ cubic foot;}$$

hence $\quad w = 62\cdot 5 \times 7\cdot 248 \times \tfrac{5}{9} = 453 \times \tfrac{5}{9} = 251\tfrac{2}{3}$ lb.

EXERCISES

1. Find the weight of a block of marble of the specific gravity of 2·7, its length, breadth, and thickness being respectively $=6$ feet, 5 feet, and 18 inches. . . $=3$ tons 7 cwt. 3 qr. 5 lb. 12 ounces.

2. One of the stones in the walls of Baalbec was a block of marble 63 feet long, its breadth and thickness being each 12 feet; what is its weight, the specific gravity being 2·7?
$=683$ tons 8 cwt. 3 qr.

3. Find the weight of a log of oak 24 feet long, 3 broad, and 1 foot thick, its specific gravity being ·925. $=37$ cwt. 18 lb. 8 ounces.

4. How many male fir-planks 16 feet long, 9 inches broad, and 6 inches thick will a ship 400 tons burden carry? . . $=4344\frac{8}{33}$.

509. Problem III.—To find the cubic content of a body when its weight is given.

RULE.—Divide the weight of the body in ounces by 1000 times its specific gravity, and the quotient is the content in feet; or,

Divide twice the weight in pounds by $125s$.

By last problem, $w=1000sv$ ounces $=62\frac{1}{2}sv$ lb.;

hence $$v=\frac{w}{1000s}=\frac{w}{62\frac{1}{2}s}=\frac{2w}{125s}.$$

The first value must be used when w is given in ounces, and the last when w is given in pounds.

EXAMPLE.—Find the content of an irregular block of sandstone weighing 1 cwt., its specific gravity being 2·52.

$$v=\frac{w}{1000s}=\frac{112\times 16}{2520}=\cdot 7\tfrac{1}{9}\text{ cubic feet}=1228\cdot 8\text{ cubic inches.}$$

EXERCISES

1. How many cubic feet are in a ton-weight of male fir? $=65\cdot 1636$.

2. How many cubic feet are contained in a block of sandstone weighing 8 tons, its specific gravity being 2·52? . . $=113\frac{7}{9}$ feet.

3. Find the number of cubic feet contained in a ton of dry oak of the specific gravity ·925. . . . . $=38\cdot 746$.

510. Problem IV.—To find the quantity of either of the ingredients in a compound consisting of two, when the specific gravities of the compound and of the ingredients are given.

RULE.—Multiply the weight of the mass by the specific gravity of the body whose quantity is to be found, and by the difference between the specific gravity of the mass and the other body; divide this product by the difference of the specific gravities of the bodies, multiplied into the specific gravity of the compound mass; and the quotient will be the quantity of that body.

Let W, w, w' denote the weights of the compound and of the ingredients; and S, s, s', their specific gravities respectively, s being that of the denser ingredient;

then $$w=\frac{(S-s')s}{(s-s')S}W,\text{ and } w'=\frac{(s-S)s'}{(s-s')S}W.$$

EXAMPLE.—A composition weighing 56 lb., having a specific gravity 8·784, consists of tin and copper of the specific gravities

7·32 and 9 respectively; what are the quantities of the ingredients?

$$w=\frac{(S-s')s}{(s-s')S}W=\frac{1\cdot464\times9\times56}{1\cdot68\times8\cdot784}=\frac{13\cdot176}{14\cdot75712}\times56=50;$$

and hence $\qquad w'=W-w=56-50=6.$

Or there are 56 lb. of copper and 6 of tin.

By Art. 505, the volume of the body whose weight is w in ounces is $V=\frac{w}{1000s}$; and the same applies to the bodies whose weights are w' and W, and hence multiplying by 1000,

$$\frac{w}{s}+\frac{w'}{s'}=\frac{W}{S},\ \text{also}\ w+w'=W.$$

From these two equations are easily found the two formulæ given above; and hence the origin of the rule.

Note.—This rule in many cases of alloys gives only approximate results; for experiment shows that in these cases the density is in some instances greater, and in other instances less, than what would result from a simple mixture of the ingredients. This indicates something of the nature of chemical action.

EXERCISES

1. An alloy of the specific gravity 7·8 weighs 10 lb., and is composed of copper and zinc of the specific gravities 9 and 7·2; what is the weight of the ingredients?

=3·846 lb. of copper, and 6·154 lb. of zinc.

2. An alloy of the specific gravity 7·7, consisting of copper and tin of the specific gravities 9 and 7·3, weighs 25 ounces; what is the weight of each of the ingredients?

=6·875 ounces of copper, and 18·125 of tin.

3. A circular piece of gold and a common cork have equal weights and diameters, and the cork is $1\frac{3}{4}$ inches long. How thick is the piece of gold, the specific gravity of the gold being 19·25, and that of the cork ·25? . . . . $=\frac{1}{44}$ inch.

4. Given that the specific gravity of petroleum is 0·88, and that a quart of water weighs 40 ounces; find how many gallons of petroleum will weigh $38\frac{1}{2}$ lb. . . . . . . $=4\frac{3}{8}$ gallons.

5. Find the weight of a piece of oak 7 feet high, 3 feet wide, and $1\frac{1}{2}$ inches thick, taking the specific gravity of oak as ·93.

=152·578 lb.

6. If the specific gravity of brass be taken as 8·4, find the weight of a bar of the same material 10 inches long and 4 square inches in section. . . . . $=12\frac{11}{72}$ lb.

7. The specific gravity of mercury is 13·6; find the length of a column of water 1 inch in diameter which shall be equal to a column of mercury of the same diameter which is 30 inches in length. . . . . =34 feet.

ARCHED ROOFS

511. **Arched roofs** are either vaults, domes, saloons, or groins.

Vaulted roofs consist of two similar arches springing from two opposite walls, and meeting in a line at the top, or else forming a continuous arch.

Domes are formed by arches springing from a circular or polygonal base, and meeting in a point above.

Saloons are formed by arches connecting the side-walls with a flat roof or ceiling in the middle.

Groins are formed by the intersection of vaults with each other.

512. Arched roofs are either **circular, elliptical,** or **Gothic.** In the first kind the arch is a portion of the circumference of a circle; in the second it is a portion of the circumference of an ellipse; and in the third kind there are two arches which are portions of circles having different centres, and which meet at an angle in a line directly over the middle of the breadth, or span, of the arch.

513. By the **cubic content of arched roofs** is to be understood the content of the vacant space contained by its arches, and a horizontal plane passing through the base of the arch.

VAULTS

514. **Problem I.—To find the cubic content of a vaulted roof.**

RULE.—Multiply the area of one end, or of a vertical section, by the length.

Let Æ = the area of the end, l = the length; then V = Æl.

The areas of the ends are to be found by means of the rules in the 'Mensuration of Surfaces.'

EXAMPLE.—Find the volume of a semicircular vault, the span of which is $=20$, and its length $=60$ feet.

$$A\!\!\!R = \cdot 7854 \times 20^2 \times \tfrac{1}{2} = 157 \cdot 08,$$

and $$V = A\!\!\!R l = 157 \cdot 08 \times 60 = 9424 \cdot 8 \text{ cubic feet.}$$

EXERCISES

1. Find the cubic content of an elliptic vault whose span is $=30$, height $=12$, and length $=60$ feet. . . . $=16964 \cdot 64$ cubic feet.

2. What is the cubic content of a Gothic vault, its span being $=24$, the chord of each arch $=24$, and the distance of each arch from the middle of its chord $=9$, and the length of the vault $=30$ feet? $=17028 \cdot 1218$ cubic feet.

515. Problem II.—To find the surface of a vaulted roof.

RULE.—Multiply the length of the arch by the length of the vault.

Let $a=$ the length of the arch, $l=$ that of the vault, and $s=$ the surface; then $s=al$.

EXAMPLE.—What is the surface of a semicircular vault, the span of which is $=20$, and length $=60$?

$$a = \pi r = 3 \cdot 1416 \times 10 = 31 \cdot 416,$$

and $$s = al = 31 \cdot 416 \times 60 = 1884 \cdot 96.$$

EXERCISES

1. What is the surface of a circular vaulted roof, the span of which is $=60$ feet, and its length $=120$ feet? $=11309 \cdot 76$ square feet.

2. Find the surface of a vaulted roof, its length and that of its arch being $=106$ and $42 \cdot 4$ feet. . . . $=499 \cdot 38$ square yards.

DOMES

516. A dome with a polygonal base and circular arches, whose radii are equal to the apothem of the base, is called a **polygonal spherical dome.**

517. Problem III.—To find the cubic contents of a dome.

RULE.—Multiply the area of the base by two-thirds of the height.

Let $b=$ the base, $h=$ the height; then $V=\frac{2}{3}bh$.

EXAMPLE.—What is the solidity of a hexagonal spherical dome, a side of its base being $=20$ feet?

Here $$b = \tfrac{1}{2} \times 6sh = 3 \times 20 \times h = 60h \text{ (Art. 267)},$$

and $$V = \tfrac{2}{3}bh = \tfrac{2}{3}60h^2 = 40h^2;$$

and $h^2 = \frac{3}{4}s^2$; for (fig. to Art. 265) ACB is in this case an equilateral triangle, and $AC = s$, $AF = \frac{1}{2}s$, and $CF = h$, also $CF^2 = AC^2 - AF^2 = \frac{3}{4}s^2$;

hence $V = 40^2h = 30s^2 = 30 \times 20^2 = 12000$ cubic feet.

Exercises

1. Find the content of a spherical dome whose circular base has a diameter = 30 feet. . . . . . . = 7068·6 cubic feet.

2. What is the content of an octagonal dome, each side of its base being = 40 feet, and its height = 42 feet? = 216313·53 cubic feet.

518. Problem IV.—To find the surface of a dome.

Rule.—When the dome is hemispherical, its surface is twice the area of the base; or, $s = 2 \times \cdot 7854d^2$.

When the dome is elliptical on a circular base, multiply twice the area of the base by the height, and divide the product by the radius of the base; the quotient will be the surface.

In other cases, multiply double the area of the base by the height of the dome, and divide the product by the radius of the base for an approximation to the surface; or $s = \frac{1}{r}(2bh)$.

Example.—Find the surface of a hexagonal spherical dome, each side of its base being = 30 feet.

Here $h = r$, and $s = 2b = 2 \times 30^2 \times 2\cdot598 = 4676\cdot4$ square feet.

Exercises

1. How many square yards of painting are contained in a hemispherical dome = 50 feet diameter? . . = 436·3 square yards.

2. Find the surface of a dome with a circular base = 100 feet circumference, its height being = 20 feet. . = 2000 square feet.

SALOONS

519. The **vacuity** of a saloon is the space contained by a horizontal plane through the base of the arches, the flat ceiling, and the arches.

520. Problem V.—To find the vacuity of a saloon.

Rule.—Find the continued product of the height of the arc, its breadth or horizontal projection, the perimeter of the ceiling, and ·7854.

From a side of the room, or its diameter when circular, take a like side or diameter of the ceiling, multiply the square of the

remainder by the corresponding tabular area for regular polygons, or by 1 when the room is rectangular, or by ·7854 when circular, and multiply this product by $\frac{2}{3}$ of the height.

Multiply the area of the flat ceiling by the height of the arch, and the sum of this product, and the two preceding, will be the content.

Let h, b, and p be the height and breadth of the arc and perimeter of ceiling; S, s two corresponding sides of the room and ceiling; a, a' the areas of the ceiling and of corresponding tabular polygon (Art. 268); and A, B, C the three products; then $A = \cdot 7854bhp$, $B = \frac{2}{3}(S-s)^2a'h$, $C = ah$, and $V = A + B + C$. For a square or rectangular room take 1 for a', and for a circular room take ·7854.

EXAMPLE.—Find the cubic content of a saloon formed by a circular quadrantal arc of 2 feet radius, connecting a ceiling with a rectangular room = 20 feet long and 16 wide.

$$A = \cdot 7854bhp = \cdot 7854 \times 2 \times 2 \times 56 = 175 \cdot 93$$
$$B = \tfrac{2}{3}(S-s)^2a'h = \tfrac{2}{3}(20-16)^2 \times 1 \times 2 = 21 \cdot 33$$
$$C = ah \quad . \quad . = 16 \times 12 \times 2 \quad . = 384 \cdot$$
$$\text{Hence } V = A + B + C \quad . \quad . = 581 \cdot 26 \text{ cubic feet.}$$

EXERCISE

A circular building = 40 feet diameter, and = 25 feet high to the ceiling, is covered with a saloon, the circular quadrantal arc of which is = 5 feet radius; required the cubic contents of the room.

= 30779·46 cubic feet.

521. Problem VI.—To find the curve surface of a saloon.

RULE.—Multiply the length of the arch by the mean perimeter.

Let l = the length of the arc, and p = the mean perimeter measured along the middle of the arch; then $s = pl$.

EXAMPLE.—The breadth of the curve surface of a saloon is = 10 feet, and the mean perimeter = 150 feet; what is its curve surface?

$$s = pl = 150 \times 10 = 1500 \text{ square feet.}$$

EXERCISE

Find the curve surface of a saloon, whose breadth is = $8\frac{1}{2}$ feet, and mean perimeter = 164 feet. . . . = 1394 square feet.

GROINS

522. Problem VII.—To find the cubic contents of the vacuity of a groin.

RULE.—Multiply the area of the base by the height, and this product by ·904.

$$V = \cdot 904bh.$$

EXAMPLE.—Find the vacuity of a square circular groin, the side of its base being = 24 feet, and its height = 12 feet.

$$V = \cdot 904bh = \cdot 904 \times 24^2 \times 12 = 6248 \cdot 4 \text{ cubic feet.}$$

EXERCISE

Find the content of the vacuity of an elliptical groin with a square base, whose side is = 20 feet, and the height of the groin = 6 feet. . . . . = 2169·6 cubic feet.

523. Problem VIII.—To find the surface of a groin.

RULE.—Multiply the area of the base by 1·1416.

This rule will give very nearly the surface for circular and elliptical groins of small eccentricity.

$$s = 1 \cdot 1416b.$$

EXAMPLE.—Find the surface of a circular groin with a square base, whose side is = 12 feet.

$$s = 1 \cdot 1416b = 1 \cdot 1416 \times 12^2 = 164 \cdot 39 \text{ square feet.}$$

EXERCISE

What is the surface of a circular groin having a square base, whose side is = 9 feet? . . . . . . = 92·4696 square feet.

GAUGING

524. Gauging is the art of measuring the dimensions and computing the capacity of any vessel or any portion of it.

The vessels usually gauged are casks, tuns, stills, and ships. The dimensions of the three former kinds are generally taken in inches, as the object is to determine the number of gallons of liquid contained in them.

When the capacity of a vessel is known in cubic inches, the number of gallons contained in it could then be easily found by dividing the capacity by 277·274, the number of cubic inches in an imperial gallon. The capacities of vessels can be found by means of the rules in the 'Mensuration of Solids,' but they can be found more readily by

means of certain numbers called divisors, multipliers, and gauge-points.

PRINCIPLES AND DEFINITIONS OF TERMS

525. **For Rectilineal Figures.**—The number of cubic inches in the measure of capacity or quantity of any vessel or solid is called the **divisor** for that body.

The number of cubic inches in the capacity being divided by the divisor, will give the capacity or quantity in the required denomination. Thus, the number of cubic inches in the capacity of a vessel being divided by 277·274, gives the number of imperial gallons; by 2218·192, gives the number of imperial bushels. So the number of cubic inches contained in a quantity of dry starch being divided by 40·3, will give the number of pounds, for 40·3 is the number of cubic inches in a pound of starch.

526. The **reciprocals** of the divisors are the **multipliers.**

It is evident that if, instead of dividing by the preceding divisors, we multiply by their reciprocals, the results will be the same. These multipliers will therefore be found by dividing 1 by the preceding divisors.

527. The **square roots** of the divisors are called **gauge-points.**

The gauge-points are just the sides of squares, of which the **content at one inch deep** is the measure of capacity or of quantity—that is, 1 gallon, 1 bushel, or 1 pound of starch, soap, tallow, or glass.

528. By the **content of any given surface at one inch deep** is meant the content in cubic inches of a right prism or vessel whose height or depth is 1 inch, and base the given surface.

Thus the content of a circular area is the content of a cylinder 1 inch high, whose base is the circle; the content of a square is the content of a parallelepiped 1 inch high, whose base is the given square.

Let V = the volume of a vessel or solid in cubic inches,
c = " capacity of it in the required denomination,
m = " number of cubic inches in the measure of capacity, as in 1 gallon, 1 pound, &c.,
n = " multiplier,
g = " gauge-point;

then $$c=\frac{V}{m}=nV, \text{ for } n=\frac{1}{m},$$

also $$g^2\times 1=m, \text{ and } g=\sqrt{m};$$

hence also $$c=\frac{V}{m}=\frac{V}{g^2}.$$

529. **For Circular Areas.**—If the number of cubic inches in the measure of capacity or quantity is divided by the number ·785398 or ·7854, the quotients are the **circular divisors.**

Let m_1 = this divisor, and d = the diameter of the area;

then $V = \cdot785398d^2$; and hence, $c = \frac{V}{m} = \frac{d^2}{m_1}$, for $m_1 = \frac{m}{\cdot785398}$.

530. If the number ·785398 is divided by the number of cubic inches in the measure of capacity or quantity, the quotients are the **circular multipliers.**

It is evident that the multiplier n_1 is the reciprocal of m_1; hence $c = n_1d^2$.

531. The square roots of the circular divisors are the **circular gauge-points.**

The gauge-points are the **diameters of circles,** of which the content at 1 inch deep is the number of cubic inches in the measure of capacity or quantity.

Since $c = \frac{d^2}{m_1}$, when $c = 1$, $\frac{d^2}{m_1} = 1$, therefore $d^2 = m_1$,

or $$d = \sqrt{m_1} = g_1.$$

532. **Polygonal Areas.**—If the number of cubic inches in the measure of capacity is divided by the tabular areas of polygons (Art. 268), the quotients are the **polygonal divisors.**

Thus, if a = the area of any regular polygon, and s its side,

a_1 = " " a similar regular polygon whose side is 1,

m_2 = " polygonal divisor,

then $$m_2 = \frac{m}{a_1},\quad a = s^2a_1,\quad c = \frac{a}{m} = \frac{s^2a_1}{m_2a_1} = \frac{s^2}{m_2}.$$

533. The reciprocals of the divisors are the **multipliers.**

If n_2 = the multiplier, then $n_2 = \frac{1}{m_2}$, and hence $c = n_2s^2$.

534. The square roots of the divisors are the **gauge-points.**

The gauge-points are the sides of regular polygons whose areas are equal to the number of cubic inches in the measure of capacity.

For if g_2 = the gauge-point, then $g_2 = \sqrt{m_2}$, or $g_2^2 = m_2 = \frac{m}{a_1}$.

Hence $m = g_2^2a_1$, or g_2 is the side of the polygon, whose content is m.

535. **Spherical Areas.**—If the circular divisors are increased in the ratio of 2 to 3, the results are the **spherical divisors;** the **spherical multipliers** are the reciprocals of the divisors; and the **spherical gauge-points** are the square roots of the divisors.

By Art. 531, $c=\frac{d^2}{m_1}h$, if h = the height of the cylinder. Now, if d = the diameter of a sphere, and m_3 the divisor,

$$c=\frac{\cdot 5236d^3}{m}=\frac{2}{3}\cdot\frac{\cdot 7854d^3}{m}=\frac{2d^3}{3m_1}=d^3\div\frac{3}{2}m_1=d^3\div m_3.$$

And if n_3 is the reciprocal of m_3, $c=n_3d^3$.

Also the gauge-point $g_3=\sqrt{m_3}$ is the diameter of a sphere whose volume is $=mg_3$.

For $g_3^2=m_3=\frac{3}{2}m_1=\frac{3}{2}\cdot\frac{m}{\cdot 7854}$, or $m=\cdot 5236g_3^2$.

536. **For Conical Vessels.**—The **conical divisors** are three times those for cylinders, the **multipliers** are their reciprocals, and the **gauge-points** are the square roots of the divisors.

The reason why the divisors are three times as great as those for cylinders is, that the volume of a cylinder is three times that of a cone of the same base and height. It can also be proved, as is similarly done in the preceding articles, that the gauge-point is the diameter of a cone which at one inch of height is equal to the measure of capacity.

537. **For Prismoidal Vessels.**—If the divisors for rectilineal and cylindric figures are multiplied by 6, the products will be **prismoidal divisors**; their reciprocals, the **prismoidal multipliers**; and the square roots of the prismoidal divisors, the **prismoidal gauge-points.**

TABLES OF MULTIPLIERS, DIVISORS, AND GAUGE-POINTS

I. FOR PRISMATIC VESSELS WITH SQUARE BASES.

Measures	Divisors	Multipliers	Gauge-points
Inches in the area of unity,	1	1	1
Superficial foot, . .	144	·006944	12
A solid foot, . . .	1728	·000578	41·57
Imperial gallon, . .	277·274	·003607	16·65
" bushel, . .	2218·192	·000451	47·1
A pound of hard soap, .	27·14	·036845	5·21
" " dry starch, .	40·30	·024813	6·35
" " green glass, .	12·18	·082102	3·48

II. FOR CYLINDRIC VESSELS

Measures	Divisors	Multipliers	Gauge-points
Inches in the area of unity,	1·27324	·785398	1·128
A superficial foot, . .	183·34	·005454	13·54
A solid foot, . . .	2200·16	·000454	46·91
Imperial gallon, . .	353·04	·002833	18·79
" bushel, . .	2824·29	·000356	53·14
A pound of hard soap, .	35·65	·02805	5·97
" " dry starch, .	51·3	·019491	7·16
" " green glass, .	15·5	·064516	3·94

III. FOR REGULAR POLYGONAL PRISMATIC VESSELS

Measures	Divisors	Multipliers	Gauge-points
PENTAGONAL BASE			
Imperial gallons, . .	161·161	·006205	12·69
" bushels, . .	1289·288	·000776	35·91
HEXAGONAL BASE			
Imperial gallons, . .	106·723	·00937	10·33
" bushels, . .	853·782	·001171	29·22
HEPTAGONAL BASE			
Imperial gallons, . .	76·302	·016106	8·73
" bushels, . .	610·414	·001638	24·71
OCTAGONAL BASE			
Imperial gallons, . .	57·425	·017414	7·58
" bushels, . .	459·403	·002177	21·43

IV. FOR CONICAL VESSELS.

Measures	Divisors	Multipliers	Gauge-points
Imperial gallons, . .	1059·109	·000944	32·54
" bushels, . .	8472·87	·000118	92·049

V. FOR SPHERICAL VESSELS

Measures	Divisors	Multipliers	Gauge-points
Imperial gallons, . .	529·554	·001888	23·01
" bushels, . .	4236·434	·000236	65·09

VI. PRISMOIDAL VESSELS, FRUSTUMS, OR CYLINDROIDS

Measures	Divisors	Multipliers	Gauge-points
WITH SQUARE ENDS			
Imperial gallons, . .	1663·644	·000601	40·79
" bushels, . .	13309·15	·000075	115·36
WITH CIRCULAR ENDS			
Imperial gallons, . .	2118·217	·000472	46·02
" bushels, . .	16945·74	·000059	130·17

538. **Problem I.—To gauge regular rectilineal and circular areas one inch deep.**

RULE.—Find the square of the side or the diameter in inches, and multiply or divide it by the proper multiplier or divisor for the regular figure.

$$c = \frac{a}{m} = na, \quad c = \frac{s^2}{m} = ns^2, \text{ or } c = \frac{d^2}{m_1} = n_1 d^2.$$

EXAMPLE.—Find the content of a square cistern whose side is $=108$ inches in imperial gallons.

$$c = \frac{a}{m} = \frac{108^2}{277 \cdot 274} = 42 \cdot 067.$$

Or, $$c = ns^2 = \cdot 003607 \times 108^2 = 42 \cdot 072.$$

EXERCISES

1. If the side of a square is $=49$ inches, what is its content in imperial gallons? . . . . $=8·66$.
2. What is the content of a regular octagon whose side is $=150$ inches in imperial gallons? . . . . $=391·8$.
3. Find the content of a circular tun in imperial gallons, its diameter being $=72$ inches. . . . . $=14·684$.

539. Problem II.—To gauge areas one inch deep.

RULE.—Find the superficial content, and divide or multiply it by the proper divisor or multiplier, for the required denomination.

EXAMPLE.—Find the area of a rectangular cistern in imperial bushels, its length and breadth being $=72$ and 42 inches.

$$c=\frac{V}{m}=\frac{72\times 42}{2218\cdot 192}=1\cdot 363.$$

EXERCISES

1. Find the content in pounds of hard soap of an oblong vessel, its length being $=201$, and its breadth $=60$ inches. . . $=444\cdot 36$.
2. Find the area of a triangular vessel in imperial gallons, its base being $=100$ inches, and the perpendicular on it $=80$ inches. $=14\cdot 426$.
3. Required the content of a parallelogram in pounds of hard soap, the length being $=84$ inches, and the perpendicular breadth $=32$ inches. . . . . $=99\cdot 04$.
4. What is the area in imperial bushels of a trapezoid, the parallel sides being $=60$ and 145, and the perpendicular breadth $=80$ inches? $=3\cdot 698$.
5. Find the area of a quadrilateral in pounds of dry starch, one of its diagonals being $=80$, and the perpendiculars on it from the opposite angles being $=24\cdot 6$ and $14\cdot 4$. . . . . . $=38\cdot 7$.
6. What is the area in imperial gallons of an oval figure whose transverse diameter is $=85$ inches, and six equidistant ordinates, whose common distance is $=15$ inches, being in order $40\cdot 6$, $44\cdot 3$, $50\cdot 4$, $50\cdot 1$, $42\cdot 7$, and $38\cdot 2$, and two segments at each end, whose bases are the extreme ordinates, and heights $=5$ inches, and nearly of a parabolic form? . . . . $=13\cdot 22$.

540. Problem III.—To find the area of an ellipse when its two axes are given.

RULE.—Find its area by the rule in Art. 428, and divide or multiply it by the proper divisor or factor, and the result will be the required area; or,

Find the product of the axes, and multiply or divide it by the circular factor or divisor, and the result is the area.

EXERCISES

1. Find the area of an ellipse in imperial gallons, its axes being $=99$ and 75. . . . . $=21\cdot 03$.
2. Find the content of an ellipse in imperial gallons, its axes being $=108$ and 75. . . . . $=22\cdot 947$.

541. Problem IV.—To gauge solids whose bases are regular figures.

RULE.—Find the cubic content; then multiply or divide it by the proper multiplier or divisor corresponding to the required measure or weight; or,

Multiply the square of the given side or diameter by the depth, and divide or multiply the product by the proper tabular divisor or multiplier for the given figure of the base (Art. 537).

If V = the volume, $c = \frac{V}{m}$, or $c = nV$.

EXAMPLE.—Find the content of an octagonal prism in imperial bushels, its side being = 60 inches, and depth = 75.

By Art. 369, $V = 4 \cdot 8284 \times 60^2 \times 75 = 1303668$,

and $$c = \frac{V}{m} = \frac{1303668}{2218 \cdot 19} = 587 \cdot 7 \text{ bushels};$$

or $$c = n_3 s^2 h = \cdot 002177 \times 60^2 \times 75 = 587 \cdot 79 \text{ bushels}.$$

EXERCISES

1. Find the content in imperial gallons and bushels of a vessel with a square bottom, each side being = 30 inches, and its depth = 40. . . . . = 129·83 and 16·236.

2. What is the content in imperial bushels of a cylindric vessel whose diameter is = 48 inches, and depth = 64 inches?
= 52·2 bushels.

3. Find the content in imperial bushels of a regular pentagonal prismatic vessel, a side of its base being = 54 inches, and its depth = 80 inches. . . . . = 180·94.

4. Find the content in imperial gallons of a conical vessel, the diameter of its base being = 27 inches, and its height = 60 inches.
= 41·3.

5. What is the content in imperial bushels of a pyramidal vessel whose base is a regular hexagon, the length of its side being = 40 inches, and the height of the vessel = 72 inches? . . = 44·96.

6. Find the content of a conical vessel in imperial gallons, the diameter of its base being = 60 inches, and its height = 60 inches.
= 203·9.

7. What is the content in imperial bushels of a pyramidal vessel, whose base is a regular octagon, its side being = 105 inches, and its depth = 120 inches? . . . . = 960·5.

542. Problem V.—To find the content of a spherical vessel.

RULE.—Find the volume of the sphere, and multiply or divide

it by the proper multiplier or divisor for the required measure or weight; or,

Divide or multiply the cube of the diameter by the corresponding tabular divisor or multiplier (Art. 537).

EXERCISES

1. Find the content of a spherical vessel whose diameter is = 34 inches in imperial gallons. . . . . = 74·2.

2. Find the content in imperial bushels of a spherical vessel whose diameter is = 68 inches. . . . . = 74·2.

543. Problem VI.—To find the content of a spheroid.

RULE.—Find its volume, and multiply or divide it by the proper multiplier or divisor for the required measure or weight; or,

Multiply the square of the equatorial diameter by the polar diameter, and divide or multiply the product by the divisor or multiplier for spherical vessels. (See Art. 447.)

For, if b is the equatorial diameter and a the polar diameter of a spheroid, and a the diameter of a sphere; v the volume of the spheroid, and v' that of the sphere;

then (Art. 447), $v = \cdot 5236ab^2$, and $v' = \cdot 5236a^3$;

and hence $v : v' = b^2 : a^2$,

from which the rules are evident.

EXERCISES

1. Find the content in imperial gallons of a prolate spheroid, its polar diameter being = 72 inches, and its equatorial = 50. = 339·9.

2. What is the content in imperial bushels of a prolate spheroid whose diameters are = 70 and 90? . . . . . . = 104·1.

544. Problem VII.—To find the content of a frustum of a cone or pyramid, or of a prismoid or cylindroid.

RULE.—To the areas of the ends add four times the area of the middle section; multiply the sum by one-sixth of the height, and the product is the volume. Divide or multiply the volume by the proper divisor or multiplier for the given denomination, and the result is the content. (See Art. 389.)

$$V = \tfrac{1}{6}h(B + b + 4M), \text{ and } c = \frac{V}{m}, \text{ or } c = nV.$$

For regular figures, to the squares of a side of each end add four times the square of the side of the middle section, multiply the sum by one-sixth of the height, and this product by the multiplier for the corresponding prismoidal vessels;

$$c = \tfrac{1}{6}h\{E^2 + e^2 + 4e'^2\}n;$$

in which E = a side of the greater end, e = a side of the less end, $e' = \frac{1}{2}\{E + e\}$, and n = the prismatic multiplier for the form of the base, or the cylindric multiplier if the frustum be that of a cone.

EXAMPLES.—1. Find the content in imperial gallons of a vessel, which is a frustum of a square pyramid, the sides of its ends being = 78 and 42 inches, and its depth = 60 inches.

$$V = 10(78^2 + 42^2 + 120^2) = 222480,$$

and $$c = \frac{V}{m} = \frac{222480}{277 \cdot 274} = 802 \cdot 4 \text{ imperial gallons.}$$

2. What is the content in imperial gallons of a frustum of a regular hexagonal pyramid, the sides of its ends being = 72 and 48 inches, and its depth = 72 inches?

It is found that V = 682400·28 cubic inches,

and $$c = \frac{V}{m} = 2461 \cdot 1 \text{ imperial gallons.}$$

Here $$n = \cdot 00937, \text{ and } V = \tfrac{1}{6}h(E^2 + e^2 + 4e')A',$$

or $$c = \tfrac{1}{6} \times 72(72^2 + 48^2 + 4 \times 60^2) \times \cdot 00937$$
$$= 12 \times 21888 \times \cdot 00937 = 2461 \cdot 09 \text{ imperial gallons.}$$

EXERCISES

1. What is the content in imperial gallons of a frustum of a square pyramid, the sides of its ends being = 52 and 28, and its depth = 36 inches? . . . . = 213·996.

2. What is the content in imperial gallons of a frustum of a regular hexagonal pyramid, the sides of its ends being = 54 and 36, and its depth = 48 inches? . . . . . . = 922·9.

3. Find the content in imperial gallons of a frustum of a rectangular pyramid, the sides of its greater end being = 36 and 16, those of its smaller end = 27 and 12, and its depth = 80 inches. = 128·12.

4. What is the content of a conic frustum in imperial gallons, the diameters of its ends being = 44 and 16, and its depth = 40 inches? . . . . = 109·39.

5. What is the content in imperial gallons of a vessel of the form of an elliptic cone, the diameters of one end being = 48 and 42, and those of the other = 40 and 34 inches, the corresponding axes of the ends being parallel, and the depth = 30 inches? . = 142·55.

The contents of other solids can be found by determining their volumes by the usual rules, and then dividing by the proper number for the required measure of quantity. The contents of many solids of rather irregular figures can be calculated by means of the first

and second rules of Art. 485, which may also be used for regular figures, as portions of conoids and spheres.

545. Problem VIII.—To gauge mash-tuns, stills, and other brewing and distilling vessels.

Divide the vessel into small portions by means of planes parallel to its base; find the areas of the middle sections of these portions, and multiply these areas by the corresponding depths of the portions to which they belong: the products are the volumes of the portions, and the sum of these volumes is the whole volume; divide the whole volume by the number corresponding to the required measure or weight, and the result is the required content.

The vessel is divided into portions of 6 or of 10 inches deep, according as the sides are more or less inclined; and so that the difference of the corresponding diameters of two successive middle sections may not differ by more than 1 inch.

When the vessel is nearly circular, cross—that is, perpendicular—diameters are taken at the middle of any portion, and the mean of them is considered to be the diameter of a cylinder of the same depth as that portion, whose volume is nearly equal to it. In this case the volumes of the different portions are calculated as cylinders.

EXAMPLE.—Find the content, in imperial gallons, of an under-back, the form of which is nearly the frustum of a cone, from the following dimensions, the cross diameters being measured at the middle of the several portions into which the vessel is divided, their depths being those in the first column:—

Depth of Portions	Depth of Middle Sections	Cross Diameters		Mean Diameters
8	4	70	68·8	69·4
10	13	72	72·2	72·1
10	23	73·6	73·5	73·5
10	33	74	73·8	73·9

The whole depth is 38 inches. The area to 1 inch deep of the middle section of the first portion is found thus:—

$$a = d^2 \div m = 69{\cdot}4^2 \div 353{\cdot}04 = 13{\cdot}64.$$

In the same manner, the areas to 1 inch deep for the other middle sections is found to be, in order, 14·73, 15·3, 15·47; and each of

these being multiplied by the depths, and the sum of the results taken, it will be the content as under :—

For the 1st portion, content	$=13{\cdot}64\times\ 8$	$=109{\cdot}12$
" 2nd " "	$=14{\cdot}73\times10$	$=147{\cdot}3$
" 3rd " "	$=15{\cdot}3\ \times10$	$=153{\cdot}0$
" 4th " "	$=15{\cdot}47\times10$	$=154{\cdot}7$
Content of vessel in imperial gallons		$=564{\cdot}12$

It is usual to construct a table containing the contents of a fixed vessel, as of a mash-tun or still, for every inch in depth. The contents for the first inch of depth from the bottom of the preceding vessel is 15·47; for two inches, it is the double of this, or 30·94; for three inches, it is three times this, or 46·41; and so on. For the area of each of the first ten inches, it is 15·47; for each of the next ten inches, it is 15·3; and a table is thus easily constructed.

EXERCISES

1. Find the content in imperial gallons of a flat-bottomed copper, the mean diameters at the middle of four portions into which it is divided by horizontal sections being as under :—

Depth of Portions in Inches	Mean Diameters of their Middle Sections
12	54·4
10	51·9
10	49·6
10	47·3
Whole depth = 42	

Content in imperial gallons = 310.

The bottoms of coppers are seldom flat; they are generally rising or falling — that is, convex or concave internally. The content of a vessel with a rising or falling crown, as the bottom is in this case called, is found by calculating, as in the preceding example, the content above the centre of the crown when it is rising, and then adding the content of the space contained between the bottom and a horizontal plane touching its crown, from which the depth of the vessel is taken. The content of this portion is most easily found by measuring the quantity of water required to fill it, till the bottom is covered. A similar method is adopted for a falling crown.

2. Find the content of a still, from the dimensions below, the uppermost portion being considered a frustum of a sphere, the

cross diameters at its two ends being given, and also those at the middle of other four portions, the quantity of water required to cover its rising crown being 35 gallons :—

Depth of Portions in Inches	Cross Diameters		Content in Imperial Gallons
8	27 55·2	27 54·8	43·5
9	59·8	60·2	91·77
9	63·8	64·4	104·74
9	64	64·6	105·4
10·5	62	62·4	115·07
45·5 = whole depth.			= 495·52.

546. When the sides of the vessel are sloping and straight, though the vessel be circular or oval, if two corresponding diameters at the top and bottom are measured, those at any intermediate depth are easily found. Thus, if c denotes the excess of the top diameter above that at the bottom, and if h is the depth of the vessel, and h' the depth of any other place, reckoning from the bottom, and c' the excess of the diameter there above the bottom diameter; then

$$h : h' = c : c', \text{ and } c' = \frac{c}{h}h' ;$$

c' being thus found, if it is added to the bottom diameter, the result is the diameter at the given depth. If $h' = 10$ inches, then c' is the difference of diameters for every 10 inches; and the diameters for every 10 inches of depth are therefore easily found. The cross diameters are computed in the same manner; and then the content at every inch of depth can be found and registered in a table.

CASK GAUGING

547. Casks are usually divided into four varieties :—The **first variety** is the middle frustum of a spheroid; the **second,** the middle frustum of a parabolic spindle; the **third,** two equal frustums of a paraboloid united at their bases; and the **fourth,** two equal conic frustums united at their bases.

The rules for calculating the contents of the middle frustums of circular, elliptic, and hyperbolic spindles are too difficult for the purposes of practical gauging, and they are therefore omitted in treatises on this subject.

When the cask is much curved, it is considered to belong to the first variety; when less curved, to the second; when still less, to

the third; and when it is straight from the bung to the head, to the fourth variety.

First Variety

548. Problem IX.—To find the content of a cask of the first or spheroidal variety.

RULE.—To twice the square of the bung diameter add the square of the head diameter, multiply the sum by the length of the cask, and divide the product by 1059·108, and the quotient is the content in imperial gallons, the dimensions being all taken in inches; or,

$$C=(2B^2+H^2)L\div 1059{\cdot}108,$$

where H, B are the head and bung diameters, and L the length of the cask.

Note.—This is just the rule given in Art. 449 in the first case; only, instead of multiplying by ·2618 or $\frac{1}{3}$ of ·7854, and then dividing by 277·274 for imperial gallons, the divisor 1059·108 is taken, which is 3 times the divisor 353·036 for circular areas.

EXAMPLE.—What is the content of a cask whose bung and head diameters are $=32$ and 24, and length $=40$ inches?

$$C=(2B^2+H^2)L\div 1059{\cdot}108=(2\times 32^2+24^2)\times 40\div 1059{\cdot}108$$
$$=(2048+576)\times 40\div 1059{\cdot}108=99{\cdot}1 \text{ imperial gallons.}$$

EXERCISES

1. Find the content in imperial gallons of a cask whose bung and head diameters are $=30$ and 18, and length $=40$ inches. $=80{\cdot}2$.

2. What is the content of a cask whose bung and head diameters are $=24$ and 20, and length $=30$ inches? . . $=43{\cdot}97$.

Second Variety

549. Problem X.—To find the content of a cask of the second variety.

RULE.—To twice the square of the bung diameter add the square of the head diameter, and from the sum subtract $\frac{2}{5}$ of the square of the difference of these diameters; multiply the remainder by the length, and the product, divided by 1059·108, will give the content in imperial gallons.

$$C=\{2B^2+H^2-\tfrac{2}{5}(B-H)^2\}L\div 1059{\cdot}108.$$

The rule in this case is the same as that in Art. 473, which is easily reduced to this form.

EXAMPLE.—Let the dimensions of a cask of the second variety be the same as those given in the example for the first variety, to find its content.

$$C=\{2B^2+H^2-\tfrac{2}{5}(B-H)^2\}L\div 1059\cdot108$$
$$=(2\times32^2+24^2-\tfrac{2}{5}\times8^2)\times40\div1059\cdot108$$
$$=(2624-25\cdot6)40\div1059\cdot108=98\cdot1 \text{ imperial gallons.}$$

EXERCISES

1. Find the content of a cask whose bung and end diameters are $=48$ and 36, and length $=60$ inches. . . . . $=331\cdot24$.

2. What is the content of a cask whose bung and head diameters are $=36$ and 20, and its length $=40$ inches? . . $=109\cdot14$.

Third Variety

550. **Problem XI.—To find the content of a cask of the third variety.**

RULE.—Add the square of the bung diameter to that of the head diameter, multiply the sum by the length, and divide the product by 706·0724 for its content.

$$C=(B^2+H^2)L\div706\cdot0724.$$

The formula is the same as that in Art. 446; only, instead of multiplying by $\frac{1}{2}\times\cdot7854$, and dividing by $277\cdot274$, the equivalent divisor 706·0724 is used.

EXAMPLE.—What is the content in imperial gallons of a cask of the third variety, of the same dimensions as that in the example for the first variety?

$$C=(B^2+H^2)L\div706\cdot0724=(32^2+24^2)\times40\div706\cdot0724$$
$$=(1024+576)\times40\div706\cdot0724=90\cdot64.$$

EXERCISES

1. Find the content of a cask whose bung and head diameters are $=30$ and 24, and its length $=36$. . . . . . . $=75\cdot26$.

2. What is the content of a cask, whose bung and head diameters are $=29$ and 15, and its length $=24$ inches? . . $=36\cdot23$.

Fourth Variety

551. **Problem XII.—To find the content of a cask of the fourth variety.**

RULE.—Add together the product of the bung and head

diameters, and their squares; multiply the sum by the length, and divide the product by 1059·1086 for the content.

$$C=(B^2+BH+H^2)L\div 1059{\cdot}1086.$$

For this is the formula of Art. 386, except that, instead of the factor ·2618 or $\frac{1}{3}\times{\cdot}7854$, and the divisor 277·274, the equivalent divisor 1059·1086 is taken.

EXAMPLE.—Find the content of a cask of the fourth variety, whose bung and head diameters are=32 and 24, and length =40 inches.

$$C=(B^2+BH+H^2)L\div 1059{\cdot}1086$$
$$=(1024+768+576)\times 40\div 1059{\cdot}11=89{\cdot}43.$$

EXERCISES

1. What is the content of a cask whose bung diameter is=32 inches, end diameter=18, and length=38 inches? . . =69·04.
2. Find the content of a cask whose diameters are=40 and 20, and length=50 inches. . . . . =132·2.

MEAN DIAMETERS OF CASKS

552. The **mean diameter** of a cask is the diameter of a cylinder of the same length, whose capacity is equal to that of the cask.

The mean diameter may be found by means of the following table, the construction of which is this:—If the bung diameter be denoted by 1, and the head diameter, divided by the bung diameter, be denoted by H, the contents of the four varieties of casks will be expressed by

$$\tfrac{1}{3}(2+H^2),\ \tfrac{1}{15}(8+4H+3H^2),\ \tfrac{1}{2}(1+H^2),\ \tfrac{1}{3}(1+H+H^2),$$

multiplied by ·7854L; but if D=the mean diameters, the contents are also expressed by $\cdot7854D^2L$; hence, as each of the above expressions × ·7854L is equal to the last, therefore these expressions themselves are $=D^2$ for the four varieties, or D for these varieties is equal to the square root of each of them. For example, when H=·5, then, for the first variety, $D=\sqrt{\tfrac{1}{3}(2+H^2)}=\tfrac{1}{2}\sqrt{3}={\cdot}866$; which is the number under the first variety in the following Table opposite to ·5 or H. In the same manner, all the other numbers in the Table are found, for H=·51, ·52, ·53,... up to 1. The numbers marked H are just the ratio of the bung to the head diameter, and the numbers under the different varieties are the mean diameters when the bung diameter is=1, and the ratio H is its head diameter.

TABLE OF MEAN DIAMETERS WHEN THE BUNG DIAMETER IS=1

H	First Variety	Second Variety	Third Variety	Fourth Variety	H	First Variety	Second Variety	Third Variety	Fourth Variety
·50	·8660	·8465	·7905	·7637	·76	·9270	·9227	·8881	·8827
·51	·8680	·8493	·7937	·7681	·77	·9296	·9258	·8944	·8874
·52	·8700	·8520	·7970	·7725	·78	·9324	·9290	·8967	·8922
·53	·8720	·8548	·8003	·7769	·79	·9352	·9320	·9011	·8970
·54	·8740	·8576	·8036	·7813	·80	·9380	·9352	·9055	·9018
·55	·8760	·8605	·8070	·7858	·81	·9409	·9383	·9100	·9066
·56	·8781	·8633	·8104	·7902	·82	·9438	·9415	·9144	·9114
·57	·8802	·8662	·8140	·7947	·83	·9467	·9446	·9189	·9163
·58	·8824	·8690	·8174	·7992	·84	·9496	·9478	·9234	·9211
·59	·8846	·8720	·8210	·8037	·85	·9526	·9510	·9280	·9260
·60	·8869	·8748	·8246	·8082	·86	·9556	·9542	·9326	·9308
·61	·8892	·8777	·8282	·8128	·87	·9586	·9574	·9372	·9357
·62	·8915	·8806	·8320	·8173	·88	·9616	·9606	·9419	·9406
·63	·8938	·8835	·8357	·8220	·89	·9647	·9638	·9466	·9455
·64	·8962	·8865	·8395	·8265	·90	·9678	·9671	·9513	·9504
·65	·8986	·8894	·8433	·8311	·91	·9710	·9703	·9560	·9553
·66	·9010	·8924	·8472	·8357	·92	·9740	·9736	·9608	·9602
·67	·9034	·8954	·8511	·8404	·93	·9772	·9768	·9656	·9652
·68	·9060	·8983	·8551	·8450	·94	·9804	·9801	·9704	·9701
·69	·9084	·9013	·8590	·8497	·95	·9836	·9834	·9753	·9751
·70	·9110	·9044	·8631	·8544	·96	·9868	·9867	·9802	·9800
·71	·9136	·9074	·8672	·8590	·97	·9901	·9900	·9851	·9850
·72	·9162	·9104	·8713	·8637	·98	·9933	·9933	·9900	·9900
·73	·9188	·9135	·8754	·8685	·99	·9966	·9966	·9950	·9950
·74	·9215	·9166	·8796	·8732	1·00	1·0000	1·0000	1·0000	1·0000
·75	·9242	·9196	·8838	·8780					

553. **Problem XIII. – To find the capacity of a cask of any of the four varieties by means of their mean diameters, found by the Table.**

RULE.—Divide the head by the bung diameter, and find the quotient in the column marked H in the Table, and opposite to it and under the proper variety is the mean diameter of a similar cask, whose bung diameter is 1.

Multiply this tabular mean diameter by the given bung diameter, and the product is the required mean diameter, the square of

which, multiplied by the length, and the product, divided by 353·036, or multiplied by ·0028325, is the content in imperial gallons; or,

$$C = D^2L \div 353{\cdot}036, \text{ or } C = {\cdot}0028325D^2L.$$

For, if D = the mean diameter, the content is

$$C = \frac{{\cdot}7854}{277{\cdot}274} D^2L = D^2L \div 353{\cdot}036.$$

Instead of this divisor, the corresponding multiplier may be used, and then

$$C = {\cdot}0028325D^2L.$$

EXAMPLE.—Find the content of a cask of the first variety, whose diameters are = 30 and 24, and length = 36 inches.

$$H = \tfrac{24}{30} = {\cdot}8, \text{ and opposite to } {\cdot}8 \text{ is } {\cdot}938 = D';$$

then $D = BD' = 30 \times {\cdot}938 = 28{\cdot}14,$

and $C = {\cdot}0028325D^2L = {\cdot}0028325 \times 28{\cdot}14^2 \times 36 = 80{\cdot}7.$

EXERCISE

What are the contents of each of four casks of the four varieties, their diameters being = 32 and 24, and length = 40 inches?

For the first, 99·1; for the second, 98·11; for the third, 90·62; and for the fourth, 89·44.

CONTENTS OF CASKS WHOSE BUNG DIAMETERS AND LENGTHS ARE UNITY

554. The contents of casks may be more readily computed by means of a Table of the contents of casks whose bung diameters and lengths are = 1.

Let D' have the same meaning as in the preceding problem—that is, let it denote the numbers in the preceding Table under the different varieties, which are just the mean diameters of casks whose bung diameters are 1 and head diameters H; also, let C' be the content of a cask whose mean diameter is D' and length 1, and which may be called the *standard* cask; then

$$C' = D'^2L' \times \frac{{\cdot}7854}{277{\cdot}274} = D'^2 \div 353{\cdot}036, \text{ for } L' = 1.$$

Hence, if the numbers in the preceding Table are squared, and the square divided by 353·036, or multiplied by ·0028325, the results will be the contents C' required. The following Table can thus be constructed from the preceding one. Thus, for example, when $H = {\cdot}75$, then, by the preceding Table, D' for the second variety is ·9196, and

$$C' = D'^2 \div 353{\cdot}036 = {\cdot}9196^2 \div 353{\cdot}036 = {\cdot}0023954,$$

which is just the capacity opposite to ·75 in the following Table:—

TABLE OF CONTENTS IN IMPERIAL GALLONS OF STANDARD CASKS C'

H'	First Variety	Second Variety	Third Variety	Fourth Variety
·50	·0021244	·0020300	·0017704	·0016523
·51	·0021340	·0020433	·0017847	·0016713
·52	·0021437	·0020567	·0017993	·0016905
·53	·0021536	·0020702	·0018141	·0017098
·54	·0021637	·0020838	·0018293	·0017294
·55	·0021740	·0020975	·0018447	·0017491
·56	·0021845	·0021114	·0018604	·0017690
·57	·0021951	·0021253	·0018764	·0017891
·58	·0022060	·0021394	·0018927	·0018094
·59	·0022170	·0021536	·0019093	·0018299
·60	·0022283	·0021679	·0019261	·0018506
·61	·0022397	·0021823	·0019433	·0018715
·62	·0022513	·0021968	·0019607	·0018925
·63	·0022631	·0022114	·0019784	·0019138
·64	·0022751	·0022262	·0019964	·0019352
·65	·0022873	·0022410	·0020147	·0019568
·66	·0022997	·0022560	·0020332	·0019786
·67	·0023122	·0022711	·0020521	·0020006
·68	·0023250	·0022863	·0020712	·0020228
·69	·0023379	·0023016	·0020906	·0020452
·70	·0023510	·0023170	·0021103	·0020678
·71	·0023643	·0023326	·0021302	·0020905
·72	·0023778	·0023482	·0021505	·0021135
·73	·0023915	·0023640	·0021710	·0021366
·74	·0024054	·0023799	·0021918	·0021599
·75	·0024195	·0023954	·0022129	·0021834
·76	·0024337	·0024120	·0022343	·0022071
·77	·0024482	·0024282	·0022560	·0022310
·78	·0024628	·0024445	·0022780	·0022551
·79	·0024777	·0024610	·0023002	·0022794
·80	·0024927	·0024776	·0023227	·0023038
·81	·0025079	·0024942	·0023455	·0023285
·82	·0025233	·0025110	·0023686	·0023533
·83	·0025388	·0025279	·0023920	·0023783
·84	·0025546	·0025449	·0024156	·0024035

H′	First Variety	Second Variety	Third Variety	Fourth Variety
·85	·0025706	·0025621	·0024396	·0024289
·86	·0025867	·0025793	·0024638	·0024545
·87	·0026030	·0025967	·0024883	·0024803
·88	·0026196	·0026141	·0025131	·0025063
·89	·0026363	·0026317	·0025381	·0025324
·90	·0026532	·0026494	·0025635	·0025588
·91	·0026703	·0026672	·0025891	·0025853
·92	·0026875	·0026851	·0026150	·0026120
·93	·0027050	·0027032	·0026412	·0026389
·94	·0027227	·0027213	·0026677	·0026660
·95	·0027405	·0027396	·0026945	·0026933
·96	·0027585	·0027579	·0027215	·0027208
·97	·0027768	·0027764	·0027489	·0027484
·98	·0027952	·0027950	·0027765	·0027763
·99	·0028138	·0028137	·0028044	·0028043
1·00	·0028326	·0028326	·0028326	·0028326

555. **Problem XIV.—To find the content of a cask by means of the Table of Contents of Standard Casks.**

Divide the head by the bung diameter, and find the quotient in the column H, and opposite to it and under the proper variety is the content C′ of the standard cask; multiply this tabular content by the square of the bung diameter of the given cask, and this product by the length, both in inches, and the result will be the required content in imperial gallons.

For, by Art. 553, $C = D^2L \div 353{\cdot}036$;

and $D^2 = B^2D'^2$, also (Art. 554) $C' = D'^2 \div 353{\cdot}036$;

hence $C = C'B^2L$.

EXAMPLE.—Find the content of a cask of the first variety, whose diameters are = 30 and 24, and length = 36 inches.

$H = \frac{24}{30} = \frac{4}{5} = {\cdot}8$; and hence $C' = {\cdot}0024927$,

and $C = C'B^2L = {\cdot}0024927 \times 30^2 \times 36 = 80{\cdot}7$.

The same answer as that to the example in Art. 553.

EXERCISES

1. What are the contents of each of four casks of the four varieties, their diameters being = 32 and 24, and length = 40 inches?

   The contents will be the same as for the four casks in the exercise to the preceding problem.

2. Find the contents of each of four casks of the four varieties, their diameters being $=31\cdot5$ and $24\cdot5$, and the length $=42$ inches.

Content for the first $=102\cdot6$, the second $=101\cdot87$, the third $=94\cdot9$, and the fourth $=93\cdot98$.

3. What is the content of a pipe of wine, whose length is $=50$ inches, head diameter $=22\cdot7$, and bung diameter $=31\cdot7$, the cask being of the first variety? . . . . $=119\cdot19$.

GENERAL METHOD FOR A CASK OF ANY FORM

556. **Problem XV.—To find the content of a cask of any form, by one method, independently of tables.**

RULE.—Add together 39 times the square of the bung diameter, 25 times the square of the head diameter, and 26 times the product of the diameters; multiply the sum by the length, and divide the product by 31773·25 for the content in imperial gallons.

$$C=(39B^2+25H^2+26BH)L\div31773\cdot25.$$

EXAMPLE.—Find the content of a cask whose diameters are $=32$ and 24, and length $=40$ inches.

$$\begin{aligned} C &= (39B^2+25H^2+26BH)L\div31773\cdot25 \\ &= (39\times32^2+25\times24^2+26\times32\times24)\times40\div31773\cdot25 \\ &= 93\cdot5 \text{ imperial gallons.} \end{aligned}$$

EXERCISE

Find the content of a cask whose diameters are $=36$ and 48, and length $=60$ inches. . . . . $=315\cdot7$.

Or the capacity in imperial gallons of any cask may be found as follows :—

Let D, d = inside diameters at the heads, B = inside diameter at the bung, and L the length, all in inches;

then the capacity in imperial gallons

$$=\cdot0014162L(D-d+B^2).$$

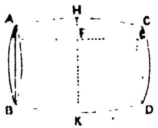
The buoyancy in pounds equals ten times the capacity in gallons minus the weight of the cask itself.

ULLAGE OF CASKS

The **ullage** of a cask is the content of the part occupied by liquor in it when not full, or of the empty part. Only two cases are usually considered—namely, when the cask is **lying**, or when it is **standing**. When the ullage of the part filled is found, that of the empty part can be obtained by subtracting the ullage found from the content of the whole cask.

557. **Problem XVI.—To find the ullage of the filled part of a lying cask in imperial gallons.**

RULE.—Divide the number of wet inches by the bung diameter, and if the quotient is under ·5, deduct from it $\frac{1}{4}$ of what it wants of ·5; but when the quotient exceeds ·5, add $\frac{1}{4}$ of that excess to it; then if the remainder in the former case, or the sum in the latter, be multiplied by the content of the whole cask, the product will be the ullage of the part filled.



Let $W = FK$ the wet inches,

$R = W \div B$,

$C' =$ the content of the cask,

$U =$ " ullage of EBDG,

$D = R \sim \cdot 5$;

then $U = (R \mp \frac{1}{4}D)C$,

using $-$ when $R < \cdot 5$, and $+$ when $R > \cdot 5$.

EXAMPLE.—The content of a lying cask is $= 98$ gallons, the bung diameter $= 32$, and wet inches $= 10$; required the ullage of the part filled.

$$R = W \div B = \tfrac{10}{32} = \cdot 3125,\ D = \cdot 5 - \cdot 3125 = \cdot 1875,\ \tfrac{1}{4}D = \cdot 0469;$$

hence $U = (R - \frac{1}{4}D)C = (\cdot 3125 - \cdot 0469) \times 98 = \cdot 2656 \times 98 = 26\cdot 03$.

Let L, $L' =$ the length of the given and experimental cask used in constructing the lines S.S. and S.L. on the gauger's rule.

C, $C' =$ their capacities; and hence $C' = 100$.

U, $U' =$ the capacity of a portion of the given cask when lying to be ullaged, and of a similar portion of experimental cask;

W, $W' =$ the wet inches for these portions.

Then $$L : W = L' : W',$$

and $$\log. L - \log. W = \log. L' - \log. W'.$$

Hence, since the slider for the line S.L. is a logarithmic line, the distance from L to W on it is equal to that from L′ to W′; and when L on the slider is opposite to C′ or 100 on S.L., W on the slider will be opposite to the same number on S.L. that W′ would be opposite to when L′ is opposite to 100 on S.L.; that is, W would be opposite to U′, the ullage of a similar portion of the experimental cask, which is therefore obtained by the above rule.

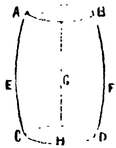
Again, C' or $100 : C = U' : U$; and since C′, C, and U′ are known, therefore U, their fourth proportional, can be found by means of the lines A, B, according to the rule in Art. 492.

Exercise

The content of a lying cask is = 90, its bung diameter = 36, and the wet inches = 27; find the ullage of the part filled. = 73·125.

558. **Problem XVII.—To find the ullage of the filled part of a standing cask in imperial gallons.**

Rule.—Divide the number of wet inches by the length of the cask, then if the quotient is less than ·5, subtract from it $\frac{1}{10}$ part of what it wants of ·5; but if it is greater than ·5, add to it $\frac{1}{10}$ of its excess above ·5; then multiply the remainder in the former case, or the sum in the latter, by the content of the cask, and the product will be the ullage.



Let $W = GH$ the wet inches,

$R = W \div L$,

and let C, U, and D have the same meaning as in last problem;

then $U = (R \mp \frac{1}{10}D)C$,

using − when $R < \cdot 5$, and + when $R > \cdot 5$.

This rule is proved in exactly the same manner as that of the preceding problem.

Example.—The content of a standing cask is = 120 gallons, its length = 48, and the wet inches = 40; required the ullage of the part filled.

$$R = W \div L = \tfrac{40}{48} = \tfrac{5}{6} = \cdot 8\dot{3}\text{; hence } D = \cdot\dot{3}\text{, and } \tfrac{1}{10}D = \cdot 0\dot{3}.$$

Hence $U = (R + \frac{1}{10}D)C = (\cdot 8\dot{3} + \cdot 0\dot{3}) \times 120 = \cdot 8\dot{6} \times 120 = 104.$

Exercise

The content of a cask is = 105 gallons, its length = 45 inches, and the wet inches = 25; what is the ullage of the part filled? = 58·9.

MALT-GAUGING

559. Barley to be malted is steeped in water in a **cistern** for not less than 40 hours. When sufficiently steeped, it is then removed to a frame called a **couch-frame,** where it remains without alteration for about 26 hours; it is then reckoned a **floor** of malt, till it is ready for the kiln.

During the steeping, the barley swells about $\frac{1}{4}$ of its original bulk, or $\frac{1}{5}$ of its bulk then; after being less than 72 hours out of the cistern, it is considered to have increased $\frac{1}{8}$ of its bulk at that time; and after being out a longer time, it is considered to have increased by $\frac{1}{2}$ of its bulk then; and hence the rule in the following problem:—

560. **Problem XVIII.—Having given the cistern, couch, or floor gauge of a quantity of malt, to find the net bushels.**

RULE.—Multiply the cistern or couch bushels by ·8, and the floor bushels by $\frac{2}{3}$, when it has been out of the cistern for less than 72 hours, or by $\frac{1}{2}$ when it has been out a longer time.

EXERCISES

1. The number of couch bushels of malt is $=420$; what are the net bushels? . . . . $=336$.

2. If the number of floor bushels, which has been 30 hours out of the cistern, is $=524$, what are the net bushels? . . $=349\frac{1}{3}$.

3. What is the number of net bushels corresponding to 636 bushels that have been out of the floor for more than 72 hours? $=318$.

During the process of malting, the malt is repeatedly gauged, and the duty is charged on the greatest gauge, after the legal deductions are made, whether that arises from the measurements taken in the couch, frame, or floor. The greatest gauge can be determined by the following problem:—

561. **Problem XIX.—Having given the greatest cistern or couch gauge, or the greatest floor-gauge, to determine which is the greatest or duty gauge.**

RULE.—Multiply the greatest cistern or couch gauge by 1·2 if the floor-gauge has been taken before the malt was 72 hours out of the cistern, or by 1·6 if taken after that time; then, if this gauge is greater than the floor-gauge, it is to be taken for the duty-gauge, otherwise the floor-gauge is to be taken.

Since the cistern and couch gauges are each to be multiplied by the same number 1·2 to obtain the floor-gauge, therefore, in this problem, the couch or cistern gauge is to be taken in preference, according as it is the greater.

EXAMPLE.—If the cistern and couch gauges, after being more than 72 hours out of the cistern, are respectively $=131·2$ and 132, and the floor-gauge $=205·2$ bushels, which will afford the greatest or duty gauge?

The couch-gauge exceeds the cistern-gauge;

hence $132 \times 1·6 = 211·2$.

So that the couch bushels produce 6 bushels more than the floor-gauge.

Exercises

1. Whether will 120 cistern bushels or 146 floor bushels, which have been less than 72 hours out of the cistern, produce the greatest gauge? . . . . The floor-gauge by 2 bushels.

2. Whether will 145 couch bushels or 230 floor bushels, after the malt has been more than 72 hours out of the cistern, produce the greatest gauge? . . . The couch-gauge by 2 bushels.

562. **Problem XX.—To find the content of a cistern, couch, or floor of malt.**

RULE.—Make the malt of as nearly a uniform depth as possible, then measure the length and breadth, and take a number of equidistant depths, the sum of which, divided by their number, will give the mean depth; multiply the length, breadth, and depth together, and their product by ·000451 for the content in bushels.

If the base is not a rectangle, find its area, and multiply it by the depth and by the proper multiplier; or calculate as in the previous rules.

EXAMPLE.—Find the quantity of malt in a rectangular floor, its length being $=48$ inches, its breadth $=32$, and depth, at six different places $=6·1$, $5·8$, $6·3$, $5·9$, $6·4$, and $5·5$.

$$\tfrac{1}{6}(6·1+5·8+6·3+5·9+6·4+5·5)=\tfrac{1}{6}\times 36=6,$$

and $c=48\times 32\times 6\times ·000451=4·156$ imperial bushels.

Exercises

1. What is the content in imperial bushels of a cistern of malt whose length and breadth are $=160$ and 108 inches, and mean depth $=4·68$ inches? . . . . $=36·47$.

2. What is the content of a floor of malt the length of which is $=280$ inches, the breadth $=144$ inches, and the depths at 5 places are $=21·6$, $22·3$, $22·9$, $23·4$, and $23·55$? . . . . . $=413·69$.

3. Find the content of a regular hexagonal cistern of malt, the length of its side being $=269$ inches, and its mean depth $=5$ inches. $=423·6$.

THE DIAGONAL ROD

563. The diagonal gauging-rod is 4 feet long and ·4 of an inch square. The four sides of it contain different lines; the principal one of which is a line for imperial gallons for gauging casks.

The use of the principal line is to determine the content of a cask

of the most common form by merely measuring with the rod the diagonal extending from the bung-hole to the opposite side of the head (BH, fig. to Art. 557)—that is, to the part where the staff opposite to the bung-hole meets the head; then the number on the rod at the bung-hole on the principal or first side is the number of gallons in the content of the cask.

The principal line is constructed thus:—It is found that for a cask of the common form, whose diagonal $d=40$ inches, the content c is $=144$ imperial gallons nearly, and therefore at 40 inches from the end of the rod is placed 144. Hence, if D, C are the diagonal and content of any other similar cask, then, since the contents of similar solids are proportional to the cubes of any two of their corresponding dimensions,

$$d^3 : D^3 = c : C;$$

hence $$C = \frac{c}{d^3}D^3 = \frac{144}{40^3}D^3 = \cdot 00225D^3 = \frac{9}{4000}D^3.$$

From this formula the numbers showing the contents can easily be calculated.

Thus, to find the content C for a diagonal D of 30 inches,

$$C = \tfrac{9}{4000}D^3 = \tfrac{9}{4000} \times 30^3 = 60 \cdot 75.$$

So that at 30 inches from the end of the rod is placed 60·75 gallons, for the content of a cask whose diagonal is 30 inches. In a similar manner, the other numbers showing the content are calculated and marked on the rod.

Another line on a different side of the rod is marked Seg. St. for ullaging a standing cask. Another side contains tables for ullaging lying casks. The remaining side contains lines for ullaging casks of known capacity—as firkins, barrels, &c., either lying or standing.

The diagonal dimension can easily be found by calculation when the usual dimensions are known—namely, the head and bung diameters, and the length. For it is easily perceived, by the figure to Art. 557, when a line is drawn through H, parallel to EF, to meet BA produced in some point P, that $PH=\frac{1}{2}L$ half the length, and $BP=\frac{1}{2}(B+H)$, or half the sum of the bung and head diameters HK, AB, and (Eucl. I. 47)

$$BH^2 = PH^2 + BP^2 = (\tfrac{1}{2}L)^2 + \{\tfrac{1}{2}(B+H)\}^2,$$

or $$D^2 = \tfrac{1}{4}L^2 + \tfrac{1}{4}M^2, \text{ if } M = B+H, \text{ and } D = \text{the diagonal}.$$

The content obtained by using the diagonal rod will not be very correct unless the cask be of the most common form—that is, intermediate between the second and third varieties.

BAROMETRIC MEASUREMENT OF HEIGHTS

564. *The difference of the heights* of mountains, or of other situations, can be determined by means of the atmospheric pressure at these places; and the *absolute height of* one of them above the level of the sea being *known by the* same or any other method, the height of the others above the same *level is also known.*

The principle on which the method is founded is, *that* when various corrections are made for difference of temperature and other variable elements, the differences of heights are proportional to the differences of the logarithms of the atmospheric pressures.

THE THERMOMETER

565. A **thermometer** is an instrument for measuring the **temperature** of bodies—that is, their state with respect to sensible heat.

Bodies are found to change their volume with a change of temperature, and the former change is adopted as a **measure** of the latter. The volumes of most bodies, for any increase or decrease of temperature, undergo a corresponding expansion or contraction. As the change of volume of fluids for a given change of temperature is greater than for solids, they are preferred for the construction of thermometers. But even a fluid expands so little for a moderate change of temperature that particular contrivances are resorted to to render more apparent the real expansion or contraction. The usual method is to enclose the fluid in a glass vessel, AB, consisting of a narrow-bored tube and a hollow bulb, B, formed on one of its extremities. Since the capacity of the bulb is many times greater than that of the tube, the rise or fall of the fluid in the tube, due to any change of volume, will be many times greater than if the tube had not a bulb. The fluid employed is coloured spirit of wine, or, more generally, mercury; and a graduated scale, ED, is attached to the stem to show the expansion. Thus, if the upper part, C, of the mercury is opposite to 57 on

the scale, the temperature is said to be 57 degrees, or 57°. Before the scale can be constructed, at least two points corresponding to two known temperatures must first be found. Two such points, called **fixed points**, can be determined as corresponding to the temperature of any fluid when freezing and boiling under given conditions. The freezing and boiling points of water are generally used.

There are three different methods in use for graduating the scale of a thermometer. When the freezing-point is marked 32°, and the boiling-point 212°, the scale is called **Fahrenheit's**; when the freezing-point is marked 0°, and the boiling-point 100°, it is called **centigrade**; and when the freezing-point is marked 0°, and the boiling-point 80°, it is called **Reaumur's.**

Reaumur's thermometer is not in use in any English-speaking country.

566. **Problem I.—To reduce degrees of temperature of the centigrade thermometer to degrees of Fahrenheit's scale; and conversely.**

RULE.—Multiply the centigrade degrees by 9, and divide the product by 5; then add 32 to the quotient, and the sum is the temperature on Fahrenheit's scale.

From the number of degrees on Fahrenheit's scale subtract 32, multiply the remainder by 5; and the product being divided by 9, will give the temperature in centigrade degrees.

Let t = the temperature on Fahrenheit's scale,

t' = " " the centigrade scale;

then $t = 32 + \frac{9}{5}t'$, and $t' = \frac{5}{9}(t - 32)$.

EXAMPLE.—Find the number of degrees on Fahrenheit's scale corresponding to 20 on the centigrade scale.

$$t = 32 + \tfrac{9}{5}t' = 32 + \tfrac{9}{5} \times 20 = 32 + 36 = 68.$$

Between the freezing and boiling points, there are on the centigrade scale 100°, and on Fahrenheit's 180°; these numbers are proportional to 5 and 9; hence for corresponding temperatures t, t' there will be the proportion

$$(t - 32) : t' = 9 : 5,$$

from which the formulæ are easily obtained.

EXERCISES

1. Find the number of degrees on Fahrenheit's scale corresponding to 25° on the centigrade scale. . . . . . . =77°.

2. Find the temperature on Fahrenheit's scale corresponding to $14°·4$ on the centigrade scale. . . . . $=57°·92$.

3. Find the temperature on the centigrade scale corresponding to $80°$ on Fahrenheit's. . . . . $=26°·\dot{6}$.

COMPARISON OF DIFFERENT LINEAL MEASURES

567. **Problem II.—To reduce metres to imperial feet; and conversely.**

RULE.—Multiply metres by 3·2808, and the product will be the equivalent number of imperial feet.

Multiply imperial feet by ·3048, and the product will be the equivalent number of metres.

Let F = the number of imperial feet,
and M = " equivalent number of metres;
then $F = 3·2808M$, and $M = ·3048F$.

EXAMPLE.—Find the number of imperial feet in 3462 metres.

$$F = 3462 \times 3·2809 = 11358·47.$$

EXERCISES

1. How many imperial feet and fathoms are contained in 6254·6 metres? . . . . . . $=20520·7$ feet, or $3420·1$ fathoms.

2. In 7645 metres how many imperial feet? . $=25082·48$ feet.

OLD AND NEW DIVISIONS OF THE CIRCLE

568. There are 100 centesimal degrees, called also **grades**, in a quadrant, 100 minutes in one of these degrees, and 100 seconds in a minute; this division was used by some French authors; the nonagesimal is the usual division of a quadrant into 90 degrees.

569. **Problem III.—To reduce the centesimal degrees of an arc to nonagesimal degrees; and conversely.**

RULE.—From the centesimal degrees subtract $\frac{1}{10}$ of them, and the remainder is the equivalent number of nonagesimal degrees.

To the nonagesimal degrees add $\frac{1}{9}$ of them, and the sum will be the equivalent number of centesimal degrees.

Let d = the number of nonagesimal degrees,
d' = " equivalent number of centesimal degrees;
then $d = d' - \frac{1}{10}d' = \frac{9}{10}d'$, and $d' = d + \frac{1}{9}d = \frac{10}{9}d$.

The given minutes and *seconds, if there are any,* are to be reduced to the decimal of a degree before applying the rule.

EXAMPLES.—1. Express 60° 45′ 24″ of the centesimal division in degrees of the nonagesimal division.

$$d=\tfrac{9}{10}d'=\tfrac{9}{10}\times 60{\cdot}4524=54°{\cdot}40716=54°\ 24'\ 25{\cdot}776''.$$

2. Convert 54° 24′ 25·776″ of the nonagesimal division into grades.

$$d'=d+\tfrac{1}{9}d=54{\cdot}40716+6{\cdot}04524=60{\cdot}4524=60°\ 25'\ 24''.$$

EXERCISES

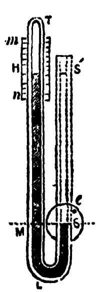
1. Convert 25° 14′ 25·4″ of the centesimal division to degrees of the nonagesimal division. . . . . . . =22° 37′ 41·83″.

2. Express 28° 40′ 28·64″ of the nonagesimal division in terms of the centesimal division. . . . . . . =31° 86′ 6·9″.

[In consequence of the 60 seconds and 60 minutes, the word *sexagesimal* is frequently used instead of nonagesimal.]

THE BAROMETER

570. The **barometer** is an instrument for measuring the weight or pressure of the atmosphere. Air is an elastic fluid, whose density is very sensitive to changes of pressure or of temperature, and is also sensibly affected by the quantity of water vapour present, though within the range of natural temperature this quantity is very small. Atmospheric air being a gravitating body, the pressure caused by it on any surface—as, for instance, a square inch—measures the weight of a column of air whose base is this surface, and whose height extends to the top of the atmosphere; and it is found, by means of the barometer, that this pressure is, at its mean state, nearly equal to the weight of a column of mercury standing on the same base, and having a height of 30 inches.



If HLS′ represent a bent tube with parallel branches standing in a vertical position, and open at both ends at S′ and T, then if mercury be poured into it till it stand at H in one branch, it will rise to S′ in the other to a level with H. But let the branch TL be closed at the top, and let all the air be removed from the region HT above the mercury surface, then it will be found that the column of mercury MH does not require for its support the column in SS′. If the tubes are long enough, it will be found possible to retain in the closed

tube a column of mercury MH about 30 inches long, although the corresponding part SS′ in the open branch is empty of mercury. In short, the column of mercury MH is supported by the pressure of the air on the surface S; and since the weight of a column of mercury 1 square inch in section and 30 inches high is 14·7 lb., it follows that this is the measure of the atmospheric pressure. A round bulb at S, with a small opening at e, is generally made on the end of the barometric tube in order that the surface of the mercury in it at S may be much greater than the surface at H. The surface S will consequently alter its position very little, while the surface H moves up or down over the range of variation corresponding to that of the atmospheric pressure, which is only between three and four inches. The atmospheric pressure changes continually from various causes, and therefore the length of the barometric column varies accordingly, its mean height being from 29·5 to 30 inches at the sea-level according to locality.

Since mercury is subject to a sensible variation of volume from change of temperature (Art. 565), the length of the barometric column must always be reduced to what it would be at some standard temperature, in order to express exactly the atmospheric pressure.

571. **Problem IV.—To reduce the height of the barometer for a given temperature of the mercury to its height for any other proposed temperature.**

Rule.—Multiply the height of the barometer by 10000, increased by the excess of the proposed temperature above 32°; and divide the product by 10000, increased by the excess of the given temperature above 32°, and the quotient will be the required height.

Let h = the required height of barometer,
h' = " given height of barometer,
t = " temperature for height h,
t' = " " " " h';

then
$$h=\frac{10000+(t-32)}{10000+(t'-32)}h'.$$

Example.—If the height of the barometer is = 30 inches when the temperature of the mercury is = 52°, what would its height be for the same atmospheric pressure if the temperature of the mercury were = 87°?

$$h=\frac{10000+(t-32)}{10000+(t'-32)}h'=\frac{10055}{10020}\times 30=30\cdot1047.$$

The volume of mercury varies $\frac{1}{5550}$ of its volume at zero for every change of one centigrade degree of its temperature, or $\frac{1}{5550}\times\frac{5}{9}=\frac{1}{10000}$ nearly for 1° Fahrenheit, the change of volume between the freezing and boiling points being assumed to be uniform for a *mercurial* thermometer.*

Hence, if h_1 = the height of barometer at 32°, its increase for $(t'-32)$ degrees is $=\frac{(t'-32)}{10000}h_1$;

and hence $$h'=h_1+\frac{(t'-32)}{10000}h_1=\frac{10000+(t'-32)}{10000}h_1.$$

Similarly, $$h=\frac{10000+(t-32)}{10000}h_1;$$

and hence $$\frac{h}{h'}=\frac{10000+(t-32)}{10000+(t'-32)};$$

and from this expression the rule is obtained.

When t, t' are within the limits of natural temperature, the more simple formula,

$$k=\frac{t-t'}{10000}h,\text{ and } h=h'+k,$$

may be used, where k is the variation of h' for $(t-t')$ degrees. The maximum error caused by using this formula will be simply ·0006 of an inch, if neither t nor t' should exceed 122°, or the error is less than $\frac{1}{1000}$ part of an inch, or less than the errors of observation in noting the height of the barometer. The preceding example, calculated by this formula, gives $h=30{\cdot}105$.

Exercises

1. Find the height of the barometer by both formulæ for the temperature of 85°, when its height at 60° is = 30·2 inches.
= 30·2753 and 30·2755.

2. If, at the temperature of 87°, the height of the barometer was observed to be = 29·75 inches, what would its height be at the temperature of 69° by both formulæ? = 29·6967 and 29·6964.

RELATION OF VOLUME AND TEMPERATURE OF AIR

572. **Problem V.—Given the volume of a quantity of air at the temperature of 32°, to find its volume at any other temperature, the pressure being the same.**

Multiply the given volume by 9 times the excess of the given

* Biot, *Traité de Physique*, vol. i.

temperature above 32°, and divide the product by 4000, and the quotient will be the increase of volume.

Let v_1 = the volume at 32,

v = " " the given temperature,

t = " given temperature;

then $v = v_1 + \frac{9}{4000}(t-32)v_1$.

It could also be proved, as in the preceding problem, that, if v' be the volume at the temperature t' of the air whose volume is v_1 at 32°,

$$\frac{v'}{v} = \frac{4000+9(t'-32)}{4000+9(t-32)} = \frac{3712+9t'}{3712+9t}.$$

EXAMPLE.—The volume of a quantity of gas at the temperature of 32° was = 1000 cubic inches; what was its volume when its temperature was raised to 52°?

$$v = v_1 + \frac{9}{4000}(t-32)v_1 = 1000 + \frac{9000}{4000} \times 20 = 1000 + 45$$
$$= 1045 \text{ cubic inches.}$$

Air, when heated from 32° to any higher temperature, expands very nearly uniformly—that is, for equal increments of temperature, there are equal increments of volume. For moderate heights in the atmosphere, the decrease of temperature may, for practical purposes, be assumed to be proportional to the increase of height. Then if h_1 is the height of a column extending to a moderate height, when 32° is the **mean temperature**—that is, the temperature at the middle point or half the sum of the extreme temperatures at the lower and upper extremities of the column—its height h, when the mean temperature has any other value t_1, can be found in the same manner as v is found from v_1; or,

$$h = h_1 + \frac{9}{4000}(t_1 - 32)h_1,$$

where t_1 = the mean temperature $= \frac{1}{2}(t+t')$, if t and t' denote the temperatures at the lower and upper ends of the column.

EXERCISES

1. If the volume of a quantity of air at the temperature of freezing is = 2500 cubic feet, what would its volume be at the temperature of 87°? . . . . = 2809·375.

2. If the height of a column of atmospheric air whose mean temperature is 32° is = 5000 feet, what would be its height were the mean temperature 57°? . . . . . . = 5281·25.

MEASUREMENT OF HEIGHTS

573. In the measurement of heights by the barometer, the thermometer by which the temperature of the air is measured is called the **detached** thermometer; and that by which the temperature of the mercury in the barometer is measured is called the **attached** thermometer. At the lower and upper stations, whose difference of level is to be determined, the pressure and temperature of the air in the shade, and the temperature of the mercury in the barometer, are observed; and from these observations the difference of level can be computed. The observations ought to be made during settled weather; and the best time of the day for doing so is between eleven and twelve o'clock—the morning and evening being unfavourable times for this purpose.

574. **Problem VI.—Given the pressure and temperature of the air, and of the mercury in the barometer, at two stations, to find their difference of level.**

METHOD I.—RULE. Reduce the barometric column at the upper station to its length for the temperature of the mercury at the lower station by Art. 571.

Find the difference of the common logarithms of this reduced column and that at the lower station; and this difference, multiplied by 10000, will give the first approximate height in fathoms.

Reduce this height, considered as the length of a column of air at the temperature of freezing, to its length for the mean temperature of the detached thermometers at the two stations by Art. 572; and this reduced length, multiplied by 6, will be the second approximate height in feet.

To this last height add the $\frac{1}{160}$ part of the second approximate height in fathoms, and the sum is the required height in feet.

Let p, p' = the barometric heights at the lower and upper station, S and S′ suppose,

t, t' = the temperatures of the air at S and S′,

T, T′ = " " " mercury at S and S′,

$\frac{1}{2}s = \frac{1}{2}(t + t')$ = the mean value of t and t',

$d = \mathrm{T} - \mathrm{T}'$ the difference of temperatures of mercury,

p_1 = the reduced value of p' to temperature T,

h'', h_1, h', h = " first approximate height in fathoms, the second

approximate height in fathoms and feet, and the required height in feet;

then $p_1 = p' + \frac{1}{10000}dp'$, or $L(p_1 - p') = Ld + Lp' - 4$;

$$h'' = 10000(Lp - Lp_1),$$

and $h = 6\{h'' + \frac{9}{4000}(\frac{1}{2}s - 32)h''\}$;

and $h = h' + \frac{1}{600}h_1$.

Instead of reducing p' to its value at the temperature T, p and p' may both be reduced to their values at any common temperatures, as their ratio will then be always the same; and instead of multiplying the difference of the logarithms of p and p_1 by 10000, the decimal point may merely be removed 4 places to the right. In the preceding formula, p and p' may be expressed in any denomination, provided it be the same; but the temperatures are according to Fahrenheit's scale.

EXAMPLE.—Find the difference of level between two places at which the barometric pressures were observed to be $= 31{\cdot}725$ and $27{\cdot}84$ inches, the temperatures of the air $= 65{\cdot}75°$ and $54{\cdot}25°$, and the temperatures of the mercury $= 60{\cdot}05°$ and $50{\cdot}75°$.

$$p = 31{\cdot}725,\ t = 65{\cdot}75,\ T = 60{\cdot}05,$$
$$p' = 27{\cdot}84,\ t' = 54{\cdot}25,\ T' = 50{\cdot}75;$$

hence $s = 120$, $d = 9{\cdot}3$,

and $p_1 = p' + {\cdot}0001dp' = 27{\cdot}84 + {\cdot}0001 \times 9{\cdot}3 \times 27{\cdot}84 = 27{\cdot}866$.

Lp . .	$= 1{\cdot}5014016$		
Lp_1 . .	$= 1{\cdot}4450746$		
	${\cdot}0563270$, or	$563{\cdot}27$	$= h''$
$\frac{9}{4000}(\frac{1}{2}s - 32)h'' = \frac{9 \times 28}{4000} \times 563 =$		$35{\cdot}47$	
		$598{\cdot}74$	h_1
Then $6h_1$	$=$	$3592{\cdot}44$	$= h'$
And $\frac{1}{100}$ of h_1 . . .	$=$	$5{\cdot}99$	
Required height h . .	$=$	$3598{\cdot}43$ feet.	

575. The preceding rule has been derived from the formula

$$h = 18336\{1 + \tfrac{2}{1000}(t + t')\}(Lp - Lp_1),$$

in which the coefficient is expressed in metres, and the temperatures on the centigrade scale, and p, p_1, are the barometric heights reduced to a common temperature, as in the preceding article. When reduced to imperial fathoms the coefficient is $= 10025$ very nearly, and $\frac{1}{2}(t + t')$ becomes on Fahrenheit's scale $\frac{1}{2}s - 32$, where s = the sum of the temperatures of the air; also $\frac{9}{4000}$ (Art. 572)

must be used for $\frac{1}{250}$ or $\frac{4}{1000}$; so that $\frac{1}{250} \times \frac{1}{2}(t+t')$, or $\frac{2}{1000}(t+t')$, becomes $\frac{9}{4000}\{\frac{1}{2}s - 32\}$, and the formula is

$$h = 10025\{1 + \tfrac{9}{4000}(\tfrac{1}{2}s - 32)\}(\text{L}p - \text{L}p_1).$$

The term $(\text{L}p - \text{L}p_1)$, multiplied by 10000, will give the approximate height in fathoms to within $\frac{1}{15}$ of the whole at its mean value, and the result is therefore h''. To h'' is then to be added the term $\frac{9}{4000}(\frac{1}{2}s - 32)h''$, and the sum is h_1, which, multiplied by 6, will give h' the second approximate height in feet. Since $25 = \frac{1}{400}$ of 10000, there ought now, for the omission of 25 in the coefficient, to be added to h' $\frac{1}{400}$ of h' to give h. But when the formula is complete there is a term in the denominator $= 1 - \cdot 0027 \cos 2l$, dependent on the variation of gravity with the latitude l, and for the mean latitude of Britain, or 54° 30′, this term is nearly $= 1 + \frac{1}{1200}$; hence the coefficient ought, for this latitude, to be reduced about $\frac{1}{1200}$ part and increased $\frac{1}{400}$; and $\frac{1}{400} - \frac{1}{1200} = \frac{1}{600}$; that is, the whole increase of h' ought to be $\frac{1}{600}$ part of h', or merely $\frac{1}{100}$ of h_1 is to be added to h' for the required height in feet.

The preceding method is applicable, with sufficient accuracy, for the height of any mountain in Great Britain. In other cases one of the two following methods must be used :—

576. METHOD 2.—The second method is by means of Tables containing the logarithms of all the possible values of the terms of the formula for every integral value of s, d, and l, the latitude. This method is the most simple, concise, and expeditious.

RULE.—Using the same notation as in the last method, opposite to the value of d in Table II. find the value of B; then let

$$\text{R} = \text{L}p \quad \text{L}p' - \text{B}.$$

Then, opposite to the value of s in Table I. find the value of A; and opposite to the latitude l in Table III. find the corresponding value of C; and let

$$\text{L}h' = \text{A} + \text{C} + \text{L.R}.$$

When the height does not exceed two or three thousand feet, the value of h' thus found will be sufficiently correct; but when the height is considerably greater, find in Table IV. the number of thousands in h', denoted by n in the first horizontal line, and under it is a correction c; also, when the value of s is different from 64, find its value in Table V., in the first horizontal line, and under it is a number k, which, multiplied by n, gives a second correction $c' = nk$, and then the required height is $h = h' + c + c'$.

577. When the lower station is some thousands of feet above the level of the sea, a third correction c'' will be found by multiplying the value of k_1 in Table VI., which corresponds to n', the number

of thousands in this height, by n, the number of thousands in the computed height h, and dividing the product by 10; that is, $c''=\frac{1}{10}nk_1$. This correction is always additive, and

$$h=h'+c+c'+c''.$$

TABLE I

s	A	*s*	A	*s*	A	*s*	A
44	4·76943	75	4·78465	105	4·79890	135	4·81268
45	993	76	513	106	936	136	314
46	4·77042	77	562	107	983	137	359
47	092	78	610	108	4·80030	138	404
48	142	79	658	109	076	139	449
49	192	80	706	110	122	140	494
50	242	81	754	111	168	141	539
51	291	82	801	112	215	142	584
52	341	83	849	113	262	143	629
53	390	84	897	114	308	144	674
54	440	85	945	115	354	145	719
55	489	86	993	116	400	146	764
56	538	87	4·79040	117	447	147	808
57	588	88	088	118	494	148	853
58	637	89	136	119	539	149	898
59	686	90	183	120	584	150	942
60	735	91	231	121	630	151	987
61	784	92	278	122	676	152	4·82031
62	833	93	326	123	722	153	076
63	882	94	373	124	768	154	120
64	931	95	420	125	814	155	164
65	980	96	467	126	859	156	209
66	4·78029	97	515	127	905	157	253
67	077	98	562	128	951	158	298
68	126	99	609	129	997	159	342
69	175	100	656	130	4·81042	160	386
70	223	101	702	131	088	161	430
71	272	102	749	132	133	162	474
72	320	103	796	133	178	163	518
73	369	104	843	134	223	164	561
74	417						

TABLE II

d	B	d	B	d	B	d	B
0	·00000	13	·00056	26	·00113	39	·00170
1	4	14	61	27	117	40	174
2	9	15	65	28	122	41	178
3	13	16	69	29	126	42	183
4	17	17	74	30	·00130	43	187
5	22	18	78	31	134	44	192
6	26	19	83	32	139	45	196
7	30	20	87	33	144	46	200
8	35	21	91	34	148	47	205
9	39	22	96	35	152	48	209
10	43	23	100	36	156	49	213
11	48	24	104	37	161	50	217
12	52	25	109	38	165		

TABLE III

l	C	l	C	l	C	l	C
0	·00117	33	·00048	46	9·99996	59	9·99944
3	116	34	44	47	92	60	41
6	114	35	40	48	88	63	31
9	111	36	36	49	84	66	22
12	107	37	32	50	80	69	13
15	101	38	28	51	76	72	05
18	095	39	24	52	72	75	9·99899
21	087	40	20	53	68	78	893
24	·00078	41	16	54	64	81	889
27	69	42	12	55	60	84	886
30	59	43	08	56	56	87	884
31	55	44	04	57	52	90	883
32	52	45	00	58	48		

TABLE IV

$n=$	1	2	4	6	8	10	12	14	16	18	20
$c=$	2·5	5·2	10·7	16·7	23	29·8	36·9	44·4	52·2	60·5	69·2

TABLE V

$s=$	44	84	104	124	144	164
$k=$	−·06	·06	·11	·17	·22	·26

TABLE VI

$n'=$	1	2	3	4	5	6	7	8	9	10
$k_1=$	1	1·9	2·9	3·8	4·8	5·8	6·7	7·7	8·6	9·6

EXAMPLE.—Find the altitude for the observations in the example of the first method, supposing the latitude to be $=55°$.

$p=31·725$, $t=65·75°$, $\mathrm{T}=60·05°$,
$p'=27·84$, $t'=54·25°$, $\mathrm{T}'=50·75°$,

Hence . . $s=120$, $d=9·3$, and $l=55°$.

L.p . . $=1·50140$ And . L.R $=2·75074$
L.p' . . $=1·44467$ A $=4·80584$
·05673 C $=9·99960$
B $=$ ·00040 Therefore, L.$h'=3·55618$

Hence, . . . R $=$ ·05633 And . . $h'=3599$
By Table IV., the value of c . . $=$ 9·6
" " V., " " c' . . $=$ ·6

Therefore, ... $h=3609·2$

Had the lower station been 6000 feet above the level of the sea, it would have been necessary to add to this value of h the correction from Table VI.; namely, $k_1=5·8$, under $n'=6$, multiplied by $\frac{1}{10}\times 3·6$, and then the value of h would have been

$$3609·2+2·1=3611·3.$$

The value of $h'=3599$ is just $\frac{9}{10}$ of a foot greater than that found by the first method. But as the formula in the first method is adapted to the latitude 54° 30′, had the latitude in the example been assumed considerably different, the results would also have differed considerably. When either the latitude differs consider-

ably from 54° 30′, or the height exceeds three or four thousand feet, the second or third method must be used.

578. METHOD 3.—The altitude may also be computed independently of the preceding Tables, and with equal accuracy, though with more calculation, by means of the complete formula,

$$h=\frac{60160}{1-\cdot 0027\cos 2l}\cdot\left(1+\frac{9}{4000}t_1\right)\left\{\mathrm{L}\frac{p}{p_1}+2\mathrm{L}\left(1+\frac{h}{r}\right)\right\}\left(1+\frac{h}{r}\right),$$

in which h is expressed in imperial feet, t_1 is $=\frac{1}{2}s-32$, p_1 is the reduced value of p' (Art. 571), and r is the radius of the earth in feet at the lower station. The quantity h in the last two factors may, without sensible error, be taken equal to the approximate value of h, as found from the two preceding factors.

The above formula is that given by Poisson, with merely an adaptation to imperial measures and to Fahrenheit's scale; but p and p_1 may be in any denomination provided it is the same in both cases.

579. **Principles of the Method.**—It is proved in the principles of pneumatics that if the altitudes, reckoning vertically from the surface of the earth, are taken in arithmetical progression, the pressures of the air are in a diminishing geometrical progression, supposing the temperature uniform, and the tension of the vapour in it proportional to the pressure. Now, at the height h, let the pressure be p,

and let $h,\ h+k,\ h+2k\ \ .\ \ .\ \ h+nk\ \ .\ \ .$ [1],

and $p,\ rp,\ r^2p\ \ .\ \ .\ \ .\ \ r^np\ \ .\ \ .\ \ .$ [2],

be an arithmetical and a geometrical series, such that any term in the latter (as r^np) denotes the pressure at the height, denoted by the corresponding term of the former (as $h+nk$); then the difference between any two terms of the former series will be proportional to the difference of the logarithms of the corresponding terms of the latter.

For let a be such a number that $a^h=p$, and let m be such a quantity that $h=mk$, then the term $h-mk$ of series [1] has corresponding to it the term $r^{-m}p$ in [2], and the unit of measure of pressure being assumed equal to that corresponding to a height $h=o$—that is, to $h-mk$—it follows that $r^{-m}p=1$, and therefore $p=r^m$, and also $p^{n/m}=r^n$. But since $h=mk$, therefore $nk=nh/m$, and consequently $a^{h+nk}=a^{h+nh/m}=p\,.\,p^{n/m}=r^np$; and hence if $h=\mathrm{L}'.\,p$ for the system whose base is a, then is $h+nk=\mathrm{L}'.\,r^np$ for the same system.

If $h+n'k$ and $r^{n'}p$ are another two corresponding terms of these two series, it is similarly proved that

$$h+n'k=\mathrm{L}'.\,r^{n'}p\,;$$

and therefore if the heights $h+nk$ and $h+n'k$ are denoted by h_1 and h_2, and the pressures r^np, $r^{n'}p$ by p_1 and p_2, then is

$$h_2 - h_1 = \mathrm{L}'.\, p_2 - \mathrm{L}'.\, p_1.$$

And, since the logarithms of the same numbers in different systems are always proportional, if L denote common logarithms, then there is some number M′, such that if N is any number, $\mathrm{L}'.\,\mathrm{N} = \mathrm{M}'\mathrm{L}\,.\,\mathrm{N}$; hence

$$h_2 - h_1 = \mathrm{M}'(\mathrm{L}\,.\,p_1 - \mathrm{L}\,.\,p_2).$$

As the pressures diminish while the heights increase, the exponents of a—that is, the logarithms—will be negative; but this circumstance does not affect the preceding reasoning.

By means of this formula, then, the difference of level of the two stations could be computed, were the air always in the same state, and the mercury in the barometer at the freezing-point of water. But as this is not the case, the value of the pressure must be corrected, as in Art. 571, which introduces the term $p_2(1+\frac{1}{10000}d)$, where $d = \mathrm{T} - \mathrm{T}'$. Again, the mean temperature of the air being different from 32, the column of air must be corrected for this temperature $t_1 = \frac{1}{2}(t+t') - 32$ (by Art. 572); and hence the factor $(1+\frac{9}{4000}t_1)$ is formed. Again, the force of gravity at the mean latitude 45° being considered 1, at any other latitude l it is represented by $1 - \cdot 0027 \cos 2l$, and this expression in the investigation becomes a factor of the denominator. The same investigation introduces also the last factor of the formula $(1+h/r)$ and the term preceding it, or $2\mathrm{L}(1+h/r)$.

When the higher station is situated on a nearly level surface, such as tableland, and its height above the level of the sea is required, instead of the fraction h/r, only $5h/8r$ is to be taken; and for an insulated mountain, $\frac{3}{4}h/r$ would probably be more correct than h/r. The fifth of the following exercises affords an application of the former remark.

When the term h/r of the formula, and that depending on the latitude, are omitted, and the coefficient increased from 18336 metres to 18393 metres—that is, from 10025 fathoms to 10060—the results obtained will be sufficiently correct, if they do not exceed seven or eight thousand feet. This coefficient was empirically determined by Ramond, from numerous barometric and trigonometric measurements of mountains in the Pyrenees. The formula is then

$$h = 10060(1+\tfrac{9}{4000}t_1)(\mathrm{L}p - \mathrm{L}p_1).$$

This coefficient, however, is rather large for the mountains and latitude of Britain.

580. The number M′, which is termed the **barometric modulus**, is found by combined observation and theory to be 18336 metres, or 10025 fathoms nearly, supposing the temperature that of freezing, and the air in a medium hygrometric state. As the mean temperature of the two stations, however, is generally considerably above freezing, and the hygrometric state indicates more than a medium humidity, the air will be of less specific gravity than if it contained only the mean quantity of water vapour at freezing; and if the consequently greater expansion of the aerial column is not taken into account, the computed altitudes will be rather less than the real, and at an average by about $\frac{1}{310}$ part. The expansion from this cause can be computed by means of the principle explained in Art. 572. But the diminution of the temperature in ascending is greater nearer the earth, and consequently the mean temperature $\frac{1}{2}s = \frac{1}{2}(t + t')$ will exceed the temperature at the mean height. Were t_1 only the half of a degree too great, the excess of the computed above the true height would be about $\frac{1}{960}$ part of the latter. It appears, then, that these two errors, being of opposite kinds, tend to compensate each other.

In consequence of these two sources of error, great heights would be more accurately determined by taking observations at intermediate stations, and computing separately the difference of level of every two succeeding stations.

In the following exercises, the heights in Britain are computed by the first two methods, and the corrections from Table IV. are added to the heights found by the first method.

Exercises

1. Find the height of Arthur Seat from these observations:— At Leith Pier the height of the barometer was $= 29 \cdot 567$ inches, attached thermometer $= 55 \cdot 25°$, detached thermometer $= 54°$; on the summit of Arthur Seat the barometer was $= 28 \cdot 704$, attached thermometer $= 51 \cdot 75°$, and detached thermometer $= 50 \cdot 5°$; and $l = 56°$. . . . . $= 801 \cdot 3$ feet.

2. Find the height of the Erzegeberg, near Ilfeld, from these observations:—

$$p = 326 \cdot 5 \text{ lines, } T = 7 \cdot 6°,\ t = 7 \cdot 8°,$$
$$p' = 317 \cdot 8 \quad " \quad T' = 6 \cdot 4°,\ t' = 6 \cdot 2°, \text{ and } l = 51°\ 34'.$$

The height of the barometer is expressed in French lines, and the temperature in Reaumur's scale. . . . . $= 724 \cdot 6$ feet.

3. Required the height of Ben Lomond from these data :—

$p = 30{\cdot}295$ inches, $T = 75{\cdot}5°$, $t = 75{\cdot}5°$,
$p' = 27{\cdot}064$ „ $T' = 60{\cdot}1°$, $t' = 60{\cdot}2°$, and $l = 56°$.

The height of the summit above the upper barometer—that is, above the surface of its cistern—being $= 2$ feet, the height of the lower barometer above the lake $= 2$ feet, and the height of the lake above the sea $= 32$ feet . . . $= 3181{\cdot}3$, or $3182{\cdot}6$ feet.

4. Find the height of the Pic de Bigorre from these observations, the temperatures being on the centigrade scale :—

$p = 73{\cdot}558$ centimetres, $T = 18{\cdot}625°$, $t = 19{\cdot}125°$,
$p' = 53{\cdot}72$ „ $T' = 9{\cdot}75°$, $t' = 4°$, and $l = 43°$.

$= 8579{\cdot}9$ feet.

The altitude of the Pic de Bigorre was found trigonometrically by Ramond to be $= 2613{\cdot}137$ metres, or $8573{\cdot}4$ feet, or $6{\cdot}5$ feet less than the preceding result.

5. Required the height of Guanaxuato from these observations made by Humboldt :—

Centimetres	Centigrade Degrees
$p = 76{\cdot}315$,	$T = t = 25{\cdot}3°$,
$p' = 60{\cdot}095$,	$T' = t' = 21{\cdot}3$, and $l = 21°$.

The height is $6846{\cdot}2$ feet ; but if only $\frac{5}{8}$ of the correction $c + c'$ is taken (Art. 579), the height is $= 6838{\cdot}4$ feet, which is $\cdot 8$ feet more than the height found by Poisson (*Mécanique*, ii. 631).

6. Required the height of Mont Blanc from these observations of Saussure :—

French Inches	On Reaumur's Scale
$p = 27{\cdot}267$,	$T = t = 22{\cdot}6°$,
$p' = 16{\cdot}042$,	$T' = t' = -2{\cdot}3°$, and $l = 45° 45'$;

the summit of the mountain being $= 3{\cdot}3$ feet above the upper station, the lower station $= 116{\cdot}6$ feet above the Lake of Geneva, and the lake $= 1228{\cdot}8$ feet above the level of the sea.

The height is $= 15818{\cdot}1$ feet. The same found geometrically by Corabeuf is $= 15783$, or $35{\cdot}1$ feet less.

7. Find the height of Chimborazo from these observations of Humboldt :—

Centimetres	Centigrade Degrees
$p = 76{\cdot}2$,	$T = 25{\cdot}3°$, $t = 25{\cdot}3°$,
$p' = 37{\cdot}727$,	$T' = 10°$, $t' = -1{\cdot}6°$, and $l = 1° 45'$;

the height of the summit above the upper station being $= 2000$ feet.

$= 21293$ feet.

MEASUREMENT OF DISTANCES BY THE VELOCITY OF SOUND

581. Since the velocity with which sound passes through the atmosphere has been determined with considerable precision, at least to within about a two-hundredth part, the formula expressing that velocity can therefore be employed to determine the distance of the source of any sound, such as the report of a gun or a peal of thunder. All we need to know is the time elapsed between the flash of the powder or of the lightning and the perception of the sound. But this time can easily be found; for the velocity with which the light of the flash is conveyed (namely, 186,000 miles per second) is so great compared with the rapidity of the propagation of sound, that the time required for the former conveyance is practically insensible; and therefore the time elapsed between the perception of the flash and the perception of the sound is just to be reckoned the time required for the propagation of the sound alone.

SOUND

Velocity of sound in air at 32° F.	=	1090	feet per second.	
" " water	=	4900	"	"
" " wet sand	=	825	"	"
" " contorted rock	=	1090	"	"
" " discontinuous granite	=	1306	"	"
" " solid granite	=	1664	"	"
" " iron	=	17500	"	"
" " copper	=	10378	"	"
" " wood	=	11000 to 16700	"	"

Distant sounds may be heard on a still day :—

Human voice,	160 yards.
Rifle,	5350 "
Military band,	5200 "
Artillery—field-guns,	35000 "

582. **Problem.—Given the time required for the conveyance of sound from one place to another, to determine their distance.**

RULE I.—To 1090 add the product of 1·14 multiplied by the excess of the temperature above 32° of Fahrenheit's scale, or subtract it if below 32°, and the sum or difference multiplied by the seconds in the given time will be the distance in feet.

Let $t_1 = t - 32$, t being the temperature of the air,
and v = the required velocity;
then $v = 1090 + 1{\cdot}14t_1$.

Also, let t = the number of seconds observed,
and d = " required distance in feet;
then $d = vt$.

583. RULE II.—If much accuracy is not required, the velocity of sound may be considered as constant and $= 1125$, the velocity obtained from the preceding formula, for $t = 62\frac{3}{4}°$.

For let $t_1 = 62\frac{3}{4} - 32 = 30{\cdot}75$,
then $v = 1090 + 1{\cdot}14 \times 30{\cdot}75 = 1125$ nearly.

EXAMPLE.—Find the distance of a ship, having observed that the report of a gun fired on board of it was heard 10 seconds after the flash was seen; the temperature of the air being 52°.

Here $t_1 = 52° - 32° = 20°$, and $t = 10$ s.;
hence $v = 1090 + 1{\cdot}14 \times 20 = 1090 + 22{\cdot}8 = 1112{\cdot}8$,
and $d = vt = 1112{\cdot}8 \times 10 = 11128$.

584. When there is wind, it will affect the velocity of the conveyance of sound. If the direction of the wind is perpendicular to the direction of conveyance of the sound, it will not materially affect this velocity; although it is found by experience that, from some peculiar influence, it is sensibly altered. If i = the inclination of the direction in which the wind is blowing to the direction in which the sound is moving, v' = the wind's velocity, and v'' = the alteration produced on the velocity of the sound by the wind, then it is easily proved that $v' : v'' = 1 : \cos i$, and $v'' = v' \cos i$;
and hence v becomes $v + v'' = v + v' \cos i$.
When $i > 90°$ its cosine is negative, and v becomes $v - v''$. These conclusions assume that the wind alters the velocity of sound by the quantity of the constituent of its motion, reckoned in the direction of the sound; but it is found by experiment that this is not strictly the case.

The above rules have been deduced from a great variety of experiments made by different philosophers; see Herschel's Treatise

on Sound in the *Encyclopædia Metropolitana.* The velocity of sound is also slightly affected by the hygrometric state of the atmosphere, but the results can only be taken into account in delicate philosophical experiments.

Exercises

1. Find the distance of a thunder-cloud when the time elapsed between the flash of lightning and the thunder is = 6 seconds (by Art. 583). . . . . = 1 mile 490 yards.

2. An echo of a sound was reflected from a rock in 4 seconds after the sound, the temperature of the air being = 60°; required the distance of the rock and the velocity of the sound by the first method. . . . The velocity = 1121·92, and distance = 2243·84.

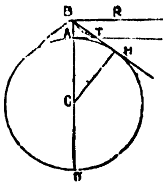
3. Find the velocity of sound for a temperature = 69°, and the distance of a gun when the sound is heard 12 seconds after seeing the flash. . Velocity = 1132·18, and distance = 13586·16 feet.

4. When the temperature of the atmosphere was = 25° centigrade, a peal of thunder was heard 13 seconds after seeing the flash; find the velocity of the sound and the distance of the thunder in miles. Velocity = 1141·3, and distance = 2·81 miles.

MEASUREMENT OF HEIGHTS AND DISTANCES

585. **Problem I.—To find the diameter of the earth when the height of a mountain and the depression of the horizon from its top are given, supposing its form to be spherical.**

Let ADH be the earth, AB the height of the mountain, and BH a line drawn from it to the horizon at H.



Measure AB, the height of the mountain, and the angle of depression of H, or its complement ABH.

Draw AT and BR perpendicular to AB; draw HC from H to the centre C of the earth, and produce the vertical BA through C to D.

The angle HBR of depression of H at B being known, its complement ABH = 90° − HBR is known.

In the triangle ABT, right-angled at A, the side AB and angle B are known; hence find AT and BT thus:—

$$\frac{AT}{AB} = \tan B, \quad \frac{BT}{AB} = \sec B.$$

But $AT = TH$;
hence $BH = BT + TH = BT + AT$.

Then in the triangle BCH, BH is known, and also angle B; hence find CH thus:—$\frac{CH}{BH} = \tan B$; and HC being now found, the diameter $AD = 2HC$.

EXAMPLE.—If from a point 2 miles above the surface of a globe, the angle of depression of the horizon was found to be $= 2° 2'$, required the diameter of the globe.

Angle $ABH = 90°$ $RBH = 90 - 2° 2' = 87° 58'$.

1. To find AT and BT in triangle ABT

L, tan B 87° 58',	= 11·4497317	L, sec B 87° 58', .	= 11·4500052
L, AB 2, . .	= 0·3010300	L, AB 2, . .	= 0·3010300
	11·7507617		11·7510352
	10		10
L, AT 56·33284, .	= 1·7507617	L, BT 56·36834,	= 1·7510352

and $BH = AT + BT = 112·7012$.

2. To find CH in triangle BCH

L, tan B 87° 58', ...	=	11·4497317
L, BH 112·7012, ...	=	2·0519285
		13·5016602
		10
L, CH 3174·39, ...	=	3·5016602

And diameter $AD = 2CH = 6348·78$.

EXERCISE

If the height of the Peak of Teneriffe is $= 12350$ feet, and the depression of the horizon from its summit $= 1° 58' 10''$, required the diameter of the earth. . . . . . . $= 7914·826$ miles.

586. **Problem II.**—The converse problem may now be easily solved—namely, **To find the height of a mountain when the depression of the horizon from its summit and the diameter of the earth is given.**

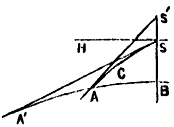
For in the triangle BCH, right-angled at H, the side CH, the earth's radius is known, and also angle CBH the complement of the depression, and hence CB can be computed.

Then $AB = BC - AC =$ required height.

REFRACTION

587. The elevation of objects at a considerable distance is sensibly increased by atmospheric refraction; for instead of a ray of light from any object moving in a straight line through the atmosphere, its path deviates a little from a rectilineal direction, and in ordinary states of the air it is a curve line, the concavity of which is turned towards the earth.

Thus, let BS be the altitude of a mountain, then its summit S is seen from a point A, by means of a ray of light moving in a curvilineal direction SCA; and if AS′ is a tangent to this curve at A, the summit S will appear to be at S′; so that its apparent altitude exceeds its real altitude, or its angle of elevation is increased by the angle SAS′, supposing AS joined by a straight line.



In a similar manner the point A, when seen from S, appears to lie in the direction SA′, and the real angle of depression HSA exceeds the apparent depression HSA′ by the angle ASA′.

The distance of the horizon, seen from any point above the earth's surface, is greater in consequence of refraction than it would be were there no refraction. The former may be called the **actual**, and the latter the **tangential** distance of the horizon.

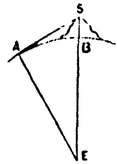
When the actual distance of the horizon is known, the tangential distance is found by subtracting $\frac{1}{12}$ of the former from it; and when the tangential distance is known, the actual distance is found by adding $\frac{1}{11}$ of itself to it.

Hence, also, a height just visible at a given distance when there is refraction would be lower than one just visible at the same distance when there is no refraction. Therefore, if a height be calculated by the preceding method (Art. 586) when the actual distance is given, the height thus computed must be diminished by about $\frac{1}{6}$ part of itself in order to obtain the true height.

588. **Problem III.—Given the diameter of the earth and the height of a mountain, to find the distance of the visible horizon from its summit when the effect of refraction is considered.**

Let S be the summit of the mountain, E the earth's centre, AB its surface, and A the horizon seen from S, supposing no refraction.

In the triangle ASE the side ES is known, for ES = EB + BS, and EB and BS are known; also the side AE is known; and angle A is a right angle; hence find angle E, and then AS; then if $\frac{1}{11}$ of AS be added to it, the sum is the visible distance of the horizon.



EXAMPLE.—Given the height BS of a mountain = 8456 feet, and the diameter of the earth = 7912 miles, to find the distance of the horizon from its summit.

$$BS = 8456 \text{ feet} = \tfrac{8456}{5280} \text{ miles} = 1\cdot6015;$$

hence $$ES = EB + BS = \tfrac{1}{2} \times 7912 + 1\cdot6015 = 3957\cdot6015.$$

1. To find angle E

L, ES 3957·6015, .	= 3·5974321
L, EA 3956, . .	= 3·5972563
	10·
L, cos E 1° 37′ 48″,	= 9·9998242

2. To find AS

	10·
L, sin E 1° 37′ 48″,	8·4540028
L, ES 3957·6, .	3·5974321
L, AS 112·57, .	2·0514349

The distance may also be found thus (Eucl. III. 36):—

Let D = the earth's diameter,

then $AS^2 = BS(D + BS) = 1\cdot6015(7912 + 1\cdot6015) = 1\cdot6015 \times 7913\cdot6015$;

hence $$AS = \sqrt{12673\cdot6328} = 112\cdot57.$$

The tangential distance of the horizon AS = 112·57
$+ \frac{1}{11}$ AS = 10·23
The actual distance . . . . . = 122·8

EXERCISE

At what distance from the summit of Mount Etna is the apparent horizon, the height of the mountain being = 10963 feet?

= 139·846 miles.

589. **Problem IV.—Given the distance at which a mountain is visible at sea, to find its height, the diameter of the earth being = 7912 miles.**

From the given distance deduct $\frac{1}{12}$ of it, and the remainder will be AS (fig. to Art. 588); then in the triangle EAS, EA, the radius of the earth, is known, and AS is given; hence angle E and ES can be found. Then BS = ES − BE is known.

EXAMPLE.—If the distance at which a mountain is visible at sea be = 180 miles, required its height.

The tangential distance of S from the horizon is $\frac{1}{12}$ less, or = 180 − 15 = 165; hence, in the triangle AES, AS must be considered to be only 165 miles.

1. To find angle E

L, AE 3956, . .	=	3·5972563
L, AS 165, . .	=	2·2174839
		10·
L, tan E 2° 23′ 18·12″,	=	8·6202276

2. To find ES

L, sin E, . .	=	8·6198504
		10·
L, AS 165, . .	=	2·2174839
L, ES 3959·437	=	3·5976335
BE = 3956·		
BS = 3·437 miles = 18147 ft.		

EXERCISES

1. The distance at which a mountain is visible at sea is = 142 miles; required its height. . . . . . . = 2·143 miles.

2. The distance at which a mountain is visible at sea is = 120 miles; required its height. . . . . . = 1·852 miles.

590. It can easily be proved that, if e, e' are the two angles of depression of two distant objects, taken at each other, and a the angle at the earth's centre, the refraction, supposing it the same for both, and denoting it by r, is found from the formula

$$2r = a - (e + e').$$

Thus, if a = angle at the earth's centre, found by reckoning 1′ for every geographical mile, or 6076 feet, and if it = 40′ 20″, and if e, e' be respectively 20′ 12″ and 14′ 2″, then $2r = 40' \ 20'' - (20' \ 12'' + 14' \ 2'') = 40' \ 20'' - 34' \ 14'' = 6' \ 6''$, and $r = 3' \ 3''$, or rather more than $\frac{1}{12}$ of a.

If one of the angles, instead of being one of depression, be one of elevation, its sign must be changed. Thus, if e' is an angle of elevation, then

$$2r = a - (e - e') = a + e' - e.$$

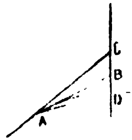
Any distance on the earth's surface may be converted into angular measure by allowing 1° for a geographical mile, or for 69·05 English miles, or 1′ for every 6076 feet.

The effect of refraction varies very much with the state of the atmosphere. In extreme cases the variation is from $\frac{1}{5}$ to $\frac{1}{18}$ of the distance; but in ordinary states of the atmosphere it varies from $\frac{1}{10}$ to $\frac{1}{14}$, of which the mean is $\frac{1}{12}$. By French mathematicians it is reckoned at about ·079 of this distance.

When great accuracy is required, small angles of elevation must be diminished or small angles of depression increased by $\frac{1}{12}$ of the distance, or 5″ for 6076 feet; that is, 1″ for 1215·2 feet, or 1″ for every 405 yards nearly.

591. Another correction is also necessary, when great accuracy is required, on account of the earth's curvature. Thus, if CD

be the vertical height of an object, and AB a horizontal line from A, the angle of elevation CAB ought to be increased by BAD, which is half the angle at the earth's centre, subtended by the arc AD. Hence half-a-minute must be added for every 6076 feet of distance, or 1″ for every 202·5 feet, or $67\frac{1}{2}$ yards. The angle ADC also exceeds a right angle by the same quantity, or it is $= 90° +$ half the angle subtended by the distance AD.



Thus, if AD = 5280 feet, or 1 English mile, and the angle of elevation CAB 12° 4′, it must be increased by $\frac{5280}{202\cdot5}\times 1'' = 26''$, and it becomes 12° 4′ 26″ = angle CAD; also angle ADC = 90° 0′ 26″. There are then two angles of the triangle ADC—namely, A and D—known, and the side AD; hence CD can be calculated. But when CAB is a small angle it must be diminished, on account of refraction, by 1″ for every 405 yards, or $\frac{1760}{405}\times 1'' = 4\cdot3''$.

592. The following example is here given as an illustration of the application of these corrections.

Example.—Given the angle of elevation = 15′ 0″, and the distance AD = 20·17 nautical miles, to find the elevation of C above A.

The angle CAB of true elevation is found by deducting the refraction from the observed elevation of C above A; and as there are 60 nautical miles in 1°, therefore

AD = 20·17 miles, ...	=	20′ 10″
Refraction = $\frac{1}{12}$ of AD, ...		1 41
Observed elevation, ...		15 0
True elevation CAB, ...	=	13 19
Also BAD = $\frac{1}{2}$(20′ 10″), ...	=	10 5
Hence CAD, ...	=	23 24

and ADC = 90° + 10′ 5″ = 90° 10′ 5″.

To find CD in triangle ACD

C = 180° − (A + D) = 180° − 90° 33′ 29″ = 89° 26′ 31″.

L, cosec C 89° 26′ 31″, . . .	=	10·0000206
L, sin A 0° 23′ 24″, . . .	=	7·8329386
L, AD 20·17, ...	=	1·3047059
L, CD in nautical miles, . . .	=	$\bar{1}$·1376651
L, 6076 feet (in a mile), . . .	=	3·7836178
L, CD in feet, ...	=	2·9212829

Hence, ... CD = 834·2.

The relative height of the point C above A is therefore 834·2 feet; but if the latter point were, say, 240 feet above the level of the sea, then the absolute height of C above the sea would be $=834{\cdot}2+240=1074{\cdot}2$.

CONCISE FORMULÆ FOR HEIGHTS

593. The distance at which the summit of an object may be seen at sea, when its height is known, and the height of an object when the distance at which its summit can be seen at sea is known, may be found more simply thus:—

Let D = the diameter of the earth,
h = " height of the object,
d = " distance at sea at which the summit of the object is visible;

then $d^2=(D+h)h=Dh+h^2$.

Now, h^2 will be very small compared with Dh, for h is so compared with D. If h were 3·956 miles, it would just be $\frac{1}{2000}$ part of D, and the error produced on the value of d^2 by rejecting the term h^2 would just be $\frac{1}{2000}$ part of h in defect. The formula then becomes

$$d^2=Dh, \text{ and } h=\frac{d^2}{D}, \text{ also } D=\frac{d^2}{h}.$$

When d is 100 miles, it would give an error on the value of h of about $\frac{1}{6000}$ part in excess. When h or d is less, the errors are also less.

The formula may be simplified by taking d in miles and h in feet; then since $d^2=\frac{1}{5280}Dh=\frac{7912}{5280}h$, and $\frac{7912}{5280}=\frac{3}{2}$ nearly,
therefore $d^2=\frac{3}{2}h$, and $h=\frac{2}{3}d^2$,
where the denomination of d is miles, and that of h is feet.

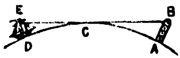
Since $\frac{7912}{5280}=1{\cdot}4985$, and $\frac{3}{2}=1{\cdot}5$, therefore $\frac{3}{2}$ is $\frac{15}{10000}=\frac{3}{2000}$ too great. The value of d^2, therefore, in miles will be too great by the $\frac{3}{2000}$ part of h in feet; and the value of h in feet will be about the $\frac{2}{3000}$ part of d^2 in miles too small.

The former simplification makes errors on the values of d and h respectively in defect and excess, and the latter in excess and defect; and they thus to some extent compensate each other.

594. When the summit of one object is just visible from that of another, the line joining them being a tangent to the surface of the sea, the distance at which each of the objects separately is visible must be calculated, and the sum of these is the whole distance at which they are mutually visible.

Thus, if AB is a lighthouse, just visible from the mast E of a

ship, the whole distance EB, or DCA, is just the sum of the distances DC and CA. Now, the height DE being given, the distance CD can be found; and AB being known, CA can be calculated; and hence the whole distance DCA can be found.



The formula $d^2 = \frac{3}{2}h$ gives the value of d, were there no refraction; and if to this value of d is added $\frac{1}{11}$ of itself, the result will be the value of d, increased by the effect of mean refraction. So the formula $h = \frac{2}{3}d^2$ gives h too great when d is the apparent distance; and if $\frac{1}{12}$ of d is subtracted from d, the formula, with this reduced value of d, will give the corrected value of h. Or the formula becomes, with this reduction, $h = \frac{2}{3}(d - \frac{1}{12}d)^2 = \frac{2}{3}(\frac{11}{12})^2 d^2 = \frac{5}{9}d^2$ nearly, and $d^2 = \frac{9}{5}h$; and $h = \frac{5}{9}d^2$. Also, since $\frac{2}{3} - \frac{5}{9} = \frac{1}{9}$, and $\frac{1}{9}d^2 = \frac{1}{9} \times \frac{3}{2}h = \frac{1}{6}h$, therefore the formula $h = \frac{5}{9}d^2$ gives a value of h about $\frac{1}{6}$ of itself less than the other, or $h = \frac{2}{3}d^2$.

EXAMPLES.—1. At what distance can an object = 24 feet high be seen at sea?

$$d^2 = \tfrac{3}{2}h = \tfrac{3}{2} \times 24 = 36, \text{ and } d = 6 \text{ miles.}$$

This is the distance were there no refraction; but the distance is increased $\frac{1}{11}$ by refraction; hence the corrected distance = 6·55 miles.

Or, $$d^2 = \tfrac{9}{5}h = \tfrac{9}{5} \times 24 = 43·2, \text{ and } d = 6·57.$$

2. From what height will the horizon be = 12 miles distant?

$$h = \tfrac{2}{3}d^2 = \tfrac{2}{3} \times 12^2 = 96 \text{ feet.}$$

But refraction makes it visible $\frac{1}{12}$ farther; hence h must be less,

or $$h = \tfrac{2}{3}\left(d - \frac{d}{12}\right)^2 = \tfrac{2}{3} \times 11^2 = 80·6 \text{ feet.}$$

Or, $$h = \tfrac{5}{9}d^2 = \tfrac{5}{9} \times 12^2 = 80 \text{ feet.}$$

CURVATURE AND REFRACTION

D	C	C−R	D	C	C−R	D	C	C−R
1	·66	·57	6	24·	20·57	12	96	82
2	2·67	2·29	7	32·67	28·00	14	130	112
3	6·	5·14	8	42·67	36·57	16	170	146
4	10·67	9·14	9	54·	46·30	18	216	185
5	16·67	14·29	10	66·67	57·14	20	266·7	228·6

D = distance in statute miles, C = curvature in feet = $\frac{2}{3}D^2$ approximately, C − R = curvature less refraction = $\frac{4}{7}D^2$ approximately.

In the following exercises the effect of refraction is taken into account.

EXERCISES

1. At what distance can an object = 54 feet high be seen at sea? = 9·8 miles.

2. At what distance can the top of a lighthouse = 216 feet high be seen at sea? . . . . = 19·7 miles.

3. Required the distance of the visible horizon from the top of Arthur Seat, which is = 820 feet high? . . . = 38·26 miles.

4. From what height will the horizon be = 36 miles distant? = 720 feet.

5. From a ship's mast at the height of 120 feet, the top of a lighthouse = 240 feet above the level of the sea was just visible; required the distance of the ship and lighthouse. . = 35·5 miles.

6. If from the summit of a mountain = 11,310 feet high, the distance of the visible horizon is = 142 miles, required the earth's diameter. . . . . = 7910 miles.

LEVELLING

595. The object of **levelling** is to determine the difference between the true and apparent level at one place in reference to another, or the difference of true level of two places.

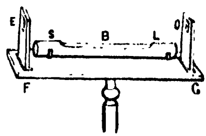
596. A line of **true level** is such that all points in it are equally distant from the centre of the earth.

597. A line of **apparent level** at any place is a horizontal line passing through that place.

Let MN be an arc of the earth's surface, and LT another concentric with MN, and LP a tangent to the arc LT at L. Then L and P are in the same apparent level when P is seen from L; also L and T are on the same true level; and PT is the difference between the true and apparent level in reference to L at a distance from it equal to LT.

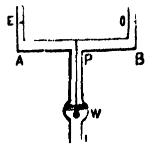
The point P, on apparent level with L, is found by means of a level placed horizontally at L.

598. The **spirit-level** (SL) consists of a glass tube nearly filled with spirit of wine, and enclosed in a brass tube, except the upper part. It is sometimes placed parallel to the axis of a telescope, and when brought to a level, a point at a distance may be found on the same level with the axis of the telescope, by looking through it to a pole or other object at a distance, and finding the point on it that appears to coincide with the intersection of two very fine wires that cross each other within the telescope.



The spirit-level is also sometimes attached to a bar of brass FG, with two upright pieces FE, GO, and small openings or **sights** at E and O, so placed as to be on a horizontal line when the level SL is horizontal, which is the case when an air-bubble at B is at the middle point.

The **plumb-line level** is furnished with sights like the spirit-level, or with a telescope. The horizontal position of the sights E, O is determined by the vertical position of the plummet PW.



The **fluid-level** consists of a tube EPO filled with some fluid to E and O, which are therefore on the same level.

Square staffs are also used in levelling. They are wooden rods, divided into feet and parts of a foot, with movable vanes; and when used, are fixed vertically in the ground.

599. Problem I.—To find the difference between the true and apparent level for any given distance.

PT (fig. to Art. 597) is the difference between true and apparent level for the distance MN, and may be found by the formula $h = \frac{2}{3}d^2$, where h is in feet and d in miles.

But if refraction is taken into account, d must be previously diminished by $\frac{1}{14}$ part, or the formula $h = \frac{4}{7}d^2$ employed. In

levelling, however, the distances are generally small, seldom more than 300 or 400 yards; and this correction for so short a distance may generally be neglected.

EXAMPLES.—1. A place at the distance of a mile from another is on the same apparent level with the latter; what is the height of the former above the point of true level with the latter?

Here $d = 1$ mile, and $h = \frac{2}{3}d^2 = \frac{2}{3} \times 1 = 8$ inches.

2. What is the difference between true and apparent level at the distance of 2022 feet?

Here $d = \frac{2022}{5280}$ miles $= \cdot 383$; hence $h = \frac{2}{3}d^2 = \frac{2}{3} \times \cdot 383^2$, or $h = \cdot 0978$ foot $= 1 \cdot 176$ inches.

3. Required the difference between true and apparent level at a distance of 4 miles.

$$h = \frac{2}{3}d^2 = \frac{2}{3} \times 4^2 = \frac{2}{3} \times 16 = 10\tfrac{2}{3} \text{ feet.}$$

EXERCISES

1. Required the difference between true and apparent level at the distance of $2\frac{1}{2}$ miles. . . . . . . $= 4$ feet 2 inches.

2. What is the difference between true and apparent level at the distance of 1240 feet? . . . . $= 0 \cdot 44$ inch.

3. Required the difference between true and apparent level at the distance of 1760 feet. . . . . $= \cdot 88\dot{8}$ inch.

4. What is the difference between true and apparent level at the distance of $1\frac{1}{4}$ miles? . . . . $= 12\frac{1}{2}$ inches.

5. If at a point in the surface of a canal it is found that for a distance of $3\frac{1}{2}$ miles the surface of the earth is on an apparent level with it, required the depth of the surface of the canal below the surface of the earth at that distance. . . $= 8$ feet 2 inches.

600. It is convenient to have formulæ when the distance is given in feet, yards, or chains, to find the difference of true and apparent level in inches.

When d is yards and h inches, $\frac{h}{12} = \frac{2}{3}\left(\frac{d}{1760}\right)^2$,

or $$h = \frac{8d^2}{8^2 \times 220^2} = \frac{d^2}{8 \times 220^2}, \quad h = \cdot 000002583d^2.$$

When d is feet and h inches, then, instead of d^2 in the preceding formula, substitute $\left(\frac{d}{3}\right)^2 = \frac{1}{9}d^2$, and $h = \cdot 000000287d^2$.

When d is imperial chains and h inches, then, since 80 chains $= 1$ mile,

$$\frac{h}{12} = \frac{2}{3}\left(\frac{d}{80}\right)^2, \text{ or } h = \frac{8d^2}{80^2} = \frac{d^2}{8 \times 10^2}, \quad h = \frac{d^2}{800} = \cdot 00125d^2.$$

The two following formulæ, in which logarithms are used, may sometimes be conveniently employed. Taking the formula when d and h are expressed in miles, or $d^2 = Dh$, it may be altered thus:—

When d and h are in feet,

$$\left(\frac{d}{5280}\right)^2 = \frac{Dh}{5280} = \frac{7912}{5280}h, \text{ or } h = \frac{d^2}{7912 \times 5280},$$

and $$Lh = 2Ld + \bar{8}\cdot3790798.$$

When h is feet and d imperial chains,

$$\left(\frac{d}{80}\right)^2 = \frac{7912}{5280}h, \text{ and } h = \frac{33d^2}{7912 \times 40},$$

and $$Lh = 2Ld + \bar{4}\cdot0181675.$$

Similarly, when h is inches and d chains,

$$Lh = 2Ld + \bar{3}\cdot0973487.$$

EXAMPLES.—1. What is the difference between true and apparent level at the distance of 3540 feet?

$$h = \cdot000000287d^2 = \cdot000000287 \times 3540^2 = 3\cdot59 \text{ inches.}$$

Or, $$Lh = 2Ld3540 + \bar{8}\cdot3790798 = 2 \times 3\cdot5490033 + \bar{8}\cdot3790798$$
$$= 7\cdot0980066 + \bar{8}\cdot3790798 = \bar{1}\cdot4770864 = L0\cdot299976 \text{ feet,}$$
or $3\cdot5997$ inches.

2. Find the difference between true and apparent level corresponding to a distance of 400 chains.

$$h = \frac{d^2}{800} = \frac{400^2}{800} = \frac{400}{2} = 200 \text{ inches} = 16 \text{ feet } 8 \text{ inches.}$$

EXERCISES

1. What is the difference between true and apparent level at the distance of 3100 feet? . . . . . . 2·76 inches.

2. Required the difference between true and apparent level for a distance of 140 chains. . . . . . . =2·0437 feet.

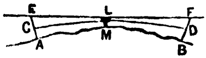
3. Required the difference of true and apparent level for a distance of 166 chains. . . . . =2·8733 feet.

601. Problem II.—To find the difference of true level of two places on the surface of the earth not far distant.

RULE.—In each of the vertical lines passing through the two places, find a point on the same true level with some

intermediate point, and the difference of the vertical heights of these two points above the given points is the difference of true level.

Thus, let A and B be the two places on the earth's surface. Place a level at L, some intermediate position, and two square staffs at A and B, and find two points E and F on the same apparent level with L; and measure the heights AE, BF, and the distances MA, MB; then calculate the distances EC and FD of true level below apparent level by last problem.



Were the instrument L placed in the middle between the two places, the points E and F that are on apparent level would evidently be also on a true level; for then CE would be equal to DF, though the distances and refraction were considerable.

Having found EC and FD, the points C and D of true level are then known; and hence AC, BD are known, and their difference is the difference of true level. If AC exceed BD, then A is evidently lower than B.

EXAMPLE.—Let the distance of the level from the two stations A and B be = 240 yards and 300 yards; let AE and BF be = 10 and 6 feet respectively; what is the difference of true level of A and B?

$$EC = h = \cdot 000002583d^2 = \cdot 147 \text{ inch},$$
$$DF = h = \cdot 000002583d^2 = \cdot 232 \text{ inch};$$

hence $AC = AE - EC = 10 \text{ ft.} - \cdot 148 \text{ in.} = 9 \text{ ft. } 11\cdot 852 \text{ in.}$
$BD = BF - DF = 6 \text{ ft.} - \cdot 232 \text{ in.} = 5 \text{ ft. } 11\cdot 768 \text{ in.}$

Therefore $AC - BD = 119\cdot 852$ inches $- 71\cdot 768$ inches $= 48\cdot 084$ inches $= 4$ feet $\cdot 084$ inch = the height of the point B above A.

Were the instrument L in the middle between A and B, then E and F would be in the same true level, and the difference of level of A and B would be $= 10 - 6 = 4$ feet.

EXERCISES

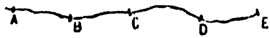
1. Find the difference between the true level of two places A and B, having given the distances AM, MB, 1040 and 1820 feet, and the heights AE, BF, of apparent level with L, 5 feet and 6 feet respectively. . . . . $= 11\cdot 36$ inches.

2. Let the distances AM, MB be = 12 and 18 chains, the heights AE, BE = 3 feet 2 inches and 5 feet 8 inches; find the difference of true level of A and B. . . . . . . $= 2$ feet $5\cdot 77$ inches.

602. Problem III.—To find the difference of true level of two places at a considerable distance.

Let A and E be the two places.

Take intermediate places B, C, D, so that the distances of any two successive places may not exceed a quarter of a mile. By last problem, find the difference of true level of A and B by means of observations taken with a level at some convenient station between A and B. Find in a similar manner the difference of true level of B and C, C and D, D and E; then the difference of level of A and B is easily found thus:—



When one place is higher than the next succeeding, reckon the difference of level positive; and when lower, negative; find the sum of the positive and also of the negative, and then the difference of these sums is the difference of level of the first and last places. The first place is higher than the last, when the sum of the positive numbers exceeds that of the negative, and lower when the contrary is the case.

Example.

Let A be	4 feet	3 inches	higher than	B, or	+4 feet	3 inches,
B "	3 "	2 "	lower "	C, "	−3 "	2 "
C "	2 "	6 "	higher "	D, "	+2 "	6 "
D "	3 "	8 "	lower "	E, "	−3 "	8 "

Sum of positive, ... =6 feet 9 inches.
" negative, ... 6 " 10 "

Difference, ... =0 feet 1 inch.

Hence A is 1 inch lower than E.

Exercises

1. Let A be = 10 feet above B, B = 8 feet below C, and C = 12 feet above D; find the difference of level of A and D.

A is = 14 feet above D.

2. Let A be = 12 feet 4 inches above B, B = 8 feet 3 inches below C, C = 10 feet 11 inches above D, and D = 3 feet 2 inches below E; what is the difference of level of A and E?

A is = 11 feet 10 inches above E.

603. Problem IV.—To find the difference of level between two objects when the observations are taken nearly in the middle between every two successive stations.

A **back** observation is one taken on a staff behind the station, and a **fore** observation is one taken on a staff before the station—that is, in the direction in which the observer is advancing with his operations.

In this method the effects of refraction and of the earth's curvature are the same for each pair of back and fore observations taken at the same station, so that the points of apparent level for these two observations are also points of true level; and thus no correction is necessary for either curvature or refraction.

RULE.—Find the sum of the back and fore observations separately; the excess of the former above the latter will show the ascent from the first to the last station, or the excess of the latter above the former will show the descent.

EXAMPLE.—From stations at nearly equal distances between the points A and B, B and C, C and D, D and E, the observations were as in the following Table; find the difference of level of A and E.

Number of Station	Distance of Station from	Distance of Station from	Back Observation	Fore Observation
1	A 200	B 200	4·2	1·5
2	B 345	C 342	2·3	5·7
3	C 500	D 504	2·1	3·9
4	D 1285	E 1280	9·5	4·2
	2330	2326	18·1	15·3
	2326		15·3	
	4656		2·8	

Hence the height of E above A is 2·8 feet, and the distance is =4656 feet.

EXERCISES

1. What is the difference of level of A and D, and their distance, taking the data from the last example?

D is = 2·5 feet lower than A, and the distance is = 2091 feet.

2. Required the height of the point A above E, and their dis-

tance from the data in the subjoined Table, arranged as in the preceding example.

Number of Station	Distance of Station from	from	Back Observation	Fore Observation
1	A 150	B 150	3·5	2·5
2	B 542	C 542	4·3	3·2
3	C 253	D 253	2·7	8·5
4	D 751	E 753	7·4	9·6

A above E 5·9 feet, and their distance = 3394 feet.

STRENGTH OF MATERIALS AND THEIR ESSENTIAL PROPERTIES.

604. The properties of matter are almost innumerable, but they may be divided into two classes—(1) Essential properties; (2) Contingent properties. The essential properties are those without which matter cannot possibly exist. The contingent properties are those which we find matter possessing, but without which we could conceive it to exist.

Essential Properties.—(1) **Extension** means that property by which every body must occupy a certain bulk or volume. When we say that one body has the same volume as another, we do not mean that it has an equal quantity of matter, but only that it occupies an equal space.

(2) **Impenetrability** means that every body occupies space to the exclusion of every other body, or that two bodies cannot exist in the same space at the same time.

Contingent Properties.—(1) **Divisibility** means that matter may be divided into a great but not an infinite number of parts. The ultimate particles of matter are termed *atoms*, derived from a Greek word signifying indivisible.

(2) **Porosity** signifies that every body contains throughout its mass minute spaces or interstices to a greater or less extent. This has been proved to be the case with many substances, and there is evidence that leads us to believe it to be true for all.

(3) **Density** is that property by which one body differs from another in respect of the quantity of matter which it contains in a given volume. The density of a substance is either the number of units of mass in a unit of volume, in which case it is equal to the heaviness (that is, weight of unit volume of substance in standard units of weight); or it is the ratio of the mass of a given volume of the substance to the mass of an equal volume of water, in which case it is equal to the specific gravity.

(4) **Cohesion** is that property by which particles of matter mutually attract each other at insensible or indefinitely small distances. It is generally regarded as differing from gravitation, which acts at all distances. It is, however, conceivable that the two kinds of attractive forces may be fundamentally the same.

(5) **Compressibility** and **dilatability** are properties common to all bodies, by which they are capable of being compressed like a sponge, or extended like a piece of india-rubber, in a greater or less degree.

(6) **Rigidity** signifies the stiffness to resist change of shape when acted on by external forces. Unpliable materials which possess this property in a large degree are termed *hard*, whilst those which readily yield to pressure are called *soft*. Substances which cannot resist a change of shape without breaking are termed *brittle*, whilst those that do resist, and at the same time change their form, are said to be *tough*.

(7) **Tenacity** is the resistance (due to cohesion) which a body offers to being pulled asunder, and it is measured by the tensile strength in lb. per square inch of the cross section of the body.

(8) **Malleability** is that property by which certain solids may be rolled, pressed, or beaten out from one shape to another without fracture. It is therefore a property depending upon the softness, toughness, and tenacity of the material.

(9) **Ductility** is that property by which some metals may be drawn through a die-plate into wires or tubes. A metal is said to be *homogeneous* when it is of the same density and composition throughout its mass. It is *isotropic* when it has the same elastic properties in all directions.

(10) **Elasticity** is that property possessed by all substances in a greater or less degree of regaining their original size and shape after the removal of the force which caused a change of form.

When a solid does not return to its original form or shape after the force has been removed, it has been stretched beyond the elastic limit of the material.

(11) **Fusibility** is that property whereby metals and many other substances, such as resins, tallows, &c., become liquid on being raised to a certain temperature.

The following Table shows in round numbers the melting-points of a few of the commoner metals:—

MELTING-POINTS OF METALS IN DEGREES FAHRENHEIT

Mercury,	- 38	Copper,	2000
Tin,	+440	German silver,	2000
Bismuth,	500	Gold,	2000
Lead,	600	Cast-iron,	2200
Zinc,	700	Steel,	2500
Antimony,	800	Nickel, also Aluminium,	2800
Brass,	1800	Wrought-iron,	3300
Silver,	1850	Platinum,	3500

605. It is convenient to introduce here the definition and properties of the moment of inertia and the radius of gyration. If the mass of every particle of a body be multiplied by the square of its distance from a given axis, the sum of the products is called the **moment of inertia** of the body about that axis.

If M be the mass of a body, and k be such a quantity that Mk^2 is the moment of inertia about a given axis, then k is called the **radius of gyration** of the body about that axis.

Thus, $Mk^2 = I =$ moment of inertia;

or, $k^2 = \frac{I}{M} =$ square of radius of gyration.

A cylinder can be conceived as made up of a great number of circular discs threaded together on the same axis, and the moment of inertia will just be the sum of the moments of inertia of all the discs. Since the radius of gyration of each disc is independent of the thickness of the disc, it follows that the radius of gyration of the whole cylinder will be the same as that of one of the discs.

The term 'moment of inertia' has been defined above with respect to a solid body only, but it is easy to see that by a slight alteration in the wording of the definition it may be made to apply equally to an area or a section of a solid. Accordingly, we find the

terms 'moment of inertia' and 'radius of gyration' applied to areas as well as solids.

For instance, we speak about the moment of inertia and radius of gyration of a circle about a diameter, of a triangle about its base, and so on.

The moment of inertia of a solid, or section of a solid, about a given axis is always proportional to the mass of the solid, or to the area of the section, as the case may be.

The following rule has been stated by Routh, and will be found useful for finding the moments of inertia about an axis of symmetry :—

Moment of inertia = mass × (sum of the squares of the perpendicular semi-axes) ÷ (3, 4, or 5, according as the body is rectangular, elliptical, or ellipsoidal).

The Table on p. 329 gives the radius of gyration for certain sections, $R = \sqrt{\frac{I}{M}}$.

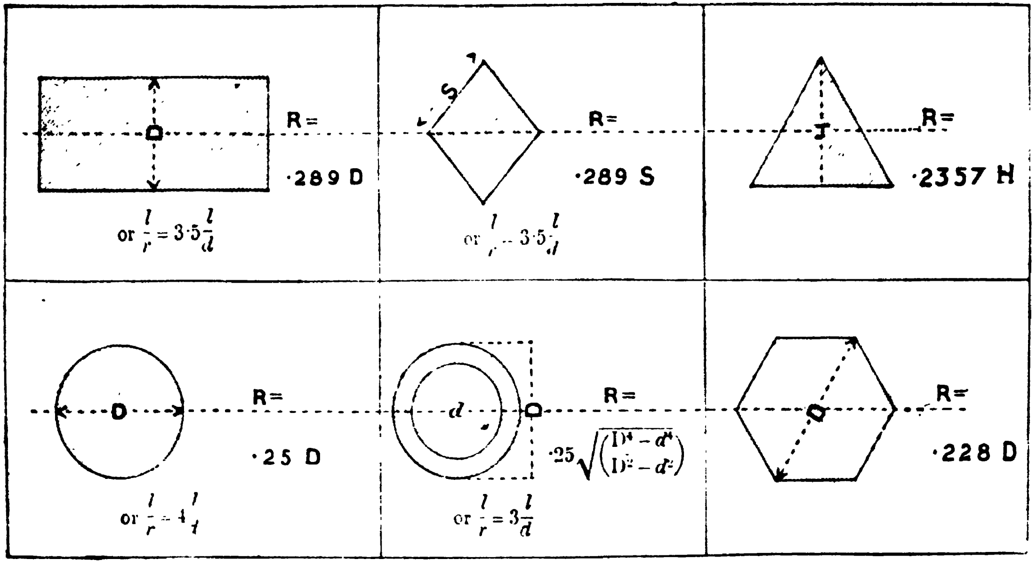
606. **Load, Stress, and Strain.**—When force is applied to a body so as to produce either elongation or compression, bending, torsion, shearing, or a tendency to any of these, the force applied is termed the *load;* the corresponding resistance, or reaction in the material, is termed the *stress* due to the load. Any alteration produced in the length, volume, or shape of the body is termed the *strain.*

Tensile Stress and Strain.—If the line of action of a load be along the axis of a bar, tie-rod, or beam, so as to tend to elongate it, the reaction per square inch of cross section is termed the **tensile stress,** and the elongation per unit of length is called the **tensile strain.**

607. **Young's modulus of elasticity** of any substance is the ratio of the tensile strength to the tensile strain. Thus Young's modulus

$$= \frac{\text{stress}}{\text{strain}} = E = \frac{P}{A} \Big/ \frac{l}{L}; \text{ or } PL = AlE,$$

where P = pull, push, or load in lb. on the bar,
A = area of cross section of the bar,
L = length of bar before the load was applied,
l = length by which the bar is extended or compressed,
p = stress or load per square inch of cross section = P/A.



The horizontal dotted lines through the figures denote the axis of rotation.

NOTE.—l=length in inches; r=radius of gyration; d=diameter in inches, or side if square; if rectangular, d=thickness.

Then, so long as $\frac{P}{A}$ does not exceed the elastic limit, l varies directly as P for the same bar; or $\frac{l}{L}$ varies directly as $\frac{P}{A}$ for different bars of the same material, and subjected to the same conditions. In other words, so long as the stress does not exceed the elastic limit, the strain will be proportional to the stress.

Hooke's law holds good for metal bars under the action of forces tending to elongate or compress them. This law states:—

(1) The amount of extension or compression for the same bar is in direct proportion to the stress.

(2) The extension or compression is inversely proportional to the cross sectional area; consequently, if the area be doubled the extension or compression will be halved, or the resistance to the load will be doubled.

(3) The extension or compression is directly proportional to the length. Since stress is reckoned by so many lb. per square inch of cross section of a material, and strain is simply a ratio, it follows that the modulus of elasticity (E) must also be reckoned by so many lb. per square inch.

EXAMPLE.—A steel bar 5 feet long and $2\frac{1}{4}$ square inches in cross section is suspended by one end; what weight hung on the other end will lengthen it by ·016 inch, if Young's modulus for steel is 30000000 lb. per square inch?

Now, the universal rule is, modulus of elasticity $=\frac{\text{stress}}{\text{strain}}$, or stress $=$ modulus $\times$ strain. For the strain is the elongation per unit of the length.

Consequently, $l=\frac{\cdot 016}{5' \times 12''}=\frac{\cdot 016}{60}=\cdot 0002\dot{6}$.

$$\therefore \text{ the stress} = \text{modulus} \times \text{strain} = 30000000 \times \cdot 0002\dot{6}$$
$$= 8000 \text{ lb. per square inch;}$$

and the total stress $=8000$ lb. $\times 2{\cdot}25$ square inches $=18000$ lb. Or, we might have applied the formula previously deduced—namely, $PL=AlE$, where P is the total pull required in lb.

$$\therefore P=\frac{AlE}{L}=\frac{2{\cdot}25 \text{ square inches} \times \cdot 016'' \times (30 \times 10^6)}{5 \times 12}=18000 \text{ lb.}$$

When the limit of elasticity is exceeded, the strain increases at a much greater rate than the stress producing it.

The constant E is termed by some writers the 'modulus' or 'coefficient of elasticity;' but such a term is inappropriate, for there are different coefficients or moduli of elasticity, according to the nature of the strain, and Young's modulus is but one among them.

TABLE A

Young's Modulus of Elasticity

E = Young's modulus, in pounds weight per square inch.
M = length corresponding with modulus.
W = weight each square inch will bear without permanent alteration in length.

Material	M. (Feet)	E. (Lb.)	W. (Lb.)
Metals			
Brass . . .	2460000	8930000	6700
Gun metal . .	2790000	9873000	10000
Iron, cast . .	5750000	18400000	15300
" wrought .	7550000	24920000	17800
Lead . .	146000	720000	1500
Steel . { from	8530000	29000000	45000
Steel . { to	12354000	42000000	65000
Tin . . .	1453000	4608000	2880
Zinc . . .	4480000	13680000	5700
Stones			
Marble . .	2150000	2520000	4900
Slate . . .	13240000	15800000	
Portland . .	1672000	1533000	1500
Timber			
Ash . . .	4970000	1640000	3796
Beech . .	4600000	1345000	3113
Elm . . .	5680000	1340000	3102
Fir . . .	8330000	2016000	4667
Larch . .	4415000	1074000	2486
Mahogany .	6570000	1596000	3694
Oak . . .	4730000	1700000	3935

608. **Limiting Stress, or Ultimate Strength.**—For every kind of material, and every way in which a load is applied, there must

be a value which, if exceeded, causes rupture or fracture of the body. The greatest stress which the material is capable of withstanding is called the **limiting stress,** or **ultimate strength** per square inch of cross section of the substance, for the particular way in which the load is applied.

Factors of Safety.—The ratio of the ultimate strength, or limiting stress, to the safe working load is called the **factor of safety.** This factor of necessity varies greatly with different materials, and even with the same material according to circumstances.

For materials which are subjected to oxidation, or to internal changes of any kind, the factor of safety must of necessity be larger than in those which are always kept dry, or are well painted and carefully handled.

TABLE B

Ultimate Strength and Working Stress of Materials when in Tension, Compression, and Shearing

Material	Ultimate Strength—Tons per Square Inch			Working Strain—Tons per Square Inch		
	Tension	Compression	Shearing	Tension	Compression	Shearing
Steel bars, . .	45	70	30	9	9	5
" plates, . .	40	—	—	8	—	—
Wrought-iron bars,	25	17	20	5	$3\frac{1}{2}$	4
" " plates,	$22\frac{1}{2}$	17	20	$4\frac{1}{2}$	$3\frac{1}{2}$	4
Iron wire cables, .	40	—	—	8	—	—
Cast-iron, . .	$7\frac{1}{2}$	48	14	$1\frac{1}{2}$	9	3
Copper bolts, .	15	25	—	3	5	—
Brass (sheet), .	14	—	—	3	—	—
Ash, ...	$7\frac{1}{2}$	4	$\frac{3}{4}$	$1\frac{1}{4}$	$\frac{2}{3}$	$\frac{1}{8}$
Beech, ...	5	4	—	1	$\frac{2}{3}$	—
Elm, ...	6	$4\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{2}{3}$	$\frac{1}{8}$
Fir, ...	5	$2\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{12}$
Oak, ...	$6\frac{1}{2}$	$3\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{6}$
Teak, ...	$6\frac{1}{2}$	5	—	1	1	—
Granite, . . .	—	$3\frac{1}{2}$	—	—	$\frac{1}{2}$	—
Sandstone, . .	—	$1\frac{1}{2}$	—	—	$\frac{1}{4}$	—
Brick in cement, .	$\frac{1}{7}$	$\frac{1}{4}$ to $\frac{7}{10}$	—	50 lb.	180 lb.	—

The breaking strain of iron and steel does not (as hitherto assumed) indicate the quality—a high breaking strain *may* be due to hard, unyielding character, or a low one may be due to extreme softness. The contraction of area at the fracture forms an essential element in estimating the strength.

609. **Examples of Stress and Strain.**—What do you understand by stress and strain respectively? If an iron rod 50 feet long is lengthened by $\frac{1}{2}$ inch under the influence of a stress, what is the strain?

Stress is the reaction per unit area of cross section due to the load.

Let P = the total tension acting on area A.

$$\text{Then stress} = \frac{P}{A}.$$

Strain is the ratio of the increase or diminution of length or volume to the original length or volume.

Let L = original length of a bar of the material, l = the amount by which the length is increased or diminished; then when the bar is subjected to stress,

$$\text{the strain} = e = \frac{l}{L}.$$

In the example given L = 50′ × 12″ = 600 inches, and $l = \frac{1}{2}$ inch.

$$\therefore \text{Strain, } e = \frac{l}{L} = \frac{\frac{1}{2}}{600} = \frac{1}{1200} = \cdot 0008\dot{3}.$$

EXERCISE

From the above question and answer, determine Young's modulus for the iron of which the rod is composed, if the load was 4366 lb., and the cross section of the rod 2 square inches.

$$(1)\ \text{Stress} = \frac{\text{total load}}{\text{cross area}};$$

or
$$p = \frac{P}{A} = \frac{4366 \text{ lb.}}{2} = 2183 \text{ lb.}$$

$$(2)\ \text{Young's modulus} = \frac{\text{stress}}{\text{strain}};$$

or
$$E = \frac{p}{e} = \frac{2183}{\cdot 0008\dot{3}} = 2500000.$$

∴ a load of 25000000 lb. would elongate a rod of the iron to double its length by tensile stress.

EXAMPLE.—A wire $\frac{1}{10}$ square inch in cross section and 10 feet

long is fixed at its upper end; a load of 1000 lb. is hung from the lower end, and then the wire is found to stretch 1 inch. (1) What is the stress? (2) What is the strain?

(1) Here $P=1000$ lb., and $A=\frac{1}{10}$ square inch.

Let p=stress, or pull per square inch in lb.

$\therefore$ the stress, or $P=\frac{P}{A}=1000/\frac{1}{10}=10000$ lb. per square inch.

(2) Original length $=L=10'=120''$, and the increase of length $=l=1$ inch.

Let e=strain, or extension per unit of length—that is, per inch in this case.

$$\therefore \text{ the strain, or } e=\frac{\text{increase of length}}{\text{original length}}=\frac{l}{L}=\frac{1''}{120''}=\cdot 0083.$$

610. **Compressive Stress and Strain.**—If the line of action of a load be along the axis of a bar, shore, strut, or pillar, so as to tend to compress or shorten the same, the reaction per square inch of cross section is termed the **compressive stress,** and the diminution per unit of length is called the **compressive strain.**

EXAMPLE.—A vertical support in the form of a hollow pillar, having 2 square inches cross section of metal, is 10 feet long. With a load of 10000 lb. resting on the top, it is found to be compressed $\frac{1}{10}$ of an inch in length. (1) What is the stress? (2) What is the strain?

(1) Here $P=10000$ lb., and $A=2$ square inches.

Let p=stress, or compression per square inch of cross section in lb.

$\therefore$ the stress, or $p=\frac{P}{A}=\frac{10000}{2}=5000$ lb. per square inch.

(2) Original length $=L=10'=120''$, and the diminution of length $=l=\frac{1}{10}''$.

Let e=strain, or compression per unit of length—that is, per inch in this case.

$$\therefore \text{ the strain, or } e=\frac{\text{diminution in length}}{\text{original length}}=\frac{\cdot 1''}{120''}=\cdot 00083.$$

611. **Work done in Stretching a Bar: Resilience.**—If a load of gradually increasing amount be applied to a bar so as to stretch it, the amount of actual stretch, or elongation of the bar, will, within the limitations already specified, be directly proportional to the load producing it.

When the bar is loaded to its elastic limit, or proof stress, as it is sometimes called, then the work done in stretching it is termed the **resilience** of the bar, and the ratio $\frac{1}{2}\frac{p^2}{E}$ (where p is the direct tensile stress) is its modulus, or coefficient of resilience.*

EXAMPLE.—If a wrought-iron tie-bar, 5 feet long and 3 inches in diameter, has a limit of elasticity of 15 tons per square inch, and a modulus of elasticity of 30000000 lb. per square inch, what is its resilience? Take $\pi = \frac{22}{7}$.

$$\text{Formula W, or resilience} = \frac{p^2}{E} \times \frac{AL}{2};$$

or

$$W = \frac{p^2}{E} \times \tfrac{1}{2} \text{ volume of the bar.}$$

A = area of the section (usually in inches),
L = original length of bar in inches,
p = direct stress = 15 tons per square inch in this case.

$$\therefore \frac{p^2}{E} \times \frac{AL}{2} = 665{\cdot}28 \text{ foot-lb.}$$

WORKING

$$p = 15 \times 2240 \text{ lb.},\ E = 30000000 \text{ lb. per square inch,}$$
$$A = \tfrac{1}{4} \times \tfrac{22}{7} \times 3^2 \text{ square inches, and } L = 5 \text{ feet.}$$

$$\therefore \text{Resilience} = \frac{(15 \times 2240)^2}{30000000} \times \frac{\frac{11}{14} \times 3^2 \times 5}{2} = 665{\cdot}28 \text{ foot-lb.}$$

612. The **shearing force** or load at *any* point or *any* transverse section of a beam is equal to the resultant or algebraical sum of all the parallel forces on *either* side of the point or section.

When the section under consideration is in the same plane as the load, the only effect the load has at that section is a tendency to shear the beam, but in the more general case, where the load acts at a distance from the given section, we have, in addition, a tendency to curve or bend the beam at the section. Hence the name **bending moment** is given to this latter effect.

The **bending moment** at any point in a beam is the algebraic sum of the moments with respect to that point of all the external forces acting on the portion of the beam on either side of that point.

* *Resilience* is derived from the Latin *re*, back, and *salio*, I leap or spring.

The resistance to bending depends only on Young's modulus and the form of the section, and has no reference to the direct resistance to crushing.

613. A column or strut under pressure may fail in three ways: firstly, by the metal being absolutely crushed; secondly, by the column bending and breaking near the centre of its length; and thirdly, by the plates composing it wrinkling, owing to their breadth being out of proportion to their thickness.

With timber, failure sometimes takes place by the fibres crushing into each other, and sometimes by their splitting apart. In the latter case, although it may be due to absolute crushing, the length has some influence on the resistance, for each fibre may be considered to be a column failing by cross breaking, assisted, however, by its adhesion to the adjacent fibres.

The absolute resistance varies from 2 to 3 tons per square inch, and the limit of elasticity may be taken as about half this, and the safe resistance as 10 cwt.

The following is the formula for failure by bending or cross breaking where l is the length of the beam, r the radius of gyration of the section (Art. 605), and E the coefficient tabulated below:—

Strain per square inch $= \dfrac{8}{E\left(\dfrac{l}{r}\right)^2}$ when the ends of the column are rounded, or of 1 strength.

When the ends are flat and fixed the formula becomes $\dfrac{8}{E\left(\dfrac{l}{r}\right)^2} \times 3.$

Cast-iron, . . .	·00018	Teak, ...	·001
Wrought-iron, .	·00010	Oak and Pitch-pine,	·002
Steel, ...	·00008	Fir, ...	·003

With timber, the value of E may be taken as ·001 all round.

Note.—When the ratio of length to radius of gyration becomes so small that the column is on the point of failing by direct crushing, the resistance to bending will be less.

A factor of safety of about $\frac{1}{4}$ the absolute resistance of the material is commonly taken, but this is really only $\frac{1}{2}$ of the elastic resistance. Now, in a long column, until the breaking weight is actually reached, except in the case mentioned, there is, theoretically, no tendency to bend, and certainly the elastic limit is not exceeded; hence so large a factor of safety is not required. Owing, however, to differences in the different parts of the same

material in the value of E, and also to the possible divergence of the line of pressure from the neutral line, some margin must be allowed, and for all purposes and conditions ¼ the absolute resistance of the material is well within the limits of safety.

The rigidity and strength of a column depends on the shape of its ends.

In calculating the strength of a column by means of the formula $\frac{8}{E\left(\frac{l}{r}\right)^2}$, three-fourths of the result so obtained will give the weight that will cause the column to deflect from the perpendicular—that is, when the column or pillar is timber. In some cases, owing to different densities, seasoning, &c., it will require four-fifths of the breaking weight to cause the column to bend at all.

In cast-iron columns, as already pointed out, there is theoretically no tendency to bend when the column's length exceeds fifteen times its diameter or thickness, and in such cases the breaking weight may be regarded as the bending weight.

With timber, the formula for the safe weight or load in tons per square inch is $\frac{4}{\cdot003\left(\frac{l}{r}\right)^2}$ or $\frac{1333}{\left(\frac{l}{r}\right)^2}$.

The safe resistance for	cast-iron	is	7 tons per square inch,
" " "	wrought-iron	"	5 " " "
" " "	timber	"	10 cwt. " "
" " "	steel	"	6½ tons " "

and a deduction for rivet-holes must be made in calculating the sectional area.

Relative Strength of Round and Flat Ends in Long Columns

Both ends rounded, 1 strength, ...	=1
One end flat and firmly fixed, 1 strength, . .	=2
Both ends flat and firmly fixed, ...	=3

Relative Strength of Section in Long Solid Columns

Cylindrical, ...	100
Triangular, ...	110
Square, ...	93

RELATIVE STRENGTH OF MATERIAL IN LONG COLUMNS, CAST-IRON BEING ASSUMED AS 1000

Wrought-iron, ...	1745
Cast steel, ...	2518
Oak, ...	109
Red deal, ...	$78\frac{1}{2}$

A further investigation of this problem will appear towards the end of this subject.

614. Problem I.—To find the weight that a rectangular cast-iron beam, supported at both ends, can sustain at its middle.

RULE.—Find the continued product of 850, the breadth and square of the depth, both in inches, and divide this product by the length in feet, and the quotient will be the required weight.

That is, $$W = 850bd^2 \div l;$$

hence $$l = \frac{850bd^2}{W},\ b = \frac{lW}{850d^2},\ \text{and}\ d^2 = \frac{lW}{850b}.$$

For malleable iron, use 950 instead of 850. The weight of the beam must always be added to the applied weight: the weight of the beam is equivalent to $\frac{5}{8}$ of its weight applied at the middle; and any weight uniformly distributed is also equivalent to $\frac{5}{8}$ of itself applied at the middle.

EXAMPLE.—A bar of cast-iron is $=2$ inches square and 15 feet long; what weight will it be capable of supporting?

$$W = 850bd^2 \div l = \frac{850 \times 2 \times 2^2}{15} = \frac{170 \times 8}{3} = 453\tfrac{1}{3}\ \text{lb.}$$

EXERCISES

1. Find the weight that can be supported by a beam $=5$ inches square and 10 feet long. . . . . $=10625$ lb.

2. A beam of cast-iron is $=20$ feet long and $2\frac{1}{2}$ inches broad, and it has to support a load of 10000 lb.; what must be its depth? $=9{\cdot}7$ inches.

3. A cast-iron joist is $=30$ feet long, 10 inches deep, and 3 inches broad; what weight, uniformly distributed, can it sustain? $=13812{\cdot}5$ lb.

615. Problem II.—To find the weight that a beam fixed at one end can sustain at its free end.

The weight is $\frac{1}{4}$ of that found by the preceding problem.

$$W = \tfrac{1}{4} \times 850bd^2 \div l.$$

When the weight is uniformly distributed over the beam, take ½ of that found by Problem I.

Exercises

1. A beam is = 30 feet long, 8 inches deep, and 2½ broad; what weight can it support at its extremity? . . . = 1133⅓ lb.
2. What load uniformly distributed over a beam = 32 feet long, 4 inches deep, and 2 broad can it sustain? . . . = 425 lb.
3. A beam = 20 feet long and 10 inches deep supports a load of 17000 lb. at its extremity; what is its breadth? . . = 16 inches.
4. A beam = 24 feet long and 2 inches broad supports 1735 lb. uniformly distributed; required its depth. . . . = 7 inches.

STRENGTH OF SHAFTING TO RESIST VARIOUS STRESSES

616. **Problem III.—To find the weight which a solid cylinder or square shaft of cast-iron, wrought-iron, or wood can sustain when the weight is applied at the centre, or distributed, and when the cylinder or shaft is supported at both ends.**

Let D = diameter in inches, or side if square;
L = length of shaft, supported at both ends, in feet;
W = weight applied at the centre in lb.

Then, $W = \frac{K \cdot D^3}{L}$, and $D = \sqrt[3]{\frac{L \cdot W}{K}}$, or $D = \sqrt[3]{\frac{L \cdot W'}{2K}}$, where W' is weight distributed in lb.

	Round Shafts	Square Shafts
For wood, . . .	$K = 40$	$K = 70$
" cast-iron, . .	$K = 500$	$K = 850$
" wrought-iron, .	$K = 700$	$K = 1200$

$L = \frac{K \cdot D^3}{W}$, and $D^3 = \frac{L \cdot W}{K}$, for weight at centre;

and $W' = \frac{2K \cdot D^3}{L}$, $L = \frac{2K \cdot D^3}{W}$, " " distributed.

Exercises

1. What weight will a cylinder = 10 feet long and 4 inches diameter support? . . . . = 3200 lb.
2. What weight will a uniformly loaded cylinder support, its length being = 24 feet, and diameter = 10 inches? . = 41666⅔ lb.

3. What will be the diameter of a cylinder = 20 feet long, which is capable of supporting 3125 lb.? . . . . . = 5 inches.

4. What will be the limit of the length of a cylinder uniformly loaded by a weight of 100000 lb., whose diameter is = 12 inches? = 17·28 feet.

617. **Problem IV.—To find the weight which a solid cylinder or square shaft of cast-iron, wrought-iron, or wood fixed at one end can sustain at the free end.**

The weight is just the fourth of that found in the previous problem, and the formulæ the same. All that is necessary is to take one-fourth of the value of K.

Rule.—Multiply the cube of the diameter, or side if square, in inches, by the value of K÷4, and divide the product by the length in feet, and the quotient will be the weight in lb.

$$W=\frac{\frac{K}{4}\,.\,D^3}{L}, \text{ or } W=\tfrac{1}{4}\times K\times D^3\div L;$$

$$L=\frac{\frac{K}{4}\,.\,D^3}{W}, \text{ and } D^3=\frac{L\,.\,W}{\frac{K}{4}}.$$

Exercises

1. What weight will a cylinder = 10 feet long and 4 inches diameter support at its free end? . . . . . . = 800 lb.

2. What will be the diameter of a cylinder = 20 feet long that can support 781·25 lb.? . . . . = 5 inches.

3. What will be the length of a cylinder, which is = 12 inches diameter, that supports 12500 lb.? . . . . . = 17·28 feet.

618. **Problem V.—To find the exterior diameter of a hollow cylinder of cast-iron, supported at both ends, so as to sustain a weight applied at the middle, the ratio of the interior and exterior diameters being given.**

Rule.—Let the ratio of the exterior to the interior diameter be that of 1 to n; then take the difference between 1 and the fourth power of n, and multiply it by 500; find also the product of the length and the weight; divide the latter product by the former; then the quotient will be the cube of the diameter.

$$d^3=lW\div 500(1-n^4).$$

When the exterior diameter d is found, the interior diameter

will be obtained by multiplying d by n. If d' = the interior diameter, and t = the thickness of the metal, then

$$d' = nd, \text{ and } t = \tfrac{1}{2}(d - d') = \tfrac{1}{2}(1 - n)d.$$

EXAMPLE.—The weight supported by a hollow cylinder is 32000 lb., its length is = 12 feet, and the ratio of the exterior and interior diameters = 10 to 1; what are its diameters?

$$d^3 = lW \div 500(1 - n^4) = \frac{12 \times 32000}{500(1 - \cdot1^4)} = \frac{12 \times 64}{1 - \cdot0001} = \frac{768}{\cdot9999} = \frac{7680000}{9999}$$

$$= 768\cdot08, \text{ and } d = \sqrt[3]{768\cdot08} = 9\cdot15 \text{ inches};$$

hence $d' = nd = \cdot1 \times 9\cdot15 = \cdot915$ inch,

and $t = \tfrac{1}{2}(1 - n)d = \tfrac{1}{2}(1 - \cdot1) \times 9\cdot15 = \tfrac{1}{2} \times \cdot9 \times 9\cdot15 = 4\cdot1175.$

EXERCISES

1. A hollow cylinder = 10 feet long supports 2500 lb., and the ratio of its diameters is = 2 to 1; what are the diameters?
= 3·76 and 1·88, and thickness of metal ·94 inch.

2. A hollow cylinder = 9 feet long is intended to support 15000 lb., and the thickness of the metal is to be = $\frac{1}{5}$ of the exterior diameter; required its diameters. . . . . . = 6·769 and 4·061 inches.

619. Problem VI.—To find the resistance to torsion (or torque) of solid and hollow shafts.

RULE FOR SOLID SHAFTS.—Multiply the cube of the diameter by the shearing stress in lb. per square inch permissible in the material of the shaft, and the result by $\frac{3\cdot1416}{16}$.

Or, putting this in formula form,

$$\text{T.R. (torsional resistance)} = \frac{\pi}{16} D^3 f \quad . \quad . \quad . \quad [1],$$

where D = outside diameter, and f = shearing stress in lb. per square inch.

RULE FOR HOLLOW SHAFTS.—From the 4th power of the outside diameter subtract the 4th power of the inside diameter, and divide by outside diameter. Multiply this quotient by the shearing stress in lb. per square inch permissible in the material of the shaft, and by the quotient of $\frac{3\cdot1416}{16}$.

Or, putting this in formula form,

$$\text{T.R. (torsional resistance)} = \frac{\pi}{16}\left(\frac{D^4 - d^4}{D}\right)f,$$

where D = outside diameter, d = inside diameter, both in inches, and f = shearing stress in lb. per square inch.

It is instructive to compare the torsional resistances of solid and hollow shafts of the same weight and material. The result shows that, for the same length and weight, the hollow shaft having outer and inner diameters in the proportion of 2 to 1 will be 44·3 per cent. stronger than the solid one.

NOTE.—*The strength of shafts varies as the third power of their diameters, whilst their stiffness varies as the fourth power.*

EXAMPLE.—Find the torsional resistance or 'twisting moment' of a hollow shaft of cast-iron, the external and internal diameters of which are 20 inches and 8 inches respectively. Take the surface stress as 6000 lb. per square inch.

Here $D=20$ inches, $f=6000$ lb. per square inch, $d=8$ inches.

Then, as
$$\text{T.R.}=\text{T.M.}=\frac{\pi}{16}\cdot\frac{D^4-d^4}{D}\cdot f,$$

$$\therefore\ \text{T.M.}=\frac{3\cdot1416}{16}\times\frac{20^4-8^4}{20}\times6000=9183525 \text{ inch-lb.}$$

EXERCISES

1. Find the resistance to torsion of a solid shaft of cast-iron whose diameter is 5·25 inches, with a surface stress of 4500 per square inch. . . . . =127820·745 inch-lb.

2. Find the torsional moment of resistance of a wrought-iron shaft (solid) whose diameter is 6 inches, the surface stress being 8000 lb. per square inch. . . . . . =339206·5 inch-lb.*

620. In order to transmit energy through a shaft, the driving force must be applied at some distance from its centre. The driving force and its effective leverage therefore constitute what is termed a turning or twisting moment (T.M.), which puts the shaft in a state of torsion. The tendency of a purely torsional moment applied to a shaft is to cause the shaft to shear in planes normal to its axis, and this has to be met by the shearing resistance of the material, which resistance must, of course, be of the nature of a moment. The resistance the shaft offers to twisting we term its torsional resistance (T.R.), and as this balances the turning moment, we have

$$\text{T.M.}=\text{T.R.}$$

The turning moment driving a shaft may either be uniform or variable in amount. Shafts which are driven by means of gearing, and which revolve at a uniform speed, are generally considered as examples of uniform turning moment.

* See Table, Art. 625, and compare the above answer.

A typical example of variable turning moment may be recognised in the steam-engine crank-shaft, where both the driving force of the steam on the piston and its effective leverage are continually varying throughout the stroke. When the turning moment is uniform that is, when the shaft revolves uniformly at n revolutions a minute, and transmits energy at the rate of so many horse-power, we have all the data required in order to estimate T.M. The work done by a turning couple in one minute is equal to the magnitude of the turning couple multiplied by its angular displacement in the same time. Now, the turning couple, or turning moment, as it is termed, is T.M. inch-lb., or $\frac{1}{12}$ T.M. foot-lb., and the angular velocity of the shaft is $n \times 2\pi$ radians per minute. Therefore the work done =

$$\frac{\text{T.M.}}{12} \times 2\pi n \text{ foot-lb. per minute,}$$

and the horse-power (H.P.) =

$$\frac{\frac{\text{T.M.}}{12} \times 2\pi n}{33000} = \frac{n \times \text{T.M.}}{63024};$$

$$\therefore \text{ T.M.} = 63024 \cdot \frac{\text{H.P.}}{n} \quad . \quad . \quad . \quad [2].$$

EXAMPLE 1.—Supposing it was required to find the horse-power transmitted by the shaft in the first example, running, we will say, at the rate of 70 revolutions a minute, we proceed thus:—

$$\text{T.M., as there found} = 9183525 \text{ inch-lb.,}$$

$$\text{H.P.} = \frac{\text{T.M.} \times n}{63024} = \frac{9183525 \times 70}{63024} = 10200.$$

EXAMPLE 2.—If a steel shaft revolving at 60 revolutions per minute be required to transmit 220 horse-power, what should be its diameter so that the maximum stress produced in it may not exceed one-fifth of that at the elastic limit? The elastic limit in torsion is 18 tons per square inch.

Combining formulæ [1] and [2], we have

$$\text{T.R.} = \text{T.M.};$$

that is,

$$\frac{\pi}{16} D^3 f = 63024 \times \frac{\text{H.P.}}{n}.$$

$$\therefore \text{ D (outside diameter)} = 68 \cdot 5 \sqrt[3]{\frac{\text{H.P.}}{nf}} \quad . \quad . \quad [3].$$

Here H.P. = 220, $n = 60$, and $f = \frac{1}{5} \times 18 \times 2240 = 8064$ lb. per square inch.

$$\therefore \text{ D} = 68 \cdot 5 \times \sqrt[3]{\frac{220}{60 \times 8064}} = 5 \cdot 27 \text{ inches.}$$

621. **Problem VII.—To find the diameter of a shaft, the torsional moment of resistance being given, and with a shearing stress of not over 8000 lb. per square inch.**

RULE.—Divide the torsional moment of resistance in inch-lb. by the quotient of $3{\cdot}1416 \div 16 \times 8000$, and extract the cube root.

Or, expressing this in formula form,

$$D = \sqrt[3]{\frac{\text{T.M.}}{\frac{\pi}{16} \cdot f}}.$$

The shearing stress for wrought-iron is from 8000 to 10000 lb. per square inch, and for cast-iron from 4000 to 5000 lb.

The factor of safety is therefore the lowest value of f in each case.

All shafts when in motion—that is, rotating—are subjected to a combined and simultaneous bending and twisting moment.

EXAMPLE.—Supposing it be required to find the diameter of a shaft whose bending and twisting moments = 25000 inch-lb.

We employ formula [1], Problem VI., making

$$\text{T.M.} = \text{T.R.} = \frac{\pi}{16} D^3 f.$$

$$\therefore D = \sqrt[3]{\frac{\text{T.M.}}{\frac{\pi}{16} \cdot f}} = \sqrt[3]{\frac{25000}{\frac{3{\cdot}1416}{16} \times 8000}} = 2{\cdot}51 \text{ inches.}$$

EXERCISES

1. Find the diameters of the following wrought-iron shafts, whose combined bending and twisting moments are respectively in inch-lb. (take $f = 8000$ lb. per square inch) :—

42390 ;	82793 ;	120522 ;	196250.
= 3″ ;	$3\frac{3}{4}$″ ;	$4\frac{1}{4}$″ ;	5″.

2. Find the diameters of the following cast-iron shafts, whose combined bending and twisting moments are respectively in inch-lb. (take $f = 4000$ lb. per square inch) :—

84780 ;	165586 ;	241044 ;	392500.
= $4\frac{3}{4}$″, smallest ;	6″ ;	$6\frac{3}{4}$″ ;	8″.

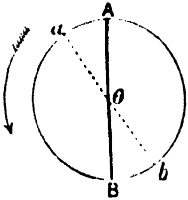
622. **Stiffness of Shafts : Angle of Twist.**—The effect of a turning moment applied to a shaft is to twist one part relatively to another. So far we have been dealing only with the resistance

the shaft offers to being twisted—that is to say, we have been concerned only with the *strength* of the shaft without regard to the question of stiffness.

In many cases, especially in light machinery, the question of the stiffness of the shafting is of greater importance than that of the strength.

The stiffness of a shaft is measured by the smallness of the angle of twist per unit length of the shaft.

This figure illustrates strain in a shaft. Let dl be the axial distance in inches between the two sections whose diameters are AB, ab, and let $d\theta$ be the circular measure of the angle between those diameters when the shaft is twisted; then the torsional or shearing strain at the surface of the shaft is =



$$\frac{D}{2} \times \frac{d\theta}{dl}.$$

D = as before, the extreme diameter of the shaft in inches.

Let f = surface stress in the material of the shaft in lb. per square inch.

Let C = modulus or coefficient of shearing elasticity, or of rigidity in lb. per square inch.

Then, as $$C = \frac{\text{stress}}{\text{strain}} = \frac{f}{\left(\frac{D}{2}\right) \cdot \frac{d\theta}{dl}},$$

$$\therefore \; d\theta = \frac{2f}{CD} \cdot dl.$$

Hence for a shaft L inches long, by a simple integration we have the angle of twist =

$$\theta = \frac{2f}{CD} \int_0^L dl = \frac{2fL}{CD}.$$

To express this result in terms of the twisting moment and the diameter of the shaft, we have

$$f = \frac{T.M.}{\frac{\pi}{16} D^3} \text{ for solid shafts,}$$

and $$f = \frac{T.M.}{\frac{\pi}{16} \frac{D^4 - d^4}{D}} \text{ for hollow shafts.}$$

Making these substitutions and simplifying, we get :—

$$\left.\begin{array}{ll}\text{For solid shafts,} & \theta=\dfrac{10\cdot 2(\text{T.M.})\text{L}}{\text{CD}^4}\ .\ \text{radians,}\\ \text{or} & \theta=\dfrac{584(\text{T.M.})\text{L}}{\text{CD}^4}\ .\ \text{degrees,}\end{array}\right\}\quad .\quad .\quad .\quad [4];$$

and for hollow shafts,

$$\left.\begin{array}{ll} & \theta=\dfrac{10\cdot 2(\text{T.M.})\text{L}}{\text{C}(\text{D}^4-d^4)}\ .\ \text{radians,}\\ \text{or} & \theta=\dfrac{584(\text{T.M.})\text{L}}{\text{C}(\text{D}^4-d^4)}\ .\ \text{degrees,}\end{array}\right\}\quad .\quad .\quad .\quad [5].$$

EXAMPLE.—The angle of twist or torsion of a shaft is limited to 1° for each 10 feet of length ; find the diameter of a solid round shaft to transmit 100 horse-power at 50 revolutions a minute, the modulus of resistance to torsion being 10000000 lb. per square inch.

Here $\theta=1^\circ$ when $\text{L}=10\times 12=120$ inches,

$\text{C}=10000000$;

also, $\text{T.M.}=63024\times\dfrac{\text{H.P.}}{n}=63024\times\dfrac{100}{50}=126024$ inch-lb.

Now, applying formula [4], the given conditions are :—

$$1^\circ=\frac{584\times 126048\times 120}{10000000\times \text{D}^4};$$

and by solving for D, we get

$$\text{D}=\sqrt[4]{\frac{584\times 126048\times 120}{10000000}}=5\cdot 45 \text{ inches.}$$

EXERCISES

1. What is the maximum horse-power which could be transmitted by a shaft 3 inches in diameter when making 150 revolutions per minute, and supposing the shearing stress in the material to be limited to 7500 lb. per square inch ? . . =94·5 horse-power.

2. If a shaft of 3 inches diameter transmits safely 33 horse-power at 100 revolutions per minute, what size of shaft will transmit safely 20 horse-power at 150 revolutions per minute ?
=2·22 inches.

623. It may be accepted generally that the coefficient of torsion in substances is about $\frac{2}{5}$ of their Young's modulus.

The limit of resistances of cast-iron, wrought-iron, and steel to

shearing may be stated as 9, 18, and 27 tons respectively, or in the ratio of 1, 2, 3.

Some authorities, however, give these materials as high a value as 14, 20, and 30 tons. Cast-iron requires to be treated with greater caution than any other material used in structures, its elastic limit being only about half its breaking weight. A test-piece does not elongate perceptibly to ordinary observation, but breaks suddenly without giving warning. It is subject to flaws in process of manufacture, and these are generally carefully concealed, being beyond the range of detection. It therefore requires a large factor of safety.

624. Problem VIII.—To find the breaking strain of cast-iron shafts (solid cylindrical) when subjected to a torsional strain with a leverage (l), the diameter of the shaft being given.

RULE.—Multiply the cube of the diameter in inches by 2·36, and divide by the leverage (l), also in inches, and the result is the breaking strain in tons.

Or, expressing this in formula form—

$$\frac{2{\cdot}36 \times d^3}{l} = \text{W in tons.}$$

EXAMPLE.—Find the weight which must be applied at a leverage of 170 inches in order to break a solid cylindrical cast-iron shaft whose diameter is 2 inches.

$$W = \frac{2{\cdot}36 \times d^3}{l} = \frac{2{\cdot}36 \times 8}{170} = \frac{18{\cdot}88}{170} = {\cdot}11 \text{ ton, or } 246{\cdot}4 \text{ inch-lb.}$$

EXERCISE

Find the weights which must be applied at a leverage of 14½ feet in order to break the under-mentioned solid cylindrical shafts of cast-iron, whose diameters are respectively 2½", 3¼", 4".

Answers, ·22 ton, ·48 ton, ·88 ton.

Actual experiment, ·18 " , ·52 " , ·86 "

$$W = \frac{2{\cdot}36 \times d^3}{l}, \text{ or } W = \frac{18{\cdot}85r^3}{l}, \text{ or } W = \left(\frac{1{\cdot}33d}{l}\right)^3;$$

$$l = \frac{2{\cdot}36 \times d^3}{W} = \frac{18{\cdot}85r^3}{W} = \frac{(1{\cdot}33d)^3}{W};$$

r = radius of shaft.

625. Torsional Moment of Resistance for Shafts, calculated from the Formula $T.M. = \frac{\pi}{16} . f . d^3$.

M = moment of resistance to torsion = π = 3·14159.

f = stress per square inch ; d = diameter of shaft in inches.

f = 8000 to 10000 lb. for wrought-iron, and 4000 to 5000 lb. for cast-iron.

Diameter—Inches	f=8000 lb.	f=10000 lb.	Diameter—Inches	f=8000 lb.	f=10000 lb.
1	1570	1962	7¼	598293	747866
1¼	3066	3832	7½	662344	827930
1½	5299	6624	7¾	730810	913512
1¾	8414	10517	8	803840	1004800
2	12560	15700	8½	964176	1205220
2¼	17883	22354	9	1144530	1430662
2½	24531	30664	9½	1346079	1682599
2¾	32651	40814	10	1570000	1962500
3	42390	52988	10½	1817471	2271839
3¼	53895	67369	11	2089670	2612088
3½	67314	84143	11½	2387774	2984717
3¾	82793	103491	12	2712960	3391200
4	100480	125600	13	3449290	4311612
4¼	120522	150652	14	4308080	5385100
4½	143066	178833	15	5298750	6623438
4¾	168260	210325	16	6430720	8038400
5	196250	245313	17	7713410	9641762
5¼	227184	283980	18	9156240	11445300
5½	261209	326511	19	10768630	13460788
5¾	298472	373090	20	12560000	15700000
6	339120	423900	21	14539770	18174710
6¼	383300	479125	22	16717360	20896700
6½	431161	538951	23	19102190	23877738
6¾	482848	603560	24	21703680	27129600
7	538510	673138			

Note.—The bending moment of resistance is half the numbers in the Table, as $M = \frac{\pi}{32} . f . d^3$.

For cast-iron shafts half the numbers to be taken.

EXAMPLE.—Required to find a shaft for a drum having $2\frac{1}{2}$ tons pulling on it at 17-inch radius, and taking $f = 8000$ lb.

The moment of weight $= W . l = 2\frac{1}{2} \times 2240 \times 17 = 95200$ lb.; the torsional moment of resistance must be equal to or greater than this amount.

Find in the Table the number next higher, which in this case is 100480, opposite 4-inch diameter, which will be the size of shaft required in wrought-iron.

If for cast-iron shaft, and $f = 4000$ lb., then 5-inch diameter is the size, since $\frac{196250}{2} = 98125$, or $95200 \times 2 = 190400$ lb., and the next higher number in the Table $= 196250$, opposite 5-inch diameter.

626. **Problem IX.—To find the weight that could safely be supported by a column of cast-iron or other material, such as oak or deal, and resting on a horizontal plane. The column may be either square or cylindrical in shape.**

Before proceeding further it will be as well to state that the safe load in structures, including weight of structures, must be regarded as follows:—

In cast-iron columns, ...	$\frac{1}{4}$	breaking weight.
" wrought-iron structures, ...	$\frac{1}{4}$	"
" cast-iron girders for tanks, ...	$\frac{1}{4}$	"
" " " bridges and floors,	$\frac{1}{6}$	"
" timber (live load), ...	$\frac{1}{10}$	"
" " (dead load), ...	$\frac{1}{5}$	"
In stone and bricks, ...	$\frac{1}{8}$	"

The shape of the ends of the column materially affects its strength, and at all times requires to be considered when computing its strength.

Nature of Column	Ends Rounded, when L exceeds 15D	Ends Flat, when L exceeds 30D
Solid cylinder of cast-iron, . . .	$W = 14{\cdot}9 \frac{D^{3{\cdot}76}}{L^{1{\cdot}7}}$;	$W = 44{\cdot}16 \frac{D^{3{\cdot}55}}{L^{1{\cdot}7}}$
Hollow cylinder of cast-iron, . .	$W = 13 \frac{D^{3{\cdot}76} - d^{3{\cdot}76}}{L^{1{\cdot}7}}$;	$W = 44{\cdot}34 \frac{D^{3{\cdot}55} - d^{3{\cdot}55}}{L^{1{\cdot}7}}$

Nature of Column	Ends Flat, when L exceeds 30D
Solid square of Dantzic oak (dry), . .	$W=10{\cdot}95\frac{D^4}{L^2}$
Solid square of red deal (dry), . . .	$W=\ 7{\cdot}81\frac{D^4}{L^2}$

W = breaking weight in tons,
L = length of column in feet,
D = external diameter of column in inches,
d = internal diameter in inches.

Now, as it is required in the problem that the safe weight be stated in the answers, attention is directed to the factors of safety already afforded.

In *hollow columns* the strength nearly equals the difference between that of two solid columns, the diameters of which are equal to the external and internal diameters of the hollow one.

627. Strength of Short Columns in which L is less than 30D.

w = breaking weight of short columns,
W = breaking weight of long columns as found above,
C = crushing force of material (expressed in tons per square inch) of which the column is formed × sectional area of column.

$$w=\frac{WC}{W+\frac{3}{4}C}.$$

To facilitate the working of the formulæ, Tables of 3·6 and 1·7 power may be employed.

3·6 Power

No.	Power	No.	Power	No.	Power
3	52	10	3982	17	26892
4	147	11	5611	18	33035
5	328	12	7674	19	40133
6	632	13	10233	20	48273
7	1102	14	13367	21	57543
8	1783	15	17136	22	68033
9	2723	16	21619	24	93058

1·7 Power

No.	Power	No.	Power	No.	Power
5	15	18	136	30	325
8	34	20	163	35	421
10	50	22	191	40	529
12	68	25	238	50	773
15	100	28	288		

Example.—What weight can a solid cylindrical cast-iron column sustain safely when its ends are flat and its dimensions are: length 20 feet, diameter 6 inches?

We first determine the ratio of its length to the diameter, in order that we may know which formula has to be employed.

Thus, $L = 20$ feet $= 240''$, $D = 6''$; $\therefore \frac{L}{D} = \frac{240}{6} = 40$.

We therefore use formula 'When L exceeds 30D,' which in this case is

$$W = 44{\cdot}16 \frac{D^{3{\cdot}55}}{L^{1{\cdot}7}};$$

hence $$W = 44{\cdot}16 \times \left(\frac{6^{3{\cdot}6}}{20^{1{\cdot}7}}\right) = 44{\cdot}16 \times \frac{632}{163} = 167{\cdot}808 \text{ tons.}$$

But as this is the breaking weight, we take $\frac{1}{4}$ of the same for the answer, neglecting, as will be seen, the weight of the column, which should also be determined.

$\therefore$ the safe weight which this column can sustain is $167{\cdot}808 \div 4 = 41{\cdot}952$ tons, or, say, 42 tons, omitting weight of column.

Exercises

1. A cast-iron solid cylinder is 15 feet long, and 5 inches in diameter; its ends are flat, and rest on a horizontal plane; find its breaking and safe loads.

 Breaking load $= 144{\cdot}84$ tons; safe load, not including weight of cylinder $= \frac{1}{4}$ of the above.

2. What weight can a solid column of Dantzic oak (square in section) sustain safely, its ends being flat? Take $L = 10$ feet, and D (that is, a side of the section) $= 10$ inches, and let C (the crushing weight) $= 2{\cdot}61$ tons per square inch.

Safe weight for dead load $=44 \cdot 28$ tons; safe weight for live load $=22 \cdot 14$ tons, neglecting weight of column in the calculation.

3. Find the safe weight that a hollow cast-iron cylinder, rounded at both ends, can sustain; its external and internal diameters are respectively 6 and 5 inches, and its length $=10$ feet.

Safe weight $=19 \cdot 76$ tons, neglecting weight of cylinder in the calculation.

4. Find the breaking weight and safe strain of a solid cylindrical column of cast-iron whose length is 33 feet, and diameter $7\frac{1}{2}$ inches.

Use the formula $\dfrac{8}{E\left(\frac{l}{r}\right)^2}$, and consider the ends flat and fixed.

Here $\dfrac{8}{E\left(\frac{l}{r}\right)^2}=\dfrac{8}{\cdot 00018 \times 44605 \cdot 44}=\dfrac{8}{8 \cdot 0289792}=\cdot 99$, or, say, 1 ton per square inch of cross sectional area $=44 \cdot 1787$ square inches $\times 1$ ton $=44 \cdot 1787$ tons for rounded ends; but as the ends are flat and fixed in the question, we multiply by 3.

$\therefore$ the breaking weight $=132 \cdot 5361$ tons.

Note.—By referring to Art. 613 we find that $\dfrac{l}{r}=4\dfrac{l}{d}$.

Here $l=33$ feet $=396$ inches, and $d=7 \cdot 5$ inches;

$\therefore 4\dfrac{l}{d}=4 \times \dfrac{396}{7 \cdot 5}=4 \times 52 \cdot 8=211 \cdot 2$, which squared $=44605 \cdot 44$.

5. Find the breaking weight of a solid cylindrical column of cast-iron whose ends are fixed, and length $=25$ feet, diameter 8 inches.

Use formula $\dfrac{8}{E\left(\frac{l}{r}\right)^2}$, ... $=300$ tons nearly.

6. Find the breaking weight in tons per square inch of the following solid cylindrical columns of cast-iron, whose ends are flat and fixed:—

When $\frac{l}{d}=10$	42 Ans.		When $\frac{l}{d}=40$	5·2 Ans.	
" $=15$	37 "		" $=60$	2·3 "	
" $=20$	21 "		" $=120$	0·6 "	
" $=30$	9·2 "				

628. Table of Crushing Weights of a few Materials

Material		Lb. per Sq. Inch	Material		Lb. per Sq. Inch
Iron, cast, .	from	80640	Oak, English,	from	6400
	to	143360		to	10000
〃	average,	107520	Pine, ...		6000
Iron, wrought,	from	35840	Teak, ...		12000
	to	40320	Basalt, Scotch, . .		8300
〃	average,	37856	〃 greenstone, .		17200
Lead, cast, . .		6944	Granite, Aberdeen, .		11000
Steel, ...		336000	〃 Cornish, . .		14000
Steel plates, . .		201600	〃 Mt. Sorrel, .		12800
Tin, cast, . .		15008	Marble, Italian, . .		9681
Aluminium bronze, .		129920	〃 statuary, .		3216
Ash, ...		8600	Sandstone, Arbroath, .		7884
Beech, ...		7700	〃 Caithness, .		6490
Birch, ...		3300	Slate, Anglesea,	from	10000
Box, ...		10300		to	21000
Cedar, ...		5700	Brick, red, . . .		808
Deal, ...		5850	〃 fire, . . .		1717
Ebony, ...		19000	Portland cement,	from	3795
Elm, ...		10300		to	5984
Fir, spruce, . .		6500	Glass, flint, . . .		27500
Larch, ...		3200	〃 crown, . .		31000
Lignum-vitæ, . .		10000	〃 common, . .		31876
Mahogany, . .		8000			

MISCELLANEOUS FORMULÆ AND TABLES

629. Weight and Strength of Rope and Chains

Rope

C = circumference of rope in inches,
L = working load 〃 tons,
S = breaking strain 〃 〃
W = weight of rope in lb. per fathom.

$$C=\sqrt{\frac{L}{k}},\ L=C^2k,\ S=C^2x,\ W=C^2y,\ \text{or}=Lz.$$

Chains

D = diameter in eighths of an inch,
W = safe load in tons;
$D = \sqrt{9W}$,
$W = \frac{D^2}{9} = 7{\cdot}111 d^2$, where
d = diameter of iron in inches;
$\cdot 85 D^2$ = weight of chain in lb. per fathom.

TABLE OF VALUES OF k, x, y, AND z

Description of Rope	k	x	y	z
Common hemp, ...	·032	·18	·18	6·
Coir, hawser laid, ...	—	—	·131	—
" cable laid, ...	—	—	·117	—
St Petersburg tarred hemp hawser, .	·037	·22	·235	6·35
" " " cable, .	·025	·15	·207	8·28
White Manilla hawser, ...	·045	·27	·177	3·93
" " cable, ...	·033	·19	·155	4·7
Best hemp, 'cold register,' . .	·100	·60	—	—
" warm, ...	·116	·70	—	—
Iron wire rope, ...	·290	1·8	·87	2·9
Steel wire rope, ...	·450	2·8	·89	1·91

EXAMPLE.—Find the breaking strain of a 4-inch common hemp rope.

$$S = C^2 x = 4^2 \times \cdot 18 = 16 \times \cdot 18 = 2{\cdot}88 \text{ tons.}$$

EXERCISES

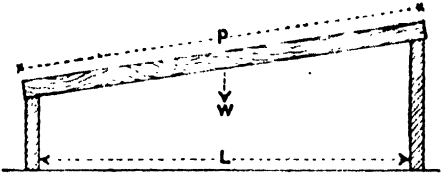
1. Find the breaking strain of the under-mentioned size ropes:— $5\frac{1}{2}''$ common hemp, $4''$ steel wire, $6''$ St Petersburg cable. =5·445, 44·8, 5·4 tons.

2. Find the circumference of a white Manilla cable that will stand a strain of 7 tons without breaking. . . =14·5 inches.

3. Find the weight of 200 fathoms of $4''$ steel wire rope. =1·27 tons.

4. Find the safe load that may be put on the following chains:— $\frac{3}{8}''$, $\frac{7}{8}''$, $1\frac{3}{8}''$. . . . . . . =1, $5\frac{4}{9}$, and $13\frac{4}{9}$ tons.

630. Breaking Weight of Beams on the Slope.



W = breaking weight for horizontal beam, as found by rule, Problem VI.—namely, $W = \frac{4bd^2S}{l}$,

L = span on horizontal line,

P = span on slope,

w = breaking weight of beam on slope;

$$w = \frac{WP}{L}.$$

EXAMPLE.—Suppose L in the above diagram = 10 feet and P = 12 feet, and that a rectangular beam of female fir 3 inches square and 12 feet long is laid on the span P; what weight applied at the centre of the beam would break it?

As a horizontal beam, its breaking weight is found by the formula,

$$W = \frac{4bd^2S}{l} = \frac{4 \times 3 \times 3^2 \times 1140}{144} = \frac{4 \times 3 \times 9 \times 1140}{144} = 855 \text{ lb.}$$

But $$w = \frac{WP}{L} = \frac{855 \text{ lb.} \times 12}{10} = 1026.$$

∴ the weight required to break the beam on the slope would be 1026 lb.

631. Beams unequally loaded.

Let W = breaking weight for load applied at the centre, as found by formula,

$$W = \frac{4bd^2S}{l}.$$

w = breaking weight for beam unequally loaded,

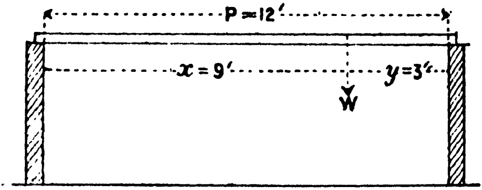
P = length of the beam or span,

x and y = distances of load from point of support;

$$w = \frac{WP^2}{4xy}.$$

Now, supposing that the beam or span P, as shown in the following sketch = 12 feet, that its breadth and depth are each 3 inches,

that it is of precisely the same material as the beam mentioned in the preceding example, and that x and y are 9 feet and 3 feet respectively, find the breaking weight.

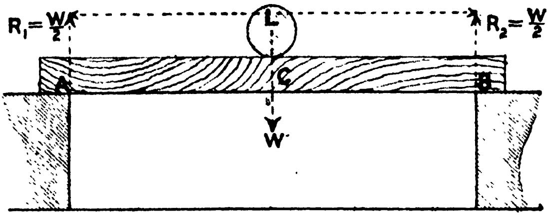


We first find the breaking weight of the beam, supposing it to be strained by a load applied at its centre.

$$W = \frac{4bd^2S}{l} = \frac{4 \times 3 \times 9 \times 1140}{144} = 855 \text{ lb., as already found.}$$

But
$$w = \frac{WP^2}{4xy} = \frac{855 \times 144^2}{4 \times 108 \times 36} = 1140 \text{ lb.}$$

632. **Pressures on and Reactions from the Supports of Beams.** If a beam is supported at its extremities and loaded at the middle, as shown by the following figure, then not only the weight of the beam, but also the load, produces pressures on and equal reactions from the supports A and B.



633. **Reactions at A and B, Load at Centre and Weight of Beam neglected.**

Let R_1 be the reaction at A, and R_2 the reaction at B; then, by taking moments about the point B, we have—

$$R_1 \times AB = W \times CB,$$

$$R_1 \times L = W \times \frac{L}{2};$$

$$\therefore R_1 = \frac{W \times L}{2 \times L} = \frac{W}{2}.$$

Also, by taking moments about the point A, we have—

$$R_2 \times BA = W \times CA,$$

$$R_2 \times L = W \times \frac{L}{2};$$

$$\therefore R_2 = \frac{W \times L}{2 \times L} = \frac{W}{2}.$$

It will be at once seen that the upward reactions are each $= \frac{1}{2}W$, and as action and reaction are equal and opposite, the pressures *downwards* at A and B (due to the load W at the centre of the beam) *must also* be equal to $\frac{1}{2}W$.

If we consider the beam as uniform throughout, and its weight as $= w$, then this force may be supposed to act at its centre of gravity, or at a distance $= \frac{1}{2}L$ from A to B. The load W also acts at a distance $\frac{1}{2}L$ from A to B. Consequently, by taking moments about B, we have—

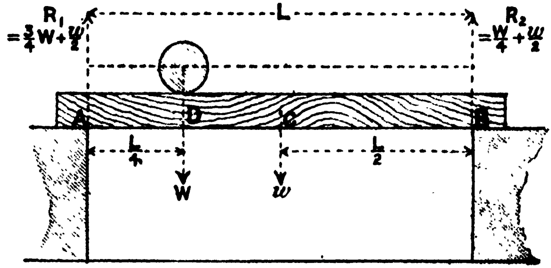
$$R_1 \times AB = W \times CB + w \times CB,$$

$$R_1 \times L = W \times \frac{L}{2} + w \times \frac{L}{2} = \frac{L}{2}(W + w);$$

$$\therefore R_1 = \frac{L}{2L}(W + w) = \frac{1}{2}(W + w) = \frac{W}{2} + \frac{w}{2}.$$

In the same way, by taking moments about A, we should find that $R_2 = \frac{W}{2} + \frac{w}{2}$; therefore the downward pressure at the points A and B must also be equal to $\frac{W}{2} + \frac{w}{2}$.

EXAMPLE 1.—A uniform beam of length L feet, and weight w lb., is supported at both ends, and carries a weight W at one-



fourth of the distance between the supports from one end; find the pressures and reactions at each point of support.

634. Pressures and Reactions at Supports A and B, due to Weight of Beam and a Load at D.—The above figure *represents the data in the question*; for the distance between the supports A and B=L, the weight w of the uniform beam acts at its centre of gravity C, or at a distance $\frac{L}{2}$ from each end, and the load W acts at D, or at a distance $\frac{L}{4}$ from one end. By taking moments about the point B, we have—

$$R_1 \times AB = W \times DB + w \times CB,$$

$$R_1 \times \ L \ = W \times \tfrac{3}{4}L + w\frac{L}{2}.$$

(Divide both sides of the equation by L.)

$\therefore$ the upward reaction at $A = R_1 = \frac{3}{4}W + \frac{1}{2}w$; *and consequently* the *downward pressure* at A, being equal and opposite to the *upward reaction* at A, must also be $= \frac{3}{4}W + \frac{1}{2}w$.

In the same way, by taking moments about the point A, we have—

$$R_2 \times BA = W \times DA + w \times CA,$$

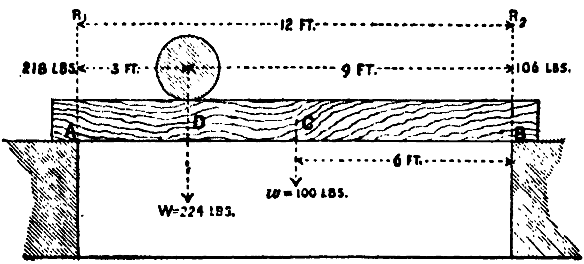
$$R_2 \times \ L \ = W \times \tfrac{1}{4}L + w \times \tfrac{1}{2}L.$$

(Divide both sides of the equation by L.)

$\therefore$ the upward reaction at $B = \frac{1}{4}W + \frac{1}{2}w$; and consequently the downward pressure at B, being equal and opposite to the upward reaction at B, must also be equal to

$$\tfrac{1}{4}W + \tfrac{1}{2}w.$$

EXAMPLE 2.—A uniform beam 12 feet long, and weighing 100 lb.,



is supported at both ends, and carries a weight of 2 cwt. at a distance of 3 feet from one end; find the pressure on each point of support.

By taking moments round B, we have—

$$R_1 \times AB = W \times DB + w \times CB,$$

$$R_1 \times 12' = 224 \times 9' + 100 \times 6'; \qquad \therefore R_1 = \frac{2616}{12} = 218 \text{ lb.}$$

To find R_2 we get—

$$R_1 + R_2 = W + w; \qquad \therefore R_2 = 224 + 100 - 218 = 106 \text{ lb.}$$

635. The following formulæ will commend themselves as being concise and less difficult to remember.

Taking the above example, and using the following formula—

$$P : W = CL : L;$$

or, expressing this proportion in words—as power is to weight, so is the counter-lever to the lever.

P = power, W = weight, CL = counter-lever, L = lever.

$$P : W = CL : L,$$

$$P : 224 :: 3 : 12,$$

$$P = \frac{672}{12} = 56 \text{ lb.} + 50 \text{ lb.} = 106 \text{ lb.},$$

and this is the power required to raise the beam from its support B. In other words, it is the pressure at the point B.

By referring to the first figure it will be easily understood why 50 is added to 56.

In order to find the pressure at the point A we proceed thus :—

$$P : W = CL : L,$$

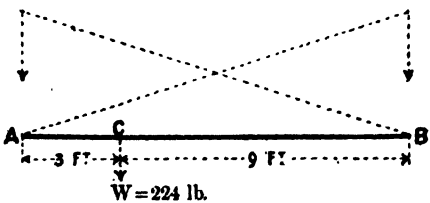
$$P : 224 :: 9 : 12,$$

$$P = \frac{2016}{12} = 168 + 50 = 218 \text{ lb.}$$

Therefore the pressure at A = 218 lb., and at B 106 lb.

It will be observed that the beam is a lever of the third order—that is, the weight is between the power and the fulcrum—and that its entire length is a lever. The counter-lever is the distance between the weight and the fulcrum.

Another solution is as follows :—



Let AB represent the beam, C the point of application of the weight 224 lb., AC and BC = distances between supports and weight.

The total distance between supports A and B = 12 feet; AC represents $\frac{3}{12}$ or $\frac{1}{4}$ of that distance, and BC = $\frac{9}{12}$ or $\frac{3}{4}$ of the same.

Then the pressure at B, neglecting weight of beam =

$$\tfrac{3}{12} \times \tfrac{224}{1} = 56 \text{ lb.},$$

and the pressure at A =

$$\tfrac{9}{12} \times \tfrac{224}{1} = 168 \text{ lb.};$$

but to each of these results we must add half the weight of the beam, viz. 50 lb.;

$$\therefore\ 56 + 50 = 106 \text{ lb.} = \text{pressure at B},$$
$$\text{and } 168 + 50 = 218 \text{ lb.} \quad " \quad " \text{ A}.$$

Exercises

1. A uniform beam 10 feet long, and weighing 1000 lb., is supported at both ends; a weight of 100 lb. is placed at a distance of 2 feet from one end; find the pressure and reaction at each point of support. . . . . = 580 lb.; 520 lb.

2. A 38-ton gun is being supported by a hydraulic jack at the breech and a tackle at the muzzle; the length of the gun is 16 feet; the point of application of the jack is 6 feet from the centre of gravity of the gun, while that of the tackle is 10 feet; find the pressure on the jack and the strain on the tackle.
= 23¾ tons on jack; 14¼ tons on tackle.

636. Stiffness of Beams (Tredgold).

B = breadth of beam in inches,
D = depth " "
L = length " feet,
W = load in lb. at the centre.

$$D = \sqrt[3]{\frac{L^2 W a}{B}}, \quad B = \frac{L^2 W a}{D^3};$$

a = ·01 fir,
= ·01 ash,
= ·013 beech,
= ·008 teak,
= ·015 elm,
= ·02 mahogany,
= ·013 oak.

When the beam is uniformly loaded, take ·625W instead of W.

637. Transverse Stress or Bending Moment of Beams.—A **transverse stress** is produced by a force or forces acting perpendicularly to the axis of a bar or beam. By axis we mean a line passing through the centres of gravity of all the transverse or cross sections of the bar or beam.

(1) Take the case of a rectangular beam where the load is applied at the centre, the beam being supported at its ends A, B, and let it be required to find the transverse stress or bending moment. *Then, neglecting the weight of the beam itself*, and confining our attention solely to the load W, we see at once that an upward reaction $=\frac{W}{2}$ is produced at A and at B. Then, by taking moments about C (the centre of the beam), we have :—

The bending moment, or B.M., at $C=\frac{W}{2}\times\frac{L}{2}=\frac{WL}{4}$.

$\therefore$ the bending moment of a beam loaded at the centre is $\frac{WL}{4}$.

Note.—This is the maximum bending moment.

(2) Should the load be uniformly distributed along its length, then the maximum bending moment is $\frac{WL}{8}$.

This shows that the bending moment at C, when the load is uniformly *distributed*, is only *half* the magnitude that it would be if the load were concentrated at the centre C. Consequently a uniform beam of certain dimensions will bear *double* the load *evenly distributed* that it can support if the load be concentrated *at or near its middle.*

(3) Should the beam be supported at both ends, and a concentrated load be placed anywhere between the points of support, the maximum bending moment is $\frac{mn}{L}W$, where m and n are the relative distances of the section from the ends, and the B.M. at any section of the beam $=\frac{\frac{mn}{L}W}{2}$ for a uniformly distributed load.

EXERCISES

1. A uniform beam 12 feet long weighs 400 lb., and is supported at its extremities; find the bending moment tending to break the beam at a point 3 feet from one end, and the shearing force.

Bending moment = 450 lb.

As previously pointed out, the shearing force or load at any point or any transverse section of the beam is equal to the resultant

or algebraical sum of all the parallel forces on either side of the point or section. Consequently the forces in this exercise on the side of A, where the shearing force is asked for, are $\frac{W}{2}$, acting vertically upwards at A, and $\frac{W}{4}$ downwards.

$\therefore$ the shearing force to the left of the section $= \frac{W}{4} = \frac{400}{4} =$ 100 lb. upwards.

The shearing force to the right of the section $= \frac{W}{4} = \frac{400}{4} = 100$ lb. downwards.

2. A uniform beam 10 feet long weighs 500 lb., and is supported at its extremities; find the bending moment tending to break the beam at a point 4 feet from one end. . . . . . $=600$ lb.

638. In order to better understand the relation of the 'shearing' and 'bending' forces, an intimate acquaintance with the science of graphic statics is necessary, and although it does not fall within the province of this work to enter into the subject at any length, the reader's attention is nevertheless directed to its important application in connection with theoretical and applied mechanics. Graphic statics is the science and art of determining by scale drawings the total stresses in the various parts of a structure. The forces transmitted through each part of a structure may be ascertained in three ways—namely, by calculation, the graphic method, and by the method of sections. The first method is extremely laborious, except in very simple problems, whereas the other methods are not only rapid, but at the same time afford self-evident means of checking the accuracy of the solution, and there is less chance of a grave error than there is in the purely analytical method.

The following works commend themselves :—*Graphics*, by Professor R. H. Smith, M.I.M.E. (Longmans, Green, & Co., London); *Principles of Graphic Statics*, by G. S. Clark (E. & F. N. Spon, London); *Elements of Graphic Statics*, by L. M. Hoskins (Macmillan & Co., London).

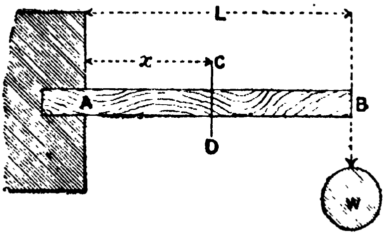
639. Problem X.—Beam fixed at one end and loaded at the other.

Let CD be a cross section anywhere within the length of the beam at a distance of x inches from the fixed end A.

To find the shearing force at section CD.

It will be observed that the only force acting to the right of the section is W lb.

$\therefore$ the shearing force $=$ W lb.



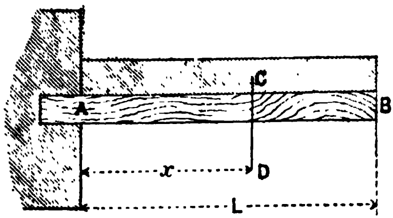
It is independent of x, and therefore the same for all such sections as CD.

The bending moment at CD is W $\times$ by its distance from the section in inches.

$\therefore$ B.M. $= \text{W} \times \text{BD} = \text{W}(\text{L} - x)$ inch-lb.

The equation is true whatever may be the position of W on the beam, so long as L denotes its distance in inches from the fixed end, and CD is between W and the support.

640. Problem XI.—Beam fixed at one end and loaded uniformly.



Regard the load on the beam as w lb. per inch run, and let it be required to find the shearing force and bending moment at any section CD at x inches from the fixed end. Consider the part of the beam to the right of CD as before. The only force is the weight of that portion of the load carried by BD ; consequently,

The shearing force (S.F.) $= w \times \text{BD} = w(\text{L} - x)$ lb. . [A].

The moment of that portion of the load on BD with respect to CD is the same as if it were all concentrated at the middle point of BD.

$\therefore$ the bending moment (B.M.) =

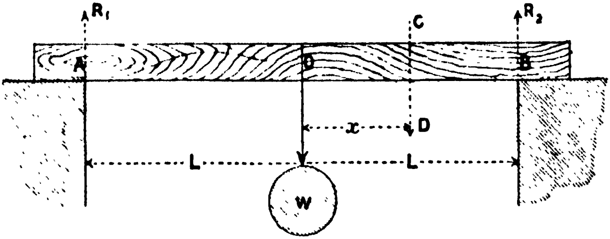
$$w \times BD \times \tfrac{1}{2}BD = \tfrac{1}{2}w \times BD^2 = \tfrac{1}{2}w(L - x)^2 \text{ inch-lb.} \qquad [B],$$

and

$$\left.\begin{aligned} S.F. &= wL \text{ lb.,} \\ B.M. &= \tfrac{1}{2}wL^2 \text{ inch-lb.,} \end{aligned}\right\} \qquad [C].$$

Equations [A] and [B] demonstrate that both the S.F. and B.M. disappear when the quantity x = that of L; and when $x = 0$ we get Equations [C].

641. Problem XII.—Beams supported at both ends and loaded at the middle.



Here we measure x from the middle point of the beam. As W is equidistant from A and B, the reactions at those points, R_1 and R_2, are equal to each other; and as their sum is W, we have—

$$R_1 = R_2 = \tfrac{1}{2}W \text{ lb.}$$

The force to the right of CD is R_2, and its leverage is BD.

$$\therefore S.F. = R_2 = \tfrac{1}{2}W \text{ lb.} \qquad [D],$$

and

$$B.M. = R_2 \times BD = \tfrac{1}{2}W(\tfrac{1}{2}L - x) \text{ inch-lb.} \qquad [E].$$

Note in this case that the *bending moment* disappears when $x = \tfrac{1}{2}L$, and increases uniformly from this until $x = 0$; it then attains its maximum value, $\tfrac{1}{4}WL$; or,

$$\text{Maximum bending moment} = \tfrac{1}{4}WL \text{ inch-lb.} \qquad [F].$$

Example.—Take a beam of length L, supported at both ends, and let it be loaded at the centre with any load W; prove that the bending moment is greatest at the middle of the beam and equal to $\tfrac{1}{4}WL$; then determine by graphic method the bending moment and shearing force at a point 6 feet from one support in a beam whose length is 25 feet between points of support, supposing it to be loaded with 5 tons at its centre.

From Equation [E] we find that, for a beam loaded as in this example, the bending moment at any distance x from its centre is

$$B.M. = \tfrac{1}{2}W(\tfrac{1}{2}L - x).$$

This is obviously greatest when $x=0$, that is, at the centre. Then, maximum bending moment $=\tfrac{1}{4}WL$, and shearing force $=\tfrac{1}{2}W$. Consequently, for the numerical values of W and L in the question before us, we have :—

Maximum B.M. $= \cdot25 \times 5 \times 25 = 31\cdot25$ foot-tons,
and shearing force $= \cdot5 \times 5 = 2\cdot5$ tons.

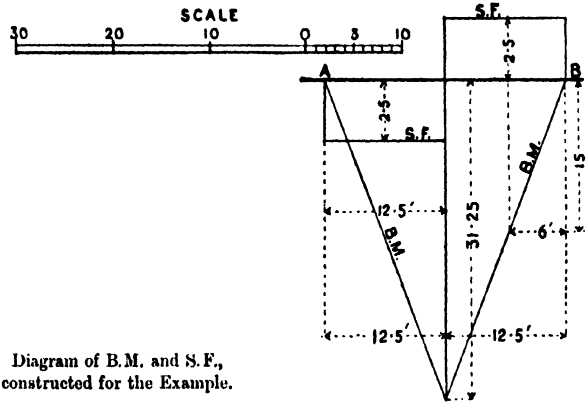


Diagram of B.M. and S.F., constructed for the Example.

At 6 feet from one end the bending moment measures 15 foot-tons. This is easily verified by means of the formula for B.M., because $x = 12\cdot5 - 6 = 6\cdot5$. Consequently the

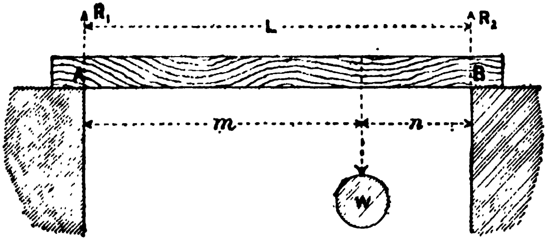
$$B.M. = \tfrac{1}{2} \times 5 \times (12\cdot5 - 6\cdot5) = 15 \text{ foot-tons.}$$

642. **Beam supported at both ends and loaded anywhere.**—The maximum bending moment with a single concentrated load will always occur immediately under the load, whether it be at the middle of the beam or not.

For the bending moment at any section at a distance x from one end is $R \times x$, and this is greatest when x is largest—that is, when the section is under the load.

To find the reactions at the supports we take moments about A and B, and get $R_2 \times L = W \times m$.

Consequently $R_2 = \frac{m}{L}W$ lb., and $R_1 = \frac{n}{L}W$ lb., and these are the



values of the shearing force (S.F.) to the right and left of W respectively—

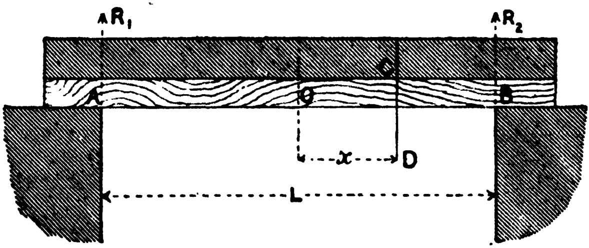
$$\text{S.F. to the right} = \frac{m}{L}W \text{ lb.,}$$

$$\text{S.F. to the left} = \frac{n}{L}W \text{ lb.}$$

Multiplying the first of these equations by n, or the latter by n, we get:—

$$\text{Maximum bending moment} = \frac{mn}{L}W \text{ inch-lb.}$$

643. **Beam supported at both ends, and loaded uniformly.**—Let the weight, as before, per inch run be denoted by w; then the total load carried by the beam will be wL lb., and

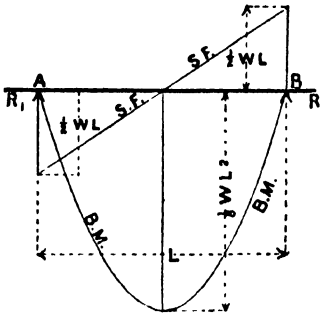


the reactions R_1 and R_2 will each be $\frac{1}{2}wL$ lb. Taking the forces to the right of the section CD, we have the S.F. $= R_2 - w \times BD$ $= \frac{1}{2}wL - w(\frac{1}{2}L - x) = wx$ lb.;

and
$$\begin{aligned}\text{B.M.} &= R_2 \times BD - w \,.\, BD \times \tfrac{1}{2}BD \\ &= \tfrac{1}{2}wL \times BD - \tfrac{1}{2}w \,.\, BD^2 \\ &= \tfrac{1}{2}w \,.\, BD(L - BD) \\ &= \tfrac{1}{2}w(\tfrac{1}{2}L - x)(\tfrac{1}{2}L + x), \\ &= \tfrac{1}{2}w(\tfrac{1}{4}L^2 - x^2) \text{ inch-lb.}\end{aligned}$$

By plotting the diagrams of S.F. and B.M. we get this figure:—

S.F. curve is a straight line.



B.M. curve is a parabola with vertex below the middle of the beam.

The limit values of S.F. and B.M. are:—

When $x = \tfrac{1}{2}L$, then S.F. $= \tfrac{1}{2}wL$ lb., and B.M. $= 0$;

" $x = 0$, then S.F. $= 0$, and maximum B.M. $= \tfrac{1}{8}wL^2$ inch-lb.

644. When a beam carries more than one load, or is loaded in more ways than one, the simplest and safest way is to consider each load separately, without regard to the others, and then combine the separate effects so as to obtain the resultant action, as follows.

EXAMPLE.—Draw the bending moment and shearing force diagrams for a beam 12 feet long, supported at both ends, and loaded with weights of 4 and 6 tons at distances of 3 and 8 feet respectively from one end of the beam.

Measuring distances from the left end of the beam, and considering each load separately, we have for the 4 tons to the left of the load—

$$\text{S.F.}_1 = \frac{n}{L}W = \frac{9}{12} \times 4 = 3 \text{ tons};$$

and to the right of it—

$$\text{S.F.}_1 = \frac{m}{L}W = \frac{3}{12} \times 4 = 1 \text{ ton}.$$

The maximum bending moment due to this load is:—

$$\text{B.M.}_1 = \frac{mn}{L}W = \frac{3 \times 9}{12} \times 4 = 9 \text{ foot-tons}.$$

It occurs immediately under the load.

Next, taking the 6 tons load, we have to the left of it—

$$\text{S.F.}_{\cdot 2} = \frac{n}{L}W = \frac{4}{12} \times 6 = 2 \text{ tons};$$

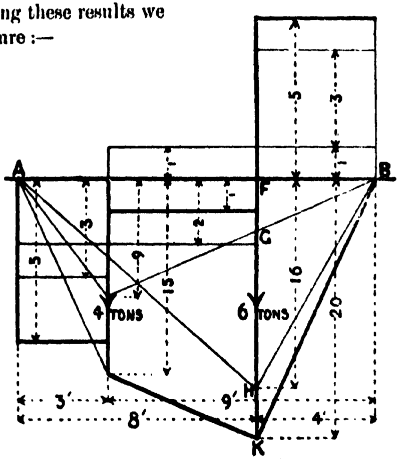
and to the right of it—

$$\text{S.F.}_{\cdot 2} = \frac{m}{L}W = \frac{8}{12} \times 6 = 4 \text{ tons}.$$

The maximum bending moment due to the 6 tons is :—

$$\text{B.M.}_{\cdot 2} = \frac{mn}{L}W = \frac{8 \times 4}{12} \times 6 = 16 \text{ foot-tons}.$$

By plotting these results we get this figure :—



S.F. and B.M. Curves for preceding Example.

The thin lines show the actions of the separate loads, and the full lines their combined results, obtained by taking the algebraic sum of the former.

The reader should carefully note the necessity of attending to the *sign* of the shearing force. Thus, between the weights we have a shearing force of 2 tons, which, on account of its sign, is drawn below the base-line; also a shearing force of 1 ton drawn above the base-line. The resultant shearing force between the loads is therefore the difference of these, and is drawn on the same side of the base-line as the greater of its components.

The bending moments everywhere along the beam are of the same sign; therefore, to obtain the combined bending moment

diagram, we have simply to add the ordinates of each separate diagram. Thus, to get the total bending moment at the section under the 6 tons load, we add FG (viz. that due to the 4 tons at that point) to FH (that due to the 6 tons). The result FK is therefore the total bending moment at that point.

It is quite sufficient to do this for the sections under each load, and then join each of the points so obtained with each other and with the ends of the beam by straight lines. If drawn to scale, the bending moment at any other point can then be obtained by measuring the corresponding ordinate.

Exercises on the Bending Moment and Shearing Force of Beams.

1. A beam 12 feet long is supported at its ends, and is loaded with a weight of 3 tons at a point 2 feet from one end; find the bending moment at the centre of the beam, and also the shearing force. . . . . . . B.M. = 36 inch-tons; S.F. = 0·5 ton.

2. A beam is 20 feet in length, and is supported at both ends; it is loaded with 1 ton evenly distributed along its length; find the bending moment at a distance of 7 feet from one end (neglect its weight). . . . . = 5096 foot-lb.

3. A uniform beam, fixed at one end and free at the other, is 10 feet long, and weighs 6 cwt.; it carries two loads, one of 2 cwt. at the free end, and the other 4 cwt. at its middle point; find the shearing forces at points $2\frac{1}{2}$ feet and 6 feet from the fixed end. = 10·5 cwt.; 6·4 cwt.

4. Find the bending moment and shearing force at a point 8 feet from the same support in the beam here mentioned; it is of uniform shape and weighs 15 cwt., and rests on supports at its ends which are 20 feet apart; it is loaded with three weights of 4, 6, and 10 cwt., at distances of 2, 7, and 12 feet respectively from one of the supports. . . . B.M. = 98 foot-cwt.; S.F. = 3 cwt.

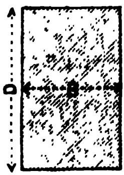
645. The following formulæ will be found useful for determining the strength of rectangular beams and girders:—

Let L = length of beam or span in inches,
B = breadth " " "
D = depth " " "
W = breaking weight in cwt.,
* K = coefficient of rupture,
M = multiplier for deflection (see 'Deflection').

* The value of K is the transverse strength of the material expressed in cwt. or lb. (see Table).

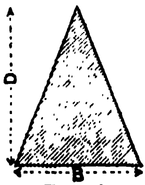
	W	K	M
One end fixed, the other loaded,	$\frac{KBD^2}{L}$	$\frac{LW}{BD^2}$	·33
One end fixed, weight distributed, ...	$\frac{2KBD^2}{L}$	$\frac{LW}{2BD^2}$	·125
Ends supported, weight at centre, ...	$\frac{4KBD^2}{L}$	$\frac{LW}{4BD^2}$	·02
Ends supported, weight distributed, ...	$\frac{8KBD^2}{L}$	$\frac{LW}{8BD^2}$	·013
Ends fixed, weight distributed,	$\frac{12KBD^2}{L}$	$\frac{LW}{12BD^2}$	·0032

To find the breaking weight of beams of the following sections, use the formula for W given above, but substituting for BD^2 the values of V for the section required. I = moment of inertia.



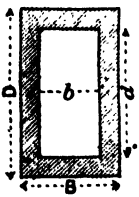
Rectangle

$$I = \frac{BD^3}{12}$$
$$V = BD^2$$



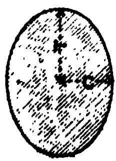
Triangle

$$I = \frac{BD^3}{36}$$
$$V = \frac{BD^2}{4}$$



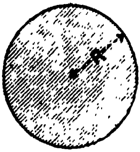
Hollow Rectangle

$$I = \frac{BD^3 - bd^3}{12}$$
$$V = \frac{BD^3 - bd^3}{D}$$



Ellipse

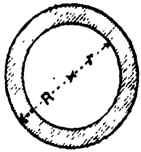
$$I = \cdot 7854CT^3$$
$$V = 4{\cdot}7CT^2$$



Circle

$I = \cdot 7854 R^4$

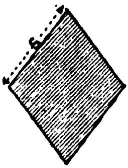
$V = 4 \cdot 7 R^3$



Hollow Circle

$I = \cdot 7854 (R^4 - r^4)$

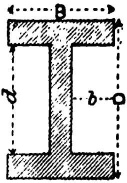
$V = 4 \cdot 7 \left(\frac{R^4 - r^4}{R} \right)$



$I = \frac{S^4}{12}$

$V = S^3$

Square



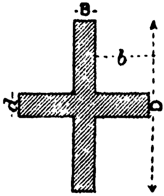
$I = \frac{BD^3 - 2bd^3}{12}$

$V = BD^2 - 2bd^2$

Semicircle

$I = \cdot 11 R^4$

$V = \cdot 38 R^3$



$I = \frac{BD^3 + 2bd^3}{12}$

$V = BD^2 + 2bd^2$

Note.—The safe weight that may be put on beams in permanent structures is from ⅓ to ¼ the breaking weight of the beam.

646. STRENGTH AND WEIGHT OF MATERIALS

Table C

Materials		Weight of a Cubic Inch	Weight of a Cubic Foot	Tensile Strength per Square Inch	Crushing Weight per Square Inch	Transverse Strength per Square Inch
Metals		Lb.	Lb.	Tons	Tons	Tons
Aluminium, sheet,		·096	166·6	12	—	—
" cast,		·092	159·8	8	—	—
Antimony, cast,		·242	419 5	·47	—	—
Bismuth, cast,		·353	613·1	1·45	—	—
Copper, bolts,		·318	552·4	17	—	—
" cast,		·31	537·3	8·4	—	—
" sheet,		·316	548·1	13·4	—	—
" wire,		·32	555	26	—	—
Gold,		·665	1150	9·1	—	—
Iron, cast,	from	·252	437	6	36	2
	to	·273	474·4	13	64	3·4
" " average,		·26	451	7·3	48	2·6
" wrought,	from	·273	474·4	16	16	3
	to	·281	486·9	29	18	5·5
" " average,		·28	485·6	22	16·9	3·8
" wire,		—	—	40	—	—
Lead, cast,		·408	708 5	·8	3·1	—
" sheet,		·41	711·6	1·5	—	—
Mercury,		·49117	848·75	—	—	—
Platinum,		·775	1343·9	—	—	—
" sheet,		—	—	—	—	—
Silver,		·377	653·8	18·2	—	—
Steel,		·288	499	52	150	—
" plates,		—	—	35	90	—
Tin, cast,		·262	455·1	2	6·7	—
Zinc, cast,		·252	437	3·3	—	—
Alloys						
Aluminium bronze, 90 to 95 per cent. copper,		·276	478·4	32	58	—
Bell metal (small bells),		·29	502·52	1·4	—	—
Brass, cast,		·3	524·37	8	—	—
" sheet,		·361	526·86	14	—	—
" wire,		·307	533·109	22	—	—
" 5 copper, 1 zinc,		·3	525·09	13 7	—	—
" 4 " 1 "		·304	527·36	14 7	—	—
" 3 " 1 "		·3	524·18	13·1	—	—
" 2 " 1 "		·299	518·06	12·5	—	—
" 1 " 1 "		·296	513·75	9·2	—	—
" 1 " 2 "		·298	517·06	19·3	—	—

Materials		Weight of a Cubic Inch	Weight of a Cubic Foot	Tensile Strength per Square Inch	Crushing Weight per Square Inch	Transverse Strength per Square Inch
		Lb.	Lb.	Tons	Tons	Tons
Brass, 1 copper, 4 zinc		·265	460·13	1·9	—	—
Gold (standard),		·638	1106·42	—	—	—
Gun metal, 10 copper, 1 tin,		·306	528·36	16·1	—	—
" 9 " 1 "		·305	528·24	15·2	—	—
" 8 " 1 "		·305	528·05	17·7	—	—
" 7 " 1 "		·305	527·89	13·6	—	—
Pewter,		—	—	—	—	—
Silver (standard),		·371	643·72	—	—	—
Speculum metal,		·264	464·87	3·1	—	—
White metal (Babbett),		·263	456·32	—	—	—
Timber						
				Lb.	Lb.	Lb.
Acacia,	from	·025	44	16000	—	1867
	to	·028	49	—	—	—
Ash,	from	·025	43	12000	8600	2000
	to	·027	47	17000	9300	3000
Beech,	from	·025	43	11000	7700	1500
	to	·025	43	22000	9300	2000
Birch,	from	·026	44	15000	3300	1900
	to	·026	45	—	6000	1930
Box,		·046	80	20000	10000	2445
Cedar, West Indian,		·026	47	5000	5700	1443
" American,		·020	35	—	—	766
" Lebanon,		·017	30	11000	5800	1300
Chestnut,		·022	38	12000	—	1770
Cork,		·008	15	—	—	—
Deal, Christiania,		·025	43	12000	5850	1562
Ebony,		·043	74	—	19000	2100
Elm, English,	from	·02	34	13200	10300	782
	to	·021	36	14000	—	1100
" Canadian,		·026	45	—	—	1920
Fir, spruce,		·018	32	10100	6500	1490
Hornbeam,		·027	47	20000	4600	—
Ironwood,		·041	71	—	—	3000
Greenheart,		·041	71	—	—	—
Larch,		·019	34	8900	3200	1330
"		·02	35	10200	5500	1660
Lignum-vitæ,		·048	83	11800	10000	3440
Lime,		·02	35	—	—	—
Mahogany, Nassau,		·024	42	—	—	1719
" Honduras,		·02	35	21000	8000	1910
" Spanish,		·031	53	—	8200	1300
Maple,		·025	42	10600	—	1694
Oak, African,		·035	62	—	—	2523
" American, red,		·03	53	10000	6000	1680

Materials		Weight of a Cubic Inch	Weight of a Cubic Foot	Tensile Strength per Square Inch	Crushing Weight per Square Inch	Transverse Strength per Square Inch
		Lb.	Lb.	Lb.	Lb.	Lb.
Oak, American, white,		·028	49	—	—	—
" English,	from	·028	48	10000	6400	1600
	to	·034	58	19000	10000	1690
Pine, red,	from	·021	36	12000	5400	1200
	to	·024	41	14000	7500	1530
" white,	from	·015	27	—	—	1229
	to	·02	34	—	—	—
" yellow,		·018	32	—	5300	1185
" Dantzic,		·023	40	8000	5400	1426
" Memel,	from	·02	34	—	—	1348
	to	·021	37	—	—	—
" Riga,	from	·017	29	—	—	—
	to	·023	41	14000	—	1383
Satinwood,		·034	60	—	—	3200
Teak,	from	·026	46	8000	12000	2110
	to	·031	54	15000	—	—
STONES, &c.						
Basalt, Scotch,		·106	184	1469	8300	—
" Greenstone,		·104	181	—	17200	—
" Welsh,		·099	172	—	16800	—
Chalk,	from	·084	145	—	501	—
	to	·094	162	—	—	—
Firestone,		·065	112	—	—	—
Granite, Aberdeen gray,		·094	163	—	10900	—
" " red,		·095	165	—	—	—
" Cornish,		·096	166	—	14000	—
" Mount Sorrel,		·096	167	—	12800	—
Limestone, compact,		·093	161	—	7700	—
" Purbeck,		·093	162	—	9160	—
" Anglesea,		—	—	—	7579	—
" Blue Lias,		·089	154	—	—	—
" Lithographic,		·093	162	—	—	—
Marble, statuary,		·098	170	722	3216	—
" Italian,		·098	170	—	9681	—
" Brabant block,		·097	168	—	9219	—
" Devonshire,		—	—	—	7428	—
Oolite, Portland stone,		·087	151	—	4100	—
" Bath stone,		·072	123	—	—	—
Sandstone, Arbroath pavement,		·089	155	1261	7884	—
" Bramley Fall,		·09	156	—	6050	—
" Caithness,		·095	165	1054	6490	857
" Craigleith,		·088	153	453	5287	—
" Derby grit,		·086	150	—	3100	—
" Red, Cheshire,		·077	133	—	2185	—

Materials	Weight of a Cubic Inch	Weight of a Cubic Foot	Tensile Strength per Square Inch	Crushing Weight per Square Inch	Transverse Strength per Square Inch
	Lb.	Lb.	Lb.	Lb.	Lb.
Sandstone, Yorkshire paving, .	·09	157	—	5714	—
Slate, Anglesea, ...	·103	179	9600 to	10000 to	1961
" Cornwall, ...	·09	157	12800	21000	
" Welsh, ...	·104	180	"	"	
" Trap, ...	·098	170	—	—	—
Miscellaneous Substances					
Asphalt, ...	·09	156	—	—	—
Brick, common, . . from	·057	100		—	—
to	·072	125	—	—	
" London stock, . .	·066	115	—	—	
" red, ...	·077	134	—	808	—
" Welsh fire, . . .	·086	150	—	—	
" Stourbridge fire, . .	·079	137	—	1717	
Cement, Portland, in powder, . from	·05	86	400	3795	—
to	·054	94	600	5984	
Cement, Roman, ...	·057	100	185	—	—
Clay, ...	·068	119	—	—	—
Coal, anthracite, ...	·055	95	—	—	—
" cannel, ...	·046	79	—	—	—
" Glasgow, ...	·046	80	—	—	—
" Newcastle, ...	·045	79	—	—	—
Coke, ...	·026	46	—	—	—
Concrete, ordinary, . . .	·068	119	—	—	—
" in cement, . . .	·079	137	—	—	—
Earth, ... from	·054	77	—	—	—
to	·072	125	—	—	—
Glass, flint, ...	·111	192	2413	27500	—
" crown, ...	·091	157	2546	31000	
" common green, . . .	·091	158	2896	31876	—
" plate, ...	·099	172	—	—	—
Gutta-percha, ...	·035	60	—	—	—
Gypsum, ...	·082	143	71	—	—
India-rubber, ...	·033	58	—	—	—
Ivory, ...	·065	114	—	—	—
Lime, quick, ...	·03	53	—	—	
Mortar, ... from	·049	86	—	—	—
to	·068	119	—	—	—
" average, ...	·061	106	—	—	—
Pitch, ...	·041	69	—	—	—
Plumbago, ...	·082	140	—	—	—
Sand, quartz, ...	·099	171	—	—	—
" river, ...	·067	117	—	—	—

647. Deflection of Beams and Girders in terms of Weight.

L = length of span in inches,
W = weight on beam in lb.,
I = moment of inertia (see Table, 'Various Sections of Beams'),
E = Young's modulus of elasticity (see Table A),
S = stress in tons per square inch on material of beam or girder,
d = deflection of beam or girder in inches,
D = effective depth.

Note.—If W is in tons, the modulus of elasticity E is, say, 8000 for cast-iron, 13000 for steel, and 11000 for wrought-iron; but if W is in lb., the value of E must be taken from Table A.

One end fixed, the other loaded, $d=\frac{W.L^3}{3EI}$;

" " uniformly " $d=\frac{WL^3}{8EI}$;

Ends supported, load at centre, $d=\frac{WL^3}{48EI}$;

" " load distributed, $d=\frac{5WL^3}{384EI}$;

" " uniform stress, $d=\frac{SL^2}{4ED}$;

" fixed, load at centre, $d=\frac{WL^3}{192EI}$;

" " weight distributed, $d=\frac{WL}{384EI}$.

Note.—The greatest deflection usually allowed in beams is 1 inch in 100 feet, or $\frac{1}{1200}$ of the span.

Example.—Find the deflection of a cast-iron beam 18 feet in length, breadth 1 inch, and depth 12 inches, when loaded at the centre with a weight of 6000 lb. The ends of the beam are supported.

Take modulus of elasticity = 18400000.

$$d=\frac{WL^3}{48EI}=\frac{6000\times 10077696}{48\times 18400000\times 144}=\cdot 47 \text{ inch.}$$

Exercises

1. Find the deflection of a cast-iron beam supported at both ends, with a weight of 12000 lb. at its centre; its length = 18 feet, breadth 2 inches, and depth 12 inches. . . . = ·475 inch.

2. What weight should be placed at the centre of a cast-iron beam of the following dimensions, length 18 feet, breadth 1 inch, depth 12 inches, in order to deflect the beam $\frac{1}{4}$ inch, the ends being supported? . . . . =3155·006 lb.

By integrating the formula we get—

$$W = \frac{d . 48 . EI}{L^3}.$$

3. A plate of steel 18 inches wide and $\frac{1}{2}$ inch thick sustains a weight of 1 ton at the centre of a 35-inch span; find the deflection (take $E = 4200000$). . . . . = ·458 inch.

4. Find the deflection of a wrought-iron bar 2 inches square, 25 inches span, with a load of 3 tons at the centre. = ·065 inch.

5. Find the breaking weight of a wrought-iron bar 1 inch square and 12 inches long, supported at the ends; the bar is made of the best material. . . . . =1·8$\dot{3}$ tons.

6. A wrought-iron solid beam of the following dimensions—length 14 feet, breadth 6 inches, and depth 9 inches; what weight uniformly distributed over it would be sufficient to break it, supposing its ends are fixed? Take the transverse strength = 3·8 tons. = 131·9 tons.

7. Suppose the beam in No. 6 exercise to be loaded up to 80 tons; find its deflection. . . . . = ·000008 inch.

8. Find the weight that should be placed as a central load on this beam in order that the usual amount of deflection be not exceeded. . . . . = 12873·2 lb.

9. Find the breaking weight of a rectangular beam of ash, also its deflection, when its ends are fixed and it is loaded uniformly (distributed load); its dimensions are—length 14 feet, breadth 6 inches, and depth 9 inches. Transverse strength = 19 cwt.

Breaking weight = 32·9 tons; deflection = ·00005 inch.

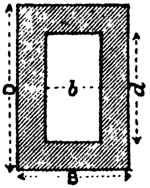
10. Find what load placed at the centre of the beam in No. 9 exercise would break it, and state the deflection just as the beam gave way. Let its ends be supported. = 10·99 tons; 4·03 inches.

11. Find the breaking weight, safe load, and deflection under the breaking load of a square beam of pine whose length = 14 feet and side 12 inches, when the ends are supported and the weight is distributed. Take coefficient K = 13 cwt.

Breaking weight = 26·74 tons, safe weight = 5·34 tons, deflection = 1·3 inches.

12. Find the breaking weight of a cast-iron beam whose length

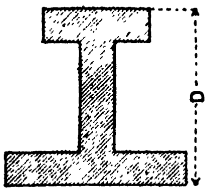
is 20 feet, and with a load at its centre; the ends are supported, and the section as shown in the fig. Take the transverse strength $=52$ cwt.



$D = 12$ inches,
$B = 8$ "
$d = 8$ "
$b = 4$ "

$= 554\frac{2}{3}$ cwt.

648. Breaking Weight of Cast-iron Girders.



D = depth of girder in inches,
A = area of bottom flange in inches,
S = span in inches,
W = breaking weight in tons.

Supported at both ends, with load at centre, $W = \frac{25AD}{S}$.

Supported at both ends, with load distributed, $W = \frac{50AD}{S}$.

If the depth $= \frac{1}{12}$ of the span, $W = A 4{\cdot}17$, } where the weight is
" " $= \frac{1}{10}$ " " $W = A \times 5$, } distributed.

Area of the top flange if the load is applied on the top $= \frac{A}{3}$.

Area of the top flange if the load is applied on the bottom flange $= \frac{A}{2}$.

Depth at the ends may equal $\frac{2D}{3}$.

Safe deflection, $\frac{1}{40}$ inch for each foot of span, under a test load of $\frac{1}{3}$ of the breaking weight.

EXAMPLE.—Find the breaking weight of a cast-iron girder of the above section from the following particulars:—Length of span 10 feet, depth of girder 10 inches, size of bottom flange $6'' \times 1\frac{1}{4}''$, with a distributed load.

$$W = \frac{50AD}{S} = \frac{50 \times 7{\cdot}5 \times 10}{10 \times 12} = 31{\cdot}25 \text{ tons.}$$

Exercise

Find the breaking weight of the following girders of the same section with a distributed load.

Span in Feet	Depth in Inches	Size of Bottom Flange	Answer
15	15	$8 \times 1\frac{1}{2}$	50 tons.
20	20	$10 \times 1\frac{1}{2}$	62·5 "
25	25	$13 \times 1\frac{3}{4}$	94·79 "
30	30	15×2	125·0 "
35	35	17×2	$141\cdot\dot{6}$ "

649. Deflection of Iron and Steel Girders, ends supported.

The usual allowance in American bridges is $\frac{1}{1200}$ after the girder is set.

S = span in feet,
P = stress on the metal by any load in tons per square inch,
E = modulus of elasticity in tons = say 10000 for iron and 13000 for steel,
D = effective depth of girder in feet,
d = deflection of girder in inches;

$d = \dfrac{3S^2P}{ED} = SK$. (For value of K, see Table, p. 380.)

Exercises

1. Find the deflection of a wrought-iron girder whose effective depth = 3 feet and span 30 feet; the stress on the metal = 5 tons per square inch. . . . . = ·45 inch.

2. What deflection would a steel girder have with a span of 50 feet, its effective depth being 4 feet, and stress = 6 tons per square inch? . . . . = ·87 inch.

It will be noticed that the deflection is too great, the girder being badly proportioned.

3. Find the deflection of a steel girder under a stress of 8 tons per square inch, the span = 32 feet, and the effective depth of the girder = 4 feet. . . . . = ·47 inch.

TABLE OF VALUES OF K

Material	Stress—Tons per Sq. Inch		Rates of Effective Depth of Girder to Span								
			1/8	1/9	1/10	1/11	1/12	1/13	1/14	1/15	1/16
Iron	·5	K	·0012	·0014	·0015	·0017	·0018	·0020	·0021	·0023	·0024
„	1	K	·0024	·0027	·0030	·0033	·0036	·0039	·0042	·0045	·0048
„	2	K	·0048	·0054	·0060	·0066	·0072	·0078	·0084	·0090	·0096
„	3	K	·0072	·0081	·0090	·0099	·0108	·0117	·0126	·0135	·0144
„	4	K	·0096	·0108	·0120	·0132	·0144	·0156	·0168	·0180	·0192
„	5	K	·0120	·0135	·0150	·0165	·0180	·0195	·0210	·0225	·0240
Steel	·5	K	·0009	·0010	·0012	·0013	·0014	·0015	·0016	·0017	·0018
„	1	K	·0018	·0021	·0013	·0025	·0028	·0030	·0032	·0035	·0037
„	2	K	·0037	·0041	·0046	·0051	·0056	·0060	·0064	·0069	·0074
„	3	K	·0055	·0062	·0069	·0076	·0083	·0090	·0097	·0104	·0110
„	4	K	·0074	·0083	·0092	·0101	·0110	·0120	·0129	·0138	·0147
„	5	K	·0092	·0104	·0115	·0127	·0138	·0150	·0161	·0173	·0184
„	6	K	·0110	·0124	·0138	·0152	·0166	·0179	·0193	·0207	·0221
„	7	K	·0129	·0145	·0161	·0177	·0193	·0209	·0225	·0242	·0258
„	8	K	·0147	·0166	·0184	·0202	·0221	·0239	·0258	·0276	·0294

650. **To find the deflection of a beam or girder of uniform section, we have the following formula:—**

$$\frac{W \cdot l^3}{24(MR)d} \times E = \frac{W \cdot l^3}{4bd} \cdot E \quad \text{for a rectangular beam,}$$

where W = weight in tons,
l = span in inches,
(MR) = moment of resistance of cross section in inches,
d = depth of beam or girder, or, rather, twice the distance of the fibres most strained from the neutral axis,
E = modulus of elasticity.

$E = \cdot00018$ for cast-iron,	$= \cdot001$ for teak.
$E = \cdot00010$ " wrought-iron,	$= \cdot002$ " oak and pitch-pine.
$E = \cdot00008$ " steel,	$= \cdot003$ " fir.

To be in a position to use the above formula, we must first determine the value of (MR) for each section. For a rectangular section, (MR), the moment of resistance or modulus of section, $= \frac{bd^2}{6}$. For other sections, see Table at the end of this subject.

651. **Determination of Moment of Inertia.**

I = moment of inertia,
N = distance of neutral axis from **lower** edge of section,
H = height of any particles from **lower** edge of section,
d = distance of any particles from the neutral axis,
B = breadth of section at any height H,
Σ = sum,
Δ = difference.

$I = \frac{2\Sigma B\Delta(d^3)}{3}$, if the neutral axis be in the centre and the figure be symmetrical; if not,

$$I = \frac{\Sigma B\Delta(H^3)}{3} - AN^2.$$

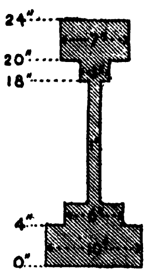
$$A = \Sigma B\Delta H. \quad N = \frac{\Sigma B\Delta(H^2)}{2A}.$$

The neutral axis, for all practical purposes, passes through the centre of gravity of any section.

The following example demonstrates the application of the formula given.

The more closely the section is divided into minute rectangles, the more accurate will be the result.

H	H^2	H^3	ΔH	ΔH^2	ΔH^3	B	$B\Delta H$	$B\Delta H^2$	$B\Delta H^3$
0	0	0	0	0	0				
4	16	64	4	16	64	10	40	160	640
6	36	216	2	20	152	6	12	120	912
18	324	5832	12	288	5616	1	12	288	5616
20	400	8000	2	76	2168	3	6	228	6504
24	576	13824	4	176	5824	7	28	1232	40768
							98 $=\Sigma B\Delta H$ $=A$	2028 $\Sigma B\Delta H^2$	54440 $\Sigma B\Delta H^3$



$$N=\frac{\Sigma B\Delta(H^2)}{2A}=\frac{2028}{196}=10\cdot346,$$

$$I=\frac{\Sigma B\Delta H^3}{3}-AN^2$$
$$=\tfrac{54440}{3}-98\times(10\cdot346)^2$$
$$=18146\cdot\dot{6}-10489\cdot8$$
$$=7656\cdot7.$$

∴ Height of neutral axis from lower edge of section $=10\cdot346=N$, and moment of inertia $=7656\cdot7$.

652. Now, the moment of resistance (MR) or modulus of the section is found as follows :—

(MR) = moment of resistance,
I = moment of inertia,
N = height of neutral axis from farthest edge of section,
$M^{r\!e}$ = modulus of rupture,
K = coefficient of fracture ;

$$(MR)=\frac{6\,.\,K\,.\,I}{N}\cdot\quad(MR)=\frac{M^{re}I}{N}.$$

The modulus of rupture M^{re} is found by multiplying the transverse strength of the material by 6. For transverse strength, see Table, 'Strength and Weight of Materials.'

EXAMPLES.—1. To find the deflection of a 60-lb. double-headed rail, $4\tfrac{1}{2}''$ deep, under a load of 1 ton at the centre of a 33" span. The moment of resistance is 6·7.

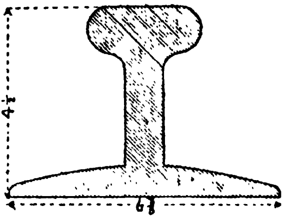
Here $W = 1{\cdot}0267$ tons, $l = 33^3 = 35937$ inches,

$(MR) = 6{\cdot}7$, $d = 4{\cdot}5$ inches,

$$\frac{Wl^3}{24(MR)d} \times E = \frac{1{\cdot}0267 \times 35937}{24 \times 6{\cdot}7 \times 4{\cdot}5} = \frac{36896{\cdot}5}{723{\cdot}6} = 50{\cdot}9.$$

$$50{\cdot}9 \times E = 50{\cdot}9 \times {\cdot}0001 = {\cdot}005 \text{ inch.}$$

If the neutral axis passes through centre of section, mean of 33 experiments (Baker) = ·005 inch.



2. Find the deflection of the 84-lb. rail shown in the fig. when loaded with 2000 lb. at the centre of a 60-inch span; find also the strain on the extreme fibres, the depth being 4½ inches.

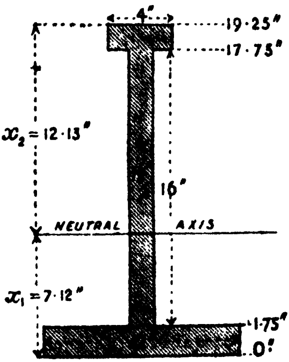
For this particular section (MR) $= 2{\cdot}49 \times 3{\cdot}5$.

Note.—The strain on the extreme fibres is given by the formula $S = \frac{Wl}{4(MR)}$, where S is in tons per square inch, W in tons, l in inches.

$$\frac{Wl^3}{24(MR)d} = \frac{93 \times 216000}{24 \times (2{\cdot}49 \times 3{\cdot}5) \times 4{\cdot}5}$$
$$= 213{\cdot}43 \times {\cdot}0001 = {\cdot}021.$$

Neutral axis assumed passing through centre of section.

$$S = \frac{Wl}{4(MR)} = \frac{93 \times 60}{4 \times 8{\cdot}715} = \frac{10{\cdot}95}{8{\cdot}715}$$
$$= 1{\cdot}25 \text{ tons per square inch.}$$



3. To determine the moment of resistance to bending of the section of the cast-iron girder as shown in the above fig. The maximum safe tensile and compressive stresses are 2½

and $7\frac{1}{2}$ tons per square inch respectively. Its dimensions are as follows:—Top flange, $4'' \times 1\frac{1}{2}''$; bottom flange, $12'' \times 1\frac{3}{4}''$; web, $16'' \times 1\frac{1}{2}''$.

Determine the moment of resistance if the girder is 20 feet long, and is supported at its two ends. Find the greatest safe load which it will carry when uniformly distributed along its length.

We first find the position of the neutral axis thus:—

H	H²	H³	ΔH	ΔH²	ΔH³	B	BΔH	BΔ(H²)	BΔ(H³)
0	0	0	0	0	0				
1·75	3·06	5·35	1·75	3·06	5·35	12	21	36·72	64·2
17·75	315·06	5592·31	16·00	312·	5586·96	1·5	24	468·00	8380·44
19·25	370·56	7188·28	1·50	55·5	1540·97	4	6	222·00	6163·88
							51	726·72	14608·52
						=	$\Sigma B\Delta H$ = A	$\Sigma B\Delta H^2$	$\Sigma B\Delta H^3$

$$N = \frac{\Sigma B\Delta H^2}{2A} = \frac{726 \cdot 72}{102} = 7 \cdot 12 \text{ inches,}$$

$$I = \frac{\Sigma B\Delta H^3}{3} - AN^2 = \frac{14608 \cdot 52}{3} - 51 \times (7 \cdot 12)^2 = 4869 \cdot 506 - 2585 \cdot 19$$
$$= 2284 \cdot 31.$$

653. The neutral axis is of fundamental importance in the theory of beams and girders, because it is the fulcrum about which both the bending and resisting couples act.

Should E not be the same for tensile and compressive stresses, then the neutral axis will not pass through the centre of the area, but will lie to the side having the greater value of E.

The greatest stress comes on the fibres farthest from the neutral axis, and is the principal effect to be considered in the question of strength.

Now, the moment of inertia for the whole section is found to be 2284·31.

For tension the modulus $= \dfrac{I}{x_1} = \dfrac{2284 \cdot 31}{7 \cdot 12} = 320.$

" compression " $= \dfrac{I}{x_2} = \dfrac{2284 \cdot 31}{12 \cdot 13} = 188.$

Tensile stress = 2·5 tons per square inch.

Compressive " = 7·5 " "

We must therefore take the lower value of the two resisting moments in fixing the load to be carried by the girder.

These are $320 \times 2{\cdot}5 = 800$ inch-tons,
and $188 \times 7{\cdot}5 = 1410$ " .

$\therefore$ bending moment = resisting moment = 800 inch-tons.

The girder will therefore carry safely a uniformly distributed load given by the equation on bending moments, Art. 649—namely, $\frac{1}{8}wL^2 = 800$.

$$\therefore W = \frac{8 \times 800}{20 \times 12} = 26\tfrac{2}{3} \text{ tons.}$$

This will make the maximum compressive stress $\frac{800}{188} = 4{\cdot}255$ tons, instead of 7·5 as given; showing that the girder is not well designed.

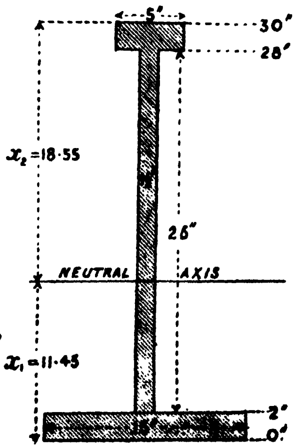
In a properly proportioned girder we should have :—

Modulus for tension × tensile stress per square inch
= " compression × compressive " "

Exercises

1. Find the breaking load (distributed) of the girder mentioned in the last example. . . . . =84·2 tons.

2. Find the breaking weight of the cast-iron girder in the accompanying fig. when loaded at the centre of a 30-foot span. What would its deflection be when carrying a load of 45 tons at its central point? Top flange $=5'' \times 2''$, web $=26'' \times 1{\cdot}5''$, bottom flange $=15'' \times 2''$. Also state its moment of inertia, and the height of neutral axis from lower edge of section. Total depth of girder $=30''$.

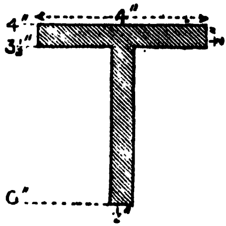


Moment of inertia = 9068·275,
Neutral axis = 11·45 inches,
Breaking load = 62·5 tons,
Deflection = ·58", or ·76".

For the deflection, see p. 387.

3. If the tensile and compressive stresses are limited to 1·5 tons and 9 tons respectively in the girder mentioned in the second exercise, find the greatest safe load that the girder will carry

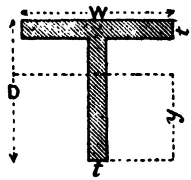
when loaded at its centre. What will the maximum compressive stress be? . . . . $=13\cdot2$ tons; $2\cdot4$ tons.



4. Determine the load that may safely be distributed on the section of wrought-iron as shown in the fig. if the span $=20$ feet, the tensile and compressive stresses being limited to 5 tons and $3\cdot5$ tons respectively. . . . . $=737$ lb.

5. If the position of the iron was reversed, and the other conditions the same as in the last question, what would be the safe load? . . . . $=590$ lb.

6. Find the position of the neutral axis and moment of inertia for the following sections of T iron.



W = width;	D = depth; in inches	t = thickness,	I (Moment of Inertia)	N.A. (Neutral Axis)
when 4	4	½	5·5641	2·816
3	4	½	5·0485	2·6731
4	3	½	2·4234	2·1731
3½	3½	½	3·635	2·4423
3½	3½	⅜	2·865	2·487

654. Should it be required to find the deflection when E and I are known, one of the formulæ ('Deflection in Terms of Weight') may be employed when W has been ascertained.

If M, the bending moment or moment of resistance, has been found, then the deflection may be determined by formula already given, or by one of those found in Table, 'Strength and Stiffness of Beams,' at the end of the subject.

Any difference in the results will be due to the value of E, as already pointed out. Another point the student or reader will do well to notice is this: for the sections in the Table, the neutral axis passes through centre of gravity of each section.

655. In the formula for deflection of beams and girders of uniform section, namely, $\frac{Wl^3}{24(MR).d}E$, the moment of resistance must first be ascertained. In the case of a cantilever of rectangular cross section loaded at the outer end, the moment of resistance $(MR)=Kbd^2$, where (MR) = resisting moment in inch-lb., and K = a constant number found by trial depending upon the nature of the material of which the beam is composed. It has been assumed that the beam or girder is of uniform section, so that I, the moment of inertia, is constant; the more general cases where I varies being rather beyond the scope of this work.

On the whole, it would be safer to adhere to the formulæ containing I as a quantity; but before closing the subject, the following examples will present the application of the formulæ more fully.

Taking the second exercise, let it be required to find the deflection of the girder in terms of the maximum bending moment, and also by formula as below.

(1) Deflection in terms of M (see Table) $=\frac{1}{12}\frac{Ml^2}{EI}$,

$M=\frac{Wl}{4}$ (see Prob. XII. p. 364), W in lb., l in inches,

$W=45$ tons $=100800$ lb.,

$l=30$ feet $=360$ inches;

$$\therefore \frac{Wl}{4}=\frac{100800\times360}{4}=9072000,$$

and
$$\frac{1\times M\times l^2}{12\times E\times I}=\frac{1\times9072000\times129600}{12\times18400000\times9068{\cdot}275}={\cdot}58 \text{ inch.}$$

This answer agrees with that already found in terms of W.

(2) The deflection (see formula $\frac{W\times l^3}{24\times(MR)\times d}\cdot E$)

$$=\frac{45\times46656000}{24\times1186{\cdot}5\times18{\cdot}55}\times{\cdot}00018={\cdot}71 \text{ inch.}$$

Here d = distance of fibres most strained from neutral axis = 18·55 inches.

We will now explain how (MR) has been obtained, for in unsymmetrical sections there are two values of the modulus of the section to be considered.

The ratio $\frac{I}{y}$ is usually noted by z; y is any distance above or below the neutral axis.

The modulus for tension $= z_t = \frac{I}{y} = \frac{9068 \cdot 275}{11 \cdot 45} = 791 \cdot 9.$

" " compression $= z_c = \frac{I}{y} = \frac{9068 \cdot 275}{18 \cdot 55} = 488 \cdot 8.$

Now, if the greatest permissible tensile and compressive stresses were limited to 1·5 tons and 9 tons respectively per square inch—and these, as already stated, are the working stresses for cast-iron—then the tensile stress $= 791 \cdot 9 \times 1 \cdot 5 = 1186 \cdot 5$ inch-tons, and the compressive stress $= 488 \cdot 8 \times 9 = 4399 \cdot 2$ inch-tons. We must therefore take the lower value of the two resisting moments (MR) in order to determine the load to be carried by the girder.

$$\therefore \text{BM} = (\text{MR}) = 1186 \cdot 5 \text{ inch-tons.}$$

The girder will therefore carry a central load given by the equation $\frac{1}{4}$WL (see Prob. XII. p. 364).

$$\therefore W = \frac{4 \times 1186 \cdot 5}{30 \times 12} = 13 \cdot 18 \text{ tons.}$$

This makes the maximum compressive stress $\frac{1186 \cdot 5}{488 \cdot 8} = 2 \cdot 4$ tons, the difference between the answers of the two formulæ being $\frac{13}{100}$ of an inch.

Note.—Should the moment of resistance be calculated by means of the formula $(\text{MR}) = \frac{6\text{KI}}{\text{N}}$ (p. 382), then $\frac{Wl^3}{24\text{M}h} \cdot \text{E}$ becomes $\frac{Wl^3}{4\text{M}h} \cdot \text{E}$.

Exercises

1. Find the greatest load that may uniformly be distributed on a cast-iron girder, having top and bottom flanges united by a web of the following dimensions. Width of upper flange 3 inches, of lower flange 9 inches; total depth 12 inches; thickness of each flange and of the web 1 inch; distance between the points of support 10 feet. The greatest admissible stress in the compression flange is 3 tons per square inch, and that in the tension flange is $1\frac{1}{2}$ tons per square inch. . . . . $= 8 \cdot 8$ tons.

2. Find the deflection of this girder by means of formula $\frac{1}{12} \cdot \frac{Ml^2}{\text{EI}}$, supposing it to be loaded at the centre with a weight of 5 tons. Take $\text{I} = 398$. . . . . $= \cdot 05$.

3. Find the deflection by formula $\frac{Wl^3}{24(\text{MR})d} \cdot \text{E}$. . . . $= \cdot 06$.

4. Find the deflection when (MR) is $= \frac{6 \cdot \text{KI}}{\text{N}}$. . . . . $= \cdot 06$.

5. A uniform beam of oak, 10 feet in length, 15 inches deep, and 10 inches wide, sustains, in addition to its own weight, a load of 5000 lb. placed at its centre; find the greatest bending moment and the greatest stress in the fibres.

Take the specific gravity of oak as 0·934.

Here the greatest bending moment takes place at the centre of the beam, and is made up of two parts—(1) that due to the beam's own weight, which is uniformly distributed along its length; and (2) that due to the 5000 lb. concentrated at its middle.

The greatest stress in the fibres is ascertained by formula

$$f = \frac{\text{B.M. or (MR)}}{\text{I}} \times y.$$

f stands for either the tensile or compressive stress, at any distance y above or below the neutral axis.

B.M. = 159072 inch-lb.; greatest stress = 424·1 lb. per square inch.

656. To find the strength of thin wrought-iron girders.

The formulæ for the moment of resistance are very simple, for here the flanges are thin in comparison with their distance apart, the bending resistance of the web being disregarded as a provision against the shearing force acting at the section.

Let A_t = area of flange in tension,
A_c = " " compression,
H = distance between centres of flanges,
f_t = mean stress in tension flange,
f_c = " " compression flange.

Distance between centre of tension flange and the neutral axis is

$$y_t = \left(\frac{A_c}{A_t + A_c}\right)H.$$

The moment of inertia of the flanges with respect to the neutral axis is

$$I = A_t\left(\frac{A_c}{A_t + A_c}\right)^2 H^2 + A_c\left(\frac{A_t}{A_t + A_c}\right)^2 H^2;$$

or

$$I = \{A_t \times A_c^2 + A_c \times A_t^2\} \times \left(\frac{H}{A_t + A_c}\right)^2;$$

$$f = \frac{M}{I}y.$$

$$\therefore f_t = \frac{M}{A_t \times A_c^2 + A_c \times A_t^2\left(\frac{H}{A_t + A_c}\right)^2} \times \left(\frac{A_c}{A_t + A_c}\right)H.$$

Hence $$f_t = \frac{M}{A_t \times H}.$$

Similarly, $$f_c = \frac{M}{A_c \times H}.$$

EXAMPLE.—A wrought-iron girder of I section has a top flange of 9 square inches in sectional area, and a bottom flange of 8 square inches. The distance between the centres of gravity of the flanges is 12 inches, and the ends of the beam rest on abutments 16 feet apart. The girder is loaded uniformly with a load equal to 1 ton per lineal foot (including the weight of the girder). What would be the mean stress per square inch on the metal in each flange at the dangerous section?

The resistance of the web to bending is neglected.

By 'dangerous section' is here meant the middle section of the girder, where the maximum bending moment occurs.

Maximum B.M. $= \frac{1}{8}(\frac{1}{12}) \times (16 \times 12)^2 = 32 \times 12$ inch-tons.

$\therefore$ mean stress in tension flange—

$$f_t = \frac{32 \times 12}{8 \times 12} = 4 \text{ tons per square inch;}$$

and mean stress in compression flange—

$$f_c = \frac{32 \times 12}{9 \times 12} = 3 \cdot 55 \text{ tons per square inch.}$$

In compression, iron may be strained to 4 tons per square inch.

In tension, iron may be strained to 5 tons per square inch.

The regulations of the French department 'Ponts et Chaussées' allow 3·81 tons per square inch.

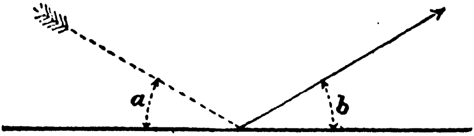
Steel may be strained to 6 tons per square inch in tension and compression.

657. Collision of Bodies.

W = weight of one body,
V = velocity of one body before impact,
Y = " " " after "
K = coefficient of restitution of the one body,
w = weight of the other body,
v = velocity of the other body before impact,
y = " " " after "
k = coefficient of restitution of the other body,
= 0 for a non-elastic body, = 1 for a perfectly elastic body.

The ideal elastic body is one for which the coefficient of restitution

is unity, and should such a body strike a plane surface, it would rebound at an angle equal to that at which it struck the plane; in other words, the angle of incidence (a) = the angle of reflection (b).



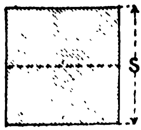
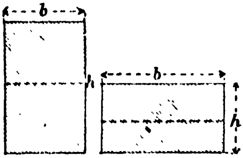
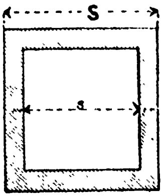
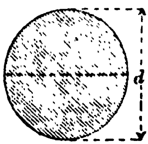
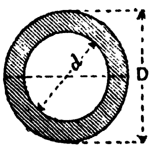
Note.—Practically this is never true, since no body is known which has its coefficient of restitution equal to unity.

For notation, see 'Collision of Bodies.'

Conditions	Non elastic Bodies	Elastic Bodies
One body in motion, . .	$y = \frac{WV}{W+w}$	$y = \frac{WV(1+k)}{W+w}$ $Y = \frac{V(W-Kw)}{W+w}$
Bodies moving in the same direction, .	$y = \frac{WV+wv}{W+w}$	$y = \frac{WV(1+k)+v(w-kW)}{W+w}$ $Y = \frac{V(W-Kw)+vw(1+K)}{W+w}$
Bodies moving in contrary directions, .	$y = \frac{WV-wv}{W+w}$	$y = \frac{WV(1+k)-v(w-kW)}{W+w}$ $Y = \frac{V(W-Kw)-vw(1+K)}{W+w}$

When the bodies are inelastic their velocities after impact will be alike, or $Y = y$.

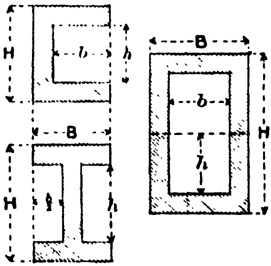
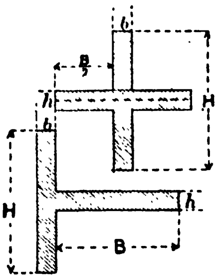
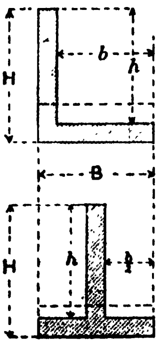
The plane of bending is supposed perpendic

Form of Section	Area of Section
	A S^2
	bh
	$S^2 - s^2$
	$\cdot 7854d^2$
	$\cdot 7854(D^2 - d^2)$

o plane of paper, and parallel to side of page

Moment of Inertia of Section about Axis through Centre of Gravity	Square of Radius of Gyration of Section $\frac{I}{A}$	Modulus of Section $\frac{I}{y}$ y = any dist. above or below N.A.
I $\frac{S^4}{12}$	$\frac{S^2}{12}$	$\frac{S^3}{6}$
$\frac{bh^3}{12}$	$\frac{h^2}{12}$	$\frac{bh^2}{6}$
$\frac{S^4 - s^4}{12}$	$\frac{S^2 + s^2}{12}$	$\frac{1}{6}\left(\frac{S^4 - s^4}{S}\right)$
$\cdot 0491 d^4$	$\frac{d^2}{16}$	$\cdot 0982 d^3$
$\cdot 0491 (D^4 - d^4)$	$\frac{D^2 - d^2}{16}$	$\cdot 0982\left(\frac{D^4 - d^4}{D}\right)$

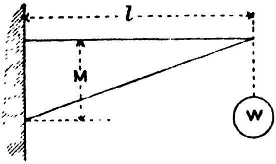
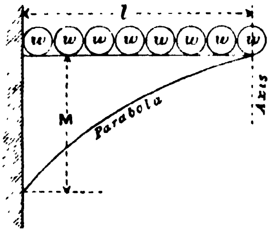
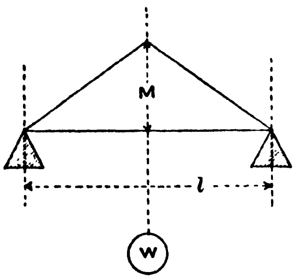
The plane of bending is supposed perpendicu

Form of Section	Area of Section
	A $BH - bh$
	$Bh + bH$
	$BH - bh$

to plane of paper, and parallel to side of page

Moment of Inertia of Section about Axis through Centre of Gravity I	Square of Radius of Gyration of Section $\frac{I}{A}$	Modulus of Section $\frac{I}{y}$ y = any dist. above or below N.A.
$\frac{BH^3 - bh^3}{12}$	$\frac{1}{12}\left(\frac{BH^3 - bh^3}{BH - bh}\right)$	$\frac{BH^3 - bh^3}{6H}$
$\frac{Bh^3 + bH^3}{12}$	—	$\frac{Bh^3 + bH^3}{6H}$
$\frac{(BH^2 - bh^2)^2 - 4BHbh(H - h)^2}{12(BH - bh)}$	—	$\frac{(BH^2 - bh^2)^2 - 4BHbh(H - h)^2}{6(BH^2 + bh^2 - 2bHh)}$

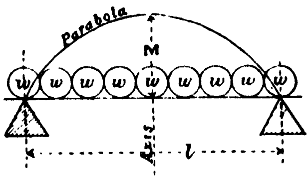
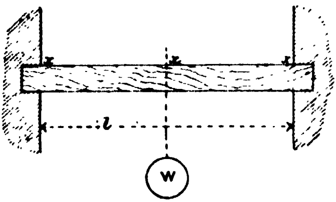
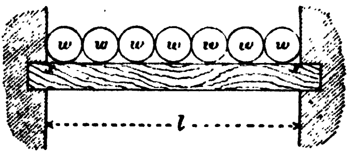
f stands for either the tensile or compressive stress

Manner of Supporting and Loading	Maximum Bending Moment M	Relative Strength
 Cantilever Loaded at End	$W \cdot l$	$\frac{1}{4}$
 Cantilever Loaded Uniformly	$\frac{Wl}{2}$	$\frac{1}{2}$
 Supported at both Ends. Loaded at Centre	$\frac{Wl}{4}$	1

at any distance y above or below the neutral axis

Deflection in terms of W	Deflection in terms of M	Deflection in terms of Stress	Relative Stiffness under same Load
$\frac{1}{3} \cdot \frac{Wl^3}{EI}$	$\frac{1}{3} \cdot \frac{Ml^2}{EI}$	$\frac{1}{3} \cdot \frac{fl^2}{Ey}$	$\frac{1}{16}$
$\frac{1}{8} \cdot \frac{Wl^3}{EI}$	$\frac{1}{4} \cdot \frac{Ml^2}{EI}$	$\frac{1}{4} \cdot \frac{fl^2}{Ey}$	$\frac{1}{6}$
$\frac{1}{48} \cdot \frac{Wl^3}{EI}$	$\frac{1}{12} \cdot \frac{Ml^2}{EI}$	$\frac{1}{12} \cdot \frac{fl^2}{Ey}$	1

f stands for either the tensile or compressive stress

Manner of Supporting and Loading	Maximum Bending Moment M	Relative Strength
 Supported at both Ends. Loaded Uniformly	$\frac{Wl}{8}$	2
 Ends Fixed. Loaded at the Centre	$\frac{Wl}{8}$	2
 Ends Fixed. Loaded Uniformly	$\frac{Wl}{12}$	3

at any distance y above or below the neutral axis

Deflection in terms of W	Deflection in terms of M	Deflection in terms of Stress	Relative Stiffness under same Load
$\frac{5}{384} \cdot \frac{Wl^3}{EI}$	$\frac{5}{48} \cdot \frac{Ml^2}{EI}$	$\frac{5}{48} \cdot \frac{fl^2}{Ey}$	$\frac{8}{5}$
$\frac{1}{192} \cdot \frac{Wl^3}{EI}$	$\frac{1}{24} \cdot \frac{Ml^2}{EI}$	$\frac{1}{24} \cdot \frac{fl^2}{Ey}$	4
$\frac{1}{384} \cdot \frac{Wl^3}{EI}$	$\frac{1}{32} \cdot \frac{Ml^2}{EI}$	$\frac{1}{32} \cdot \frac{fl^2}{Ey}$	8

PROJECTILES AND GUNNERY

658. The subject of projectiles, considered in a practical point of view, treats of the methods of determining by calculation various circumstances belonging to the motions of bodies projected in the atmosphere.

This subject is divided into two parts—namely, the parabolic and flat trajectory theories.

I.—THE PARABOLIC THEORY OF PROJECTILES

In the parabolic theory several hypotheses not strictly correct are made; but only one of them can lead to any sensible error in practice, though in some cases the error is comparatively small. This last hypothesis is, that there is no resistance from the atmosphere to the motion of a projectile; and the other two are, that gravity acts in parallel lines over a small extent of the earth's surface, and that its intensity is constant from its surface to a small height above it. The parabolic theory applies to all ordnance with high angle fire and low muzzle velocity, such as howitzers and mortars.

659. Problem I.—Of the height fallen through by a body, the velocity acquired, and the time of descent, any one being given, to find the other two.

Let h = the height fallen through,
v = " velocity acquired,
t = " time of descent,
$g = 32{\cdot}2$ feet;

then
$$h = \tfrac{1}{2}vt = \tfrac{1}{2}gt^2 = \frac{v^2}{2g},$$
$$v = gt = \sqrt{2gh} = \frac{2h}{t},$$
$$t = \frac{v}{g} = \sqrt{\frac{2h}{g}} = \frac{2h}{v}.$$

These relations of h, v, and t are proved in treatises of theoretical mechanics. Any two of these three quantities are said to be **due** to the other; thus the velocity acquired by falling from a given height is said to be due to that height, and so of the other two quantities. The acquired velocity is also called the **final** velocity. The number 32·2 is the velocity in feet that a body acquires in falling during one second. The velocity with which a body is thrown upwards or downwards is called its **initial** velocity. Should the body be thrown upwards the force of gravity imparts a negative acceleration, and if thrown downwards it imparts a positive acceleration.

Therefore the sign of g is + in the first case and − in the second case.

In solving the following exercises, such a formula is to be chosen in each case as contains the elements concerned—that is, the quantities given and sought.

EXAMPLES.—1. What is the velocity acquired in falling 10 seconds?

$$v = gt = 32{\cdot}2 \times 10 = 322.$$

2. What is the height fallen through in 5 seconds?

$$h = \tfrac{1}{2}gt^2 = \tfrac{1}{2} \times 32{\cdot}2 \times 5^2 = 402{\cdot}5.$$

EXERCISES

1. What velocity would be acquired in falling 120 feet?
=87·9 feet.

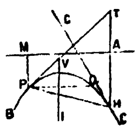
2. Required the height through which a body must fall to acquire the velocity of 1500 feet per second. . . . . =34938 feet.

3. In what time will a body acquire the velocity of 900 feet?
=27·95 seconds.

4. In how many seconds would a body fall 27000 feet?
=40·95 seconds.

660. When a body is projected in any direction except that of a vertical line, it describes a parabola.

Thus, if a body is projected in the direction PT it will describe a curvilineal path, as PVH, which will be a parabola.



661. The velocity with which the body is projected is called the **velocity of projection.**

During the time that the projectile would be carried, by the velocity of projection continued uniform, to T, it would be carried by the force of gravity

from T to H. But the distance PT is evidently proportional to the time, whereas TH is proportional to the square of the time. Since (Art. 659) $h = \frac{1}{2}gt^2 = 16 \cdot 1t^2$, therefore TH is proportional to the square of PT. And the same is true for any other line drawn, as TH, from a point in PT to the curve; and this is a property of the parabola.

662. The velocity of projection is that due to a height equal to the distance of the point of projection from the directrix of the parabola described by the projectile.

Or, the velocity at P is that acquired in falling down MP, AM being the directrix.

663. The velocity in the direction of the curve at any other point in it is equal to the velocity due to its distance from the directrix.

The velocity at any point, as H, is that due to AH; and if a body were projected with that velocity in the direction of the tangent HG, it would describe the same curve HVP, and on arriving at P, would have the velocity due to MP.

664. The height due to the velocity of projection is called the **impetus.**

Thus MP is the impetus.

665. The distance between the point of projection and any body to be struck by the projectile is called the **range,** and sometimes the **amplitude.** *When the range lies in a horizontal plane it is* called the **horizontal** range.

Thus, P being the point of projection and H the body struck, PH is the range, and PQ the horizontal range.

666. The time during which a projectile is moving to the object is called the **time of flight.**

667. The angle contained by the line of projection and the horizontal plane is called the **angle of elevation.**

Thus TPQ is the angle of elevation.

668. The inclination of the horizontal plane to the plane passing through the point of projection and the object is called the **angle of inclination.**

Thus HPQ is the angle of inclination.

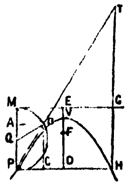
The range of a projectile may be either on a horizontal or an oblique plane.

669. Projectiles on Horizontal Planes.—The following formulæ afford rules for calculating the impetus, range, velocity of projection, time of flight, and elevation :—

Let h = the impetus MP in feet,
v = " velocity of projection in feet per second,
t = " time of flight in seconds,
r = " horizontal range = PH,
e = " angle of elevation = TPH,
r' = " greatest range,
h' = " " height = VD;

then $h = \frac{v^2}{2g}$ by Art. 659; $r = 2h \sin 2e$;
$v = \sqrt{2gh}$ " Art. 659; $r' = 2h$;
$t = 2 \sin e \sqrt{\frac{2h}{g}}$; $h' = h \sin^2 e$.

Let PT be the line of projection, and PVH the curve described.



On PM describe a semicircle MBP, and from its intersection with the tangent PT in B, draw BC parallel to the axis, and BA perpendicular to the impetus MP. Then AB = PC = $\frac{1}{4}$PH = $\frac{1}{4}r$, and BC = $\frac{1}{4}$TH, and VD = BC. Draw the radius OB, then (Eucl. III. 32) angle BPC or e = BMP = $\frac{1}{2}$POB, or POB = $2e$. Now,

$$AB/OB = \sin BOP,$$

or $$\tfrac{1}{4}r/\tfrac{1}{2}h = \sin 2e;$$

hence $\frac{1}{4}r = \frac{1}{2}h \sin 2e$, and $r = 2h \sin 2e$.

Again, the time of flight is just equal to the time of describing PT uniformly with the velocity of projection, or the time of falling through TH by gravity. Now, if TH = h'',

$$Hr/PH = \tan TPH, \text{ or } h''/r = \tan e;$$

hence $h'' = r \tan e = 2h \sin 2e \tan e = 2h \frac{2 \sin e \,.\, \cos e \,.\, \sin e}{\cos e}$;

and therefore $h'' = 4h \sin^2 e$.

But if t is the time due to h'', then

$$t = \sqrt{\frac{2h''}{g}} = \sqrt{\frac{8h \sin^2 e}{g}} = 2 \sin e \sqrt{\frac{2h}{g}},$$

which is the expression above for t.

Again, VD = BC = $\frac{1}{4}$TH, or $h' = \frac{1}{4}h'' = h \sin^2 e$.

When $e = 15°$ or $75°$, $\sin 2e = \sin 30° = \frac{1}{2}$, and $r = h$.

The greatest value of r or $2h \sin 2e$, for a given value of h, is when $\sin 2e$ is a maximum or $2e = 90$, and $e = 45$; for then $\sin 2e = 1$, and $r = 2h$.

670. **Two elevations, of which the one is as much greater than 45° as the other is less, give the same horizontal range.**

For if these angles of elevation are $45+d$ and $45-d$, then for these elevations $2e$ is $90+2d$ and $90-2d$, which are each other's supplements; and hence $\sin(90+2d)=\sin(90-2d)$, and the two values of r are

$$r=2h \sin(90+2d), \text{ and } r=2h \sin(90-2d),$$

which are equal.

671. **Problem II.—Given the velocity of projection, or the impetus and the elevation, to find the range, the time of flight, and the greatest altitude of the projectile.**

The formulæ to be used are $h=\frac{v^2}{2g}$, $r=2h \sin 2e$, $t=2 \sin e \sqrt{\frac{2h}{g}}=\frac{2v}{g}\sin e$, and $h'=h \sin^2 e$.

Or, r, t, and h' may sometimes be more easily found by logarithms; thus—

$$\begin{aligned} \mathrm{L}r &= \mathrm{L}2h + \mathrm{L}\sin 2e - 10. \\ \mathrm{L}t &= \mathrm{L}2v + \mathrm{L}\sin e - (10 + \mathrm{L}g). \\ \mathrm{L}h' &= \mathrm{L}h + 2\mathrm{L}\sin e - 20. \end{aligned}$$

Example.—A ball was discharged with a velocity of 300 feet at an elevation of 24° 36′; required the range, the time of flight, and the greatest altitude.

$$h=\frac{v^2}{2g}=\frac{300^2}{2\times 32\cdot 2}=1397\cdot 5,$$

$$r=2h \sin 2e=2795\times \cdot 756995=2116,$$

$$t=\frac{2v}{g}\sin e=\frac{600}{32\cdot 2}\times \cdot 4162808=7\cdot 76 \text{ seconds.}$$

$$h'=h \sin^2 e=242.$$

Exercises

1. A shell being discharged at an elevation of 28° 30′, and with a velocity of 230 feet in a second, what is the impetus, the range, the time of flight, and the greatest elevation?

$h=821$, $r=1378$, $h'=187$, and $t=6\cdot 82$ seconds.

2. The impetus with which a cannon-ball is fired is $=3600$, and the elevation $=75°$, and the elevation of another fired with the same impetus was $=15°$; required the ranges. . . . $=3600$.

3. Required the time of flight of a shell fired at an elevation of 32°, with an impetus of 1808 feet. . . $=11\cdot 23$ seconds.

672. Problem III.—Given the range and elevation, to find the velocity of projection.

From $r=2h \sin 2e$ is found $h=\frac{r}{2 \sin 2e}$, the 1st formula; and from $h=\frac{v^2}{2g}$ is derived $v=\sqrt{2gh}$, the 2nd formula.

Or by logarithms—

$$L2h=10+Lr-L \sin 2e,$$

and $$Lv=\tfrac{1}{2}(L2g+Lh).$$

The greatest altitude and time of flight are found as in last problem.

EXAMPLE.—A ball was projected at an elevation of 54° 20′, and was found to range 2000 feet; required the initial velocity.

$$h=\frac{r}{2 \sin 2e}=\frac{2000}{2 \times \cdot 9473966}=1055 \cdot 5,$$

and $$v=\sqrt{2gh}=\sqrt{2 \times 32 \cdot 2 \times 1055 \cdot 5}=\sqrt{67974 \cdot 2}=260 \cdot 7.$$

EXERCISES

1. A shell projected from a mortar at an elevation of 60° was found to range $=3520$ feet; required the impetus and velocity of projection. . . . . . . $h=2032 \cdot 25$, and $v=361 \cdot 77$.

2. A ball projected at an elevation of 15° or 75° was found to range over 5200 feet; what was the impetus and velocity of discharge? . . . . $h=5200$, and $v=578 \cdot 69$.

3. The elevation being $=45°$, and range $=12000$, what is the impetus? . . . . $=6000$.

673. Problem IV.—Given the impetus or projectile velocity and the range, to find the elevation.

Since $h=\frac{v^2}{2g}$, and $r=2h \sin 2e$, therefore $\sin 2e=\frac{r}{2h}=\frac{gr}{v^2}$, and the formulæ are

$$\sin 2e=\frac{r}{2h} \text{ when } h \text{ is given,}$$

and $$\sin 2e=\frac{gr}{v^2} \text{ when } v \text{ is given.}$$

Or, $$L \sin 2e=Lr+10-L2h,$$

and $$L \sin 2e=Lg+Lr+10-2Lv.$$

EXAMPLES.—1. At what elevation must a piece of ordnance be fired so as to throw a ball $=5600$ feet, the initial velocity being $=800$ feet?

$$\text{Sin } 2e = \frac{gr}{v^2} = \frac{32 \cdot 2 \times 5600}{800^2} = \cdot 28175 = \sin 16^\circ\ 21'\ 53'';$$

hence $e = 8^\circ\ 10'\ 56''$, and $90^\circ - e = 81^\circ\ 49'\ 4''$,
which are the two elevations.

The greatest height and the time of flight can now be found as in the first problem.

2. At what elevation will a mark at the distance of 5100 yards be hit with an impetus of 3000 yards?

$$\text{Sin } 2e = \frac{r}{2h} = \frac{5100}{2 \times 3000} = \cdot 85 = \sin 58^\circ\ 13';$$

hence $e = 29^\circ\ 6'\ 30''$, and $90^\circ - e = 60^\circ\ 53'\ 30''$.

Exercises

1. At what elevation must a shell be fired, with a velocity of 420 feet, so as to range $= 5400$ feet? . . . $= 40^\circ\ 9'$, or $49^\circ\ 51'$.

2. Required the elevation necessary to hit an object $= 4200$ yards distant with an impetus of 4000 yards. . . $= 15^\circ\ 50'$, or $74^\circ\ 10'$.

674. Problem V.—Given the elevation and time of flight, to find the range and velocity of projection.

The formulæ are $r = \frac{1}{2}gt^2 \cot e$, $v = \frac{gt}{2 \sin e}$.

Or,
$$\text{L } 2r = \text{L}g + 2\text{L}t + \text{L} \cot e - 10,$$
$$\text{L } 2v = \text{L}g + \text{L}t + 10 - \text{L} \sin e.$$

Example. – A ball projected at an angle of $32^\circ\ 20'$ struck the horizontal plane 5 seconds after; what was the range and projectile velocity?

$$r = \tfrac{1}{2}gt^2 \cot e = \tfrac{1}{2} \times 32 \cdot 2 \times 25 \times 1 \cdot 5798079 = 635 \cdot 87,$$

and
$$v = \frac{gt}{2 \sin e} = \frac{32 \cdot 2 \times 5}{2 \times \cdot 534844} = \frac{161}{1 \cdot 069688} = 150 \cdot 5;$$

as v is known, h can now be found by Art. 673.

The formulæ are obtained thus:—Since $h = \frac{v^2}{2g}$, and $t^2 = 4 \sin^2 e \frac{2h}{g}$; hence $h = \frac{gt^2}{8 \sin^2 e}$, and therefore (Art. 669) $r = 2h \sin 2e = \frac{gt^2}{4 \sin^2 e} \cdot \sin 2e$; but $\sin 2e = 2 \sin e \cdot \cos e$, and $\frac{\cos e}{\sin e} = \cot e$; hence $r = \frac{1}{2}gt^2 \cot e$.

Also,
$$v = \sqrt{2gh} = \sqrt{\frac{g^2t^2}{4 \sin^2 e}} = \frac{gt}{2 \sin e}.$$

EXERCISE

The time of flight of a shell projected at an elevation of 60° was $=25$ seconds; what was the initial velocity and the range?

$r=5809 \cdot 6$, and $v=464 \cdot 76$.

675. Besides the preceding theorems for projectiles on horizontal planes, many more might be given of less importance; the two following are sometimes useful:—

676. For the same impetus, the ranges are proportional to the sines of twice the angles of elevation.

Let r and r' be two ranges corresponding to the elevations e and e',

then $\quad r : r' = \sin 2e : \sin 2e'$; and therefore $r' = r \dfrac{\sin 2e'}{\sin 2e}$;

also, $$\sin 2e' = \frac{r'}{r} \cdot \sin 2e.$$

EXERCISES

1. If a shell range 1000 yards at an elevation of 45°, how far will it range at an elevation of 30° 16'? . . . $=870 \cdot 642$ yards.

2. If the range of a shell at an elevation of 45° is $=3750$, what must be the elevation for a range of 2810 feet? $=24° 16'$, or $65° 44'$.

3. A shell discharged at an elevation of 25° 12' ranges $=3500$ feet; how far will it range at an elevation of 36° 15'? . . $=4332 \cdot 2$.

677. The ranges are proportional to the impetus, or to the squares of the velocities.

Or, $r : r' = h : h'$, where h is the impetus corresponding to r, and h' to r'; hence $\quad r' = r \cdot \dfrac{h'}{h}$, and $h' = h \dfrac{r'}{r}$.

For $r = 2h \sin 2e = \dfrac{v^2}{g} \sin 2e$; hence $r \propto h \propto v^2$ when e is given.

EXERCISE

If a shell ranges 4000 feet with an impetus of 1800, how far will it range with an impetus of 1980? . . . . $=4400$.

678. The square of the time is proportional to the tangent of the elevation; also, $t^2 = \dfrac{2r}{g} \cdot \tan e$.

For $\quad t = 2 \sin e \sqrt{\dfrac{2h}{g}}$, or $t^2 = 4 \sin^2 e \cdot \dfrac{2h}{g}$, and $r = 2h \sin 2e$;

therefore $$2h = \frac{r}{\sin 2e};$$

and hence $$t^2 = 4 \sin^2 e \cdot \frac{r}{g \sin 2e} = \frac{4 \sin^2 e}{2 \sin e \cdot \cos e} \cdot \frac{r}{g} = \frac{2r}{g} \tan e.$$

EXERCISES

1. In what time will a shell range 3250 feet at an elevation of $32°$? . . . . $=11·23$ seconds.

2. What is the time of flight for the greatest range for any impetus? . . . . $t=\sqrt{\frac{2r}{g}}=\frac{1}{4}\sqrt{r}$ nearly.

II.—PRACTICAL GUNNERY

679. Although the parabolic theory of projectiles affords a tolerable approximation to fact in the case of smaller velocities not exceeding 300 or 400 feet per second for the larger kinds of shells, yet its results deviate so widely from truth for greater velocities that ranges which, calculated by this theory, exceed 20 or 30 miles are found in fact to be only 2 or 3 miles. The cause of so great a difference is, that when the velocity of a projectile exceeds 1200 or 1300 feet there is a vacuum formed behind it, because air rushes into a vacuum with a velocity of only about 1300 feet in a second; and therefore there is not merely the ordinary resistance of the air retarding the motion in this case, but also the atmospheric pressure of the air on its anterior surface, with scarcely any pressure on its posterior surface to counteract it; and even with less velocities than this, the pressure of the rarefied air on the posterior surface is so small that the unbalanced pressure on the anterior surface causes a great retardation, far exceeding that produced by the ordinary resistance, which is nearly proportional to the square of the velocity.

680. It has been found by experiment that the square of the initial velocity of a projectile varies as the charge of powder directly, and as the weight of the ball inversely. By experiments made by Dr Hutton and Sir Thomas Bloomfield, it was found that $v=1600\sqrt{\frac{2c}{b}}$, where $v=$ the initial velocity, $c=$ the charge of powder, and $b=$ the weight of the ball; but by more recent experiments performed by Dr Gregory and a select committee of artillery-officers, it has been found that

the velocity is considerably greater on account of the improved manufacture of gunpowder, and that the formula $v = 1600\sqrt{\frac{3c}{b}}$ affords a near approximation to the initial velocity. (See 'Flat Trajectory Theory.')

681. Experiments for determining the velocity of a projectile are performed by means of wire screens placed in front of the gun. The projectile in flight passes through and cuts the wire of these screens, which are placed at known distances from each other, and by an ingenious electrical arrangement connected with the wires the actual velocity is definitely recorded—that is, the 'muzzle' or 'initial' velocity.

682. **Problem VI.—Of the charge of powder, the weight of the projectile, and the initial velocity, any two being given, to find the third.**

Let v = the initial velocity,

c = " weight of the charge in lb.,

and b = " " " ball "

then $v = 1600\sqrt{\frac{3c}{b}}$ for spherical projectiles,

$v = 1600\sqrt{\frac{3 \cdot 75c}{P}}$ for elongated projectiles

(see Flat Trajectory formula);

hence $$c = \frac{b}{3}\left(\frac{v}{1600}\right)^2,$$

and $$b = 3c\left(\frac{1600}{v}\right)^2.$$

Also, the velocities are proportional to the square roots of the charges directly, and of the weights of the projectiles inversely.

For $$v \propto \sqrt{\frac{3c}{b}},$$

$$v \propto \sqrt{3c} \text{ when } b \text{ is constant,}$$

and $$v \propto \sqrt{\frac{1}{b}} \quad " \quad c \quad "$$

That is, if v, c, b are the velocity, charge, and weight of shot in one experiment, and v', c', b' the same quantities in another, then

$$v : v' = \sqrt{\frac{3c}{b}} : \sqrt{\frac{3c'}{b'}}.$$

$$v : v' = \sqrt{3c} : \sqrt{3c'} \text{ when } b \text{ is constant,}$$

and $$v' = v\sqrt{\frac{c'}{c}}.$$

$$v : v' = \sqrt{\frac{1}{b}} : \sqrt{\frac{1}{b'}} \text{ when } c \text{ is constant,}$$

or $$v : v' = \sqrt{b'} : \sqrt{b}, \text{ and } v' = v\sqrt{\frac{b}{b'}}.$$

EXAMPLES.—1. Find the initial velocity of a shell weighing= 48 lb., the charge being=3 lb.

$$v = 1600\sqrt{\frac{3c}{b}} = 1600\sqrt{\frac{9}{48}} = 1600\sqrt{\frac{3}{16}} = 400\sqrt{3}$$
$$= 400 \times 1{\cdot}732 = 692{\cdot}8.$$

2. The weight of a ball is=32 lb.; what must be the charge of powder necessary to give it a velocity of 1500 feet?

$$c = \frac{b}{3}\left(\frac{v}{1600}\right)^2 = \frac{32}{3}\left(\frac{1500}{1600}\right)^2 = \frac{32}{3} \times \frac{225}{256} = 9{\cdot}375 \text{ lb.}$$

3. *The velocity of a ball, with a charge of 10 lb. of powder, is =1200 feet; what would be its velocity with a charge of 12 lb.?*

$$v' = v\sqrt{\frac{c'}{c}} = 1200\sqrt{\frac{12}{10}} = 120\sqrt{120} = 120 \times 10{\cdot}95445 = 1314{\cdot}534.$$

EXERCISES

1. What is the velocity of a shell weighing=36 lb. when discharged with 4 lb. of powder? . . . . =923·76.

2. With what velocity will a 48-lb. ball be impelled by a charge of $2\frac{1}{2}$ lb.? . . . . =632·456.

3. The weight of a shell is=100 lb.; what charge of powder is necessary to project it with a velocity of 1000 feet? . =13·02 lb.

4. A ball is discharged with a velocity of 900 feet by a charge of 2 lb. of powder; required its weight. . . . . . =18·96 lb.

5. The velocity of a ball of 24 lb. weight is=800 feet; what would be the velocity of a ball of 18 lb. impelled with the same charge? . . . . =923·76.

683. Problem VII.—Given the range for one charge, to find the range for another charge; and conversely.

The ranges are proportional to the charges—that is, one charge is to another charge as the range corresponding to the former is to that corresponding to the latter.

Or, $$c : c' = r : r',$$

and $$r' = r \cdot \frac{c'}{c}, \text{ also } c' = c \cdot \frac{r'}{r}.$$

EXAMPLE.—If a shell range 4000 feet when discharged with 9 lb. of powder, what will be the charge necessary to project it 3000 feet?

$$c' = c\frac{r'}{r} = 9 \times \frac{3000}{4000} = 6\tfrac{3}{4} \text{ lb.}$$

It was found (Art. 677) that r is proportional to v^2, or $r \propto v^2$; and since (Art. 682) $c = \frac{b}{3}\left(\frac{v}{1600}\right)^2$, therefore c is proportional to v^2 when b is given, or $c \propto v^2$; but $r \propto v^2$; therefore $r \propto c$, or $r : r' = c : c'$.

It could be similarly shown that, when c is given or constant, $r \propto \frac{1}{b}$, or $r : r' = \frac{1}{b} : \frac{1}{b'}$, or $r : r' = b' : b$, and $r' = r \cdot \frac{b}{b'}$, also $b' = b \cdot \frac{r}{r'}$; so that the range is inversely as the weight of the ball, all other circumstances being the same.

EXERCISES

1. If a shell range 2500 feet when projected with a charge of 5 lb., what will be its range when the charge is $= 8$ lb.? . $= 4000$.

2. If a charge of 6 lb. is sufficient to impel a ball over a range of 3600 feet, what charge will be required that the range may be 4500 feet? . . . . $= 7 \cdot 5$ lb.

684. Some important problems in practical gunnery can be solved by means of the Table in Art. 689, calculated by Mr Robins, in which the actual and potential ranges for the same elevation are given in terms of the terminal velocity.

The **actual range** is the range in a resisting medium, the **potential range** is the range in a non-resisting medium or vacuum, and the **potential random** is the greatest range in a vacuum.

685. The **terminal velocity** of a projectile is that velocity which it has in a resisting medium when the resistance against it is equal to its weight, or it is the greatest velocity it can acquire in falling by its own weight through that resisting medium.

The resistance to a plane surface moving with a moderate velocity in a resisting medium is nearly equal to the weight of a column of the fluid, having the surface for its base, and a height equal to that due to the velocity in a vacuum. The resistance on a hemisphere or on the anterior surface of a ball is only half that

on a surface equal to the area of one of its great circles; and hence the resistance to a ball moving with a small velocity in the atmosphere is nearly half the weight of a column of air having a great circle of the ball for its base, and a height equal to that due to the velocity; for the resistance to a sphere is equal to only half the resistance to the end of a cylinder of the same diameter. When the velocity is not considerable the resistance is about $\frac{7}{10}$ instead of $\frac{1}{2}$ of the above column, as appears by computing the example in Art. 689, but for great velocities it is considerably greater.

Several formulæ have been given for determining the terminal velocity of a ball. One of these, due to Hutton, is as follows:—

Let r = the resistance in avoirdupois pounds, d = the diameter of the ball in inches, and v = the velocity in feet; then

$$r = (\cdot0000007565v^2 - \cdot00175v)d^2, \text{ or } r = \cdot0000044d^2v^2;$$

the former value referring to considerable, and the latter to smaller, velocities.

In order to find the terminal velocity, for which $r = w$, the weight of the ball,

$$w = \cdot5236d^3 \times \frac{\cdot578}{16} \times 7\cdot25 = \cdot137134d^3,$$

and when $r = w$, the terminal velocity v' will be found from the equation, $\cdot137134d^3 = \cdot0000044d^2v^2$,

and $$v' = \sqrt{\frac{\cdot137134}{\cdot0000044}} \cdot d = \sqrt{31167d} = 176\sqrt{d}.$$

The height due to this velocity is $h' = \frac{31167}{64}d = 487d$; and for a shell, the weight of which is $\frac{4}{5}$ of that of a ball of equal diameter,

$$v' = \sqrt{\frac{4 \times 31167}{5}d} = \sqrt{24934d} = 158\sqrt{d}.$$

686. Robins found that the resistance to a 12-lb. ball moving with a velocity of about 25·5 feet in a second was $\frac{1}{2}$ ounce avoirdupois. Now, for velocities less than 1100 feet per second, the resistance is nearly proportional to the squares of the velocities, and it is also as the squares of the diameter; hence, if c is the constant to be determined,

$$r = cd^2v^2, \text{ or } \tfrac{1}{32} \text{ lb.} = c \times 4\cdot45^2 \times 25\cdot5^2.$$

It will be found from this equation that c is $= \cdot000002427$; and the value of v' would be found as above to be $238\sqrt{d}$, and $h' = 883d$. In the Table, p. 415, Robins has taken this quantity to be $900d$, and denotes it by F—that is, F $= 900d$. This appears to be the origin of this quantity F, which has not before been accounted for.

Robins had probably found, by other experiments, that 900 would generally afford more correct results than 883.

This quantity—namely, the height due to the terminal velocity in a vacuum—may be called the **terminal height.**

687. **Problem VIII.—To find the terminal height.**

The terminal height is found by multiplying the diameter of the ball by 900.

When the ball has a different specific gravity from iron, find the height for iron; then the specific gravity of iron is to the specific gravity of the ball as the height for an iron ball is to the required height.

For iron, $F = 900d$.

For a ball of other material, whose specific gravity is s,

$$7 \cdot 25 : s = 900d : F,$$

and $$F = \frac{900sd}{7 \cdot 25} = 124sd.$$

For a shell, $F = \frac{4}{5} \times 900d = 720d.$

For lead, $s = 11 \cdot 35$, and $F = 1409d$.

688. The following Table gives the weight of a cast-iron ball when its diameter is known, and conversely. The weight is in avoirdupois pounds, and the diameter in inches :—

I

Weight	Diameter	Weight	Diameter
·136	1	17·1	5
1	1·04	18	5·09
1·10	2	24	5·61
3	2·8	29·5	6
3·7	3	32	6·21
4	3·08	42	6·75
6	3·52	47	7
8·7	4	70	8
9	4·04	100	9
12	4·45		

The weight of any solid ball may be found by multiplying the cube of its diameter by ·5236, and the result by the weight of a cubic inch of its material. Diameter to be in inches.

II

Weight of Cast-Iron Solid Cylinders in Lb.

Length of cylinder = 1 foot

Weight	Diameter—Inches	Weight	Diameter—Inches
2·4	1	89	6
9·9	2	120	7
21·9	3	156	8
39·0	4	198	9
61·0	5		

Exercises

1. Find the terminal height for an iron ball = 6 inches in diameter. = 5400.
2. Find the terminal height for a 3-lb. iron ball. . . = 2520.
3. Find the terminal height for a shell = 12 inches in diameter. = 8640.
4. Find the terminal height for a leaden ball = 2 inches in diameter. . . . . = 2818.

689. **Problem IX.—Given the actual range of a given spherical projectile, at an angle not greater than 10°, and its original velocity, to find its potential range, and the elevation to produce the actual range.**

Case 1.—When the potential random does not exceed 39000 feet.

Rule.—Divide the actual range by the terminal height, and find the quotient in one of the columns of actual ranges in the following Table; and opposite to it, in the next column of potential ranges, is a number which, multiplied by the preceding height, will give the potential range. The potential range and initial velocity being known, find the elevation by Art. 673.

Let F = the terminal height in feet,
r = " actual range in feet,
R = " potential range in feet,
r' = " actual range in the Table,
R' = " potential range in the Table,
v = " initial velocity,
h = " impetus,
e = " elevation,
d = " diameter of the projectile in inches;

then $\quad 2h =$ the potential random.

Then $\quad F = 900d,\ r' = \frac{r}{F}$, and $R = FR'$,

$$h = \left(\frac{v}{8}\right)^2 \text{ nearly, or } Lh = 2(Lv - \cdot 903090).$$

Or, $\quad Lh = 2Lv - 1 \cdot 806180.$

$$\text{Sin } 2e = \frac{R}{2h} = \frac{32R}{v^2}.$$

Or, $\quad L \sin 2e = 10 + LR - L2h$ (Art. 673).

In the following Table the first, third, and fifth columns contain the actual ranges of projectiles expressed in terms of F—that is, the F for the ball in any particular case is the unit of measure; and the second, fourth, and sixth columns contain the corresponding potential ranges—that is, with the same elevation and initial velocity—expressed in the same manner:—

Actual Range	Potential Range	Actual Range	Potential Range	Actual Range	Potential Range
·01	·0100	1·3	2·1066	3·3	13·8258
·02	·0201	1·4	2·3646	3·4	15·0377
·04	·0405	1·5	2·6422	3·5	16·3517
·06	·0612	1·6	2·9413	3·6	17·7767
·08	·0822	1·7	3·2635	3·7	19·3229
·1	·1034	1·8	3·6107	3·8	21·0006
·12	·1249	1·9	3·9851	3·9	22·8218
·14	·1468	2·0	4·3890	4·0	24·7991
·15	·1578	2·1	4·8249	4·1	26·9465
·2	·2140	2·2	5·2955	4·2	29·2792
·3	·3324	2·3	5·8036	4·3	31·8138
·4	·4591	2·4	6·3526	4·4	34·5686
·5	·5949	2·5	6·9460	4·5	37·5632
·6	·7404	2·6	7·5875	4·6	40·8193
·7	·8964	2·7	8·2813	4·7	44·3605
·8	1·0638	2·8	9·0319	4·8	48·2127
·9	1·2436	2·9	9·8442	4·9	52·4040
1·0	1·4366	3·0	10·7237	5·0	56·9653
1·1	1·6439	3·1	11·6761		
1·2	1·8669	3·2	12·7078		

In this case $2h$ does not exceed 39000, and v does not exceed 1112; for $v = 8\sqrt{h} = 8 \times 139$, and e may be found without

previously calculating h; by substituting in the last formula the value of $L2h$, it becomes

$$L \sin 2e = 11{\cdot}505150 + LR - 2Lv.$$

EXAMPLE.—At what elevation must an 18-pounder be fired, with a velocity of 984 feet, in order that its actual range on a horizontal plane may be $=2925$ feet?

$$F = 900d = 900 \times 5{\cdot}09 = 4581,$$

$$r' = \frac{r}{F} = \frac{2925}{4581} = {\cdot}64, \text{ and } R = FR' = 4581 \times {\cdot}8028 = 3678,$$

$$Lh = 2(Lv - {\cdot}903090) = 2(2{\cdot}992995 - {\cdot}903090)$$

$$= 2{\cdot}089905 \times 2 = 4{\cdot}179810, \text{ and } h = 15129.$$

$$L \sin 2e = 10 + L3678 - L2h = 13{\cdot}565612 - 4{\cdot}480840 = 9{\cdot}084772,$$

$$\text{and } 2e = 6^\circ\ 59', \text{ and } e = 3^\circ\ 29{\cdot}5'.$$

EXERCISES

1. At what elevation must a 12-lb. ball be fired, with a velocity of 700 feet, in order that it may reach an object $=2000$ feet distant? . . . . $=4^\circ\ 28{\cdot}5'$.

2. Find the elevation at which a ball $=5$ inches in diameter must be discharged, with a velocity of 800 feet, that its actual range may be $=\frac{1}{2}$ of a mile. . . . . $=2^\circ\ 53'$.

CASE 2.—When the potential random exceeds 39000 feet.

RULE.—Find two mean proportionals between 39000 and the potential random; then the less of these means is to the potential random as the potential range, found by the former case, is to the true potential range; then the elevation is found as before.

Find h as in the preceding case; then, if

R'' = the potential range found by the preceding case,

R = " true potential range,

then $R = {\cdot}00138R''\sqrt[3]{h^2}$.

Or, $LR = \bar{3}{\cdot}139977 + LR'' + \frac{2}{3}Lh$.

Instead of LR'', $LF + LR'$ may be used.

Then find e, as in the former case, or, by this formula,

$$L \sin 2e = 6{\cdot}8389469 + LR'' - \tfrac{1}{3}Lh,$$

which gives e at once, when h and R'' are found.

EXAMPLE.—At what elevation must a 24-pounder be discharged, with a velocity of 1730 feet per second, in order that its actual range may be $=7500$ feet?

$$F = 900d = 900 \times 5{\cdot}61 = 5049, \text{ and } r' = \frac{r}{F} = \frac{7500}{5049} = 1{\cdot}48;$$

hence $R'' = FR' = 5049 \times 2{\cdot}587 = 13060,$

$$Lh = 2(Lv - {\cdot}903090) = 2(3{\cdot}238046 - {\cdot}903090) = 2 \times 2{\cdot}334956$$
$$= 4{\cdot}669912, \text{ and } h = 46764,$$

and $2h = 93528$, which exceeds 39000.

Then
$$L \sin 2e = 6{\cdot}8389469 + LR'' - \tfrac{1}{3}Lh$$
$$= 6{\cdot}8389469 + 4{\cdot}1159432 - 1{\cdot}5566372$$
$$= 9{\cdot}3982529 = L \sin 14^\circ\ 29';$$

and therefore $e = 7^\circ\ 14{\cdot}5'$.

Let $a = 39000$; then $2h$ being the potential random, let x and y be two mean proportionals between a and $2h$;

then $a : x = x : y$, and $x : y = y : 2h$;

hence $y = \dfrac{x^2}{a}$, and $2h = \dfrac{y^2}{x} = \dfrac{x^3}{a^2}$;

and $x = \sqrt[3]{2a^2h}$;

also, $x : 2h = R'' : R$;

therefore, $R = \dfrac{2hR''}{x} = \dfrac{2hR''}{\sqrt[3]{2a^2h}} = {\cdot}00138R''\sqrt[3]{h^2}$.

Or, $LR = \bar{3}{\cdot}1399769 + LR'' + \tfrac{2}{3}Lh$.

Then e can be found for this potential range, and given initial velocity, as in the preceding case; or,

since $\sin 2e = \dfrac{R}{2h} = \dfrac{2hR''}{2h\sqrt[3]{2a^2h}} = \dfrac{R''}{\sqrt[3]{2a^2h}}$;

therefore $L \sin 2e = 10 - \tfrac{1}{3}L2a^2 + LR'' - \tfrac{1}{3}Lh$,

or $L \sin 2e = 6{\cdot}8389469 + LR'' - \tfrac{1}{3}Lh$.

Exercises

1. At what elevation must a ball 4·5 inches in diameter be fired, with a velocity of 1200 feet per second, in order that its actual range may be = 4500 feet? . . . . $= 4^\circ\ 45'$.

2. Required the elevation at which a 24-pounder must be fired, with a velocity of 1600 feet per second, that its actual range may be a mile. . . . . $= 4^\circ\ 29'\ 30''$.

690. **Problem X.—Given the elevation not exceeding 45°, and the velocity with which a given projectile is discharged, to determine its actual range.**

Case 1.—When the potential random does not exceed 39000 feet.

Rule.—Reduce the terminal height F, corresponding to the given projectile in the ratio of radius to the cosine of $\tfrac{2}{3}$ of the angle of elevation; find the potential range by Art. 689; divide this range by the reduced F, and find the quotient in the tabular

column of potential ranges; and opposite to it, in the preceding column of actual ranges, is a number, the product of which, by the reduced F, will give the actual range.

Let F = the terminal height found by Art. 687,

F′ = „ reduced height,

the other letters denoting as before.

Then, to find F′, $F'/F = \cos \frac{3}{4}e$,

or $LF' = LF + L\cos \frac{3}{4}e - 10$,

and h is to be found as in Art. 672.

To find R, $R/2h = \sin 2e$,

or $LR = L2h + L\sin 2e - 10$.

Then $r' = \dfrac{R}{F'}$, and $r = F'r'$,

or $Lr = LF' + Lr'$.

Example.—What is the actual range of a musket-bullet, of the usual diameter of $\frac{3}{4}$ of an inch, discharged at an elevation of 15°, with a velocity of 900 feet?

$$F = 1409d = 1409 \times \tfrac{3}{4} = 1057 \ldots \text{(by Art. 687)},$$

$$LF' = L1057 + L\cos 11°\ 15' - 10 = 3{\cdot}024075 + 9{\cdot}991574 - 10$$
$$= 3{\cdot}015649, \text{ and } F' = 1036{\cdot}7.$$

Also, $$h = \left(\frac{v}{8}\right)^2 = \left(\frac{900}{8}\right)^2 = 12656, \text{ and } 2h = 25312,$$

and (Art. 671), $R = 2h\sin 2e = 2h \times \frac{1}{2} = h = 12656$,

$$R' = \frac{R}{F'} = \frac{12656}{1036{\cdot}7} = 12{\cdot}208; \text{ hence } r' = 3{\cdot}15,$$

and $r = F'r' = 1036{\cdot}7 \times 3{\cdot}15 = 3266$ feet, the actual range.

Exercises

1. What is the actual range of a ball of 6 inches diameter, fired at an elevation of 25°, with a velocity of 1000 feet? = 10570 feet.

2. What is the actual range of a shell = 10 inches in diameter, its weight being = $\frac{2}{3}$ of that of a ball of the same diameter, when discharged at an elevation of 40°, with a velocity of 400 feet? = 3938 feet.

Case 2.—When the potential random exceeds 39000, or the impetus exceeds 19500, or the velocity exceeds 1112 feet.

Rule.—Find two mean proportionals between 39000 and the potential random, and take the less of them for the reduced potential random; then the true potential random is to the reduced potential random as the potential range to the reduced potential range. This reduced potential range, being divided by

the reduced terminal height F′, will give the tabular potential range, from which the actual range is found as in the last case.

$$LR = 3{\cdot}139977 + LR'' + \tfrac{5}{3}Lh,$$

and adding 10 to both sides, and substituting for Lh its value $2Lv - 1{\cdot}806180$ (Art. 689); then

$$LR'' = LR - \tfrac{5}{3}Lv + 4{\cdot}064143 \quad . \quad . \quad . \quad . \quad (1),$$

and $\quad LR = L2h + L \sin 2e - 10,$

or $\quad LR = 2Lv + L \sin 2e - 11{\cdot}505150 \quad . \quad . \quad . \quad (2);$

hence $\quad LR'' = \tfrac{2}{3}Lv + L \sin 2e - 7{\cdot}441007 \quad . \quad . \quad (1) + (2).$

Find F and F′, as in Art. 690.

EXAMPLE.—Required the range of the bullet in the example of the first case, discharged at the same elevation, with a velocity of 2100 feet.

In this case $v > 1112$, or $2h > 39000$.

As in the former example, $F = 1057$, and $F' = 1036{\cdot}7$.

And $LR'' = \tfrac{2}{3}Lv + L \sin 2e - 7{\cdot}441007 = 2{\cdot}214813 + 9{\cdot}698970 - 7{\cdot}441007$
$= 11{\cdot}913783 - 7{\cdot}441007 = 4{\cdot}472776;$

hence $\quad R'' = 29701.$

$$LR' = LR'' - LF' = 4{\cdot}472776 - 3{\cdot}015653 = 1{\cdot}457123,$$

and $\quad R' = 28{\cdot}65$; hence, by Table, $r' = 4{\cdot}17303$,

and $\quad r = F'r' = 1036{\cdot}7 \times 4{\cdot}17303 = 4326$ feet, the actual range.

Although the velocity in this example is more than double that in the preceding, yet the range is only 1060 feet greater.

EXERCISES

1. Find the actual range of a 42-lb. ball, discharged with a velocity of 1800 feet, at an elevation of 36°. . . = 15413 feet.

2. What will be the actual range of a 24-lb. ball, fired at an elevation of 35°, with a velocity of 1760 feet per second?

= 13695 feet.

691. It can be shown, by dynamical principles, that balls of the same density, projected at the same elevation, with velocities that are proportional to the square roots of their diameters, describe similar curves. The reason of this is, that the resistances are proportional to the masses or weights of the balls. Their velocities at their greatest height, which are horizontal, are proportional to their diameters; and any corresponding lines of their trajectories—that is, of the curves described by them—are proportional to their diameters. Their actual ranges are therefore proportional to their diameters, or to the squares of their initial velocities, but their potential ranges are in the same proportion; hence their actual and potential ranges are proportional. But the terminal heights,

being $900d$, are proportional also to their diameters, or their terminal velocities are proportional to their initial velocities. The terminal heights are therefore also proportional to their ranges, both actual and potential. Hence the quotients of the actual and potential ranges of one ball by its terminal height are respectively equal to the corresponding quotients for another ball, both being projected under the conditions stated above—that is, the tabular ranges, both actual and potential, are the same for all balls of the same density, discharged at the same elevation, with velocities proportional to the square roots of their diameters. Thus a comparatively limited set of experiments with a ball of given dimensions and density would be sufficient to determine the data for the construction of the preceding Table; by means of which the ranges of balls, of an unlimited variety of density and size, could be computed.

The weights of two balls being w, w', their diameters d, d', their velocities v, v', and the resistances to them r and r', then (Art. 686)

$$r : r' = d^2v^2 : d'^2v'^2 \text{ nearly,}$$

if the velocities are both greater or both less than 1112 feet.
And if $v : v' = \sqrt{d} : \sqrt{d'}$, then

$$r : r' = d^2d : d'^2d' = d^3 : d'^3 = w : w';$$

so that in this case the resistances are as the weights. If v and v' are the terminal velocities, then $r = w$, and $r' = w'$;

hence $$r : r' = w : w'.$$

Or, $$d^2v^2 : d'^2v'^2 = d^3 : d'^3,$$

or $$v^2 : v'^2 = d : d', \text{ or } v : v' = \sqrt{d} : \sqrt{d'}.$$

THE FLAT TRAJECTORY THEORY

692. This theory will be recognised as bearing especially on modern guns and rifles, for if the point-blank range of any gun is increased, its trajectory (or, more correctly speaking, the trajectory of its projectile) takes the form of a straight line, and less and less that of a parabola.

All guns with a high muzzle velocity are affected by the investigations made in connection with this theory.

To attain a high muzzle velocity various measures have from time to time been adopted. Careful consideration has been bestowed on the shape of projectiles, which in modern guns are elongated cylinders of iron or steel, with ogival heads, and

struck with a radius of $1\frac{1}{2}$ the diameter of the projectile. In consequence of the rotatory motion imparted to the shell by the rifling of the gun, the shell on leaving the gun 'spins' or rotates round its longer axis, thus only exposing its ogival head to the resistance of the atmosphere. Air-spacing in the powder-chamber also ensures a greater volume of gas being generated at the base of the shell, and consequently a greater pressure is set up than was the case in muzzle-loading guns without gas-checks. The twist of the rifling, and also windage, are also important factors connected with velocity, range trajectory, and time of flight. It will thus be seen that, in order to obtain a high initial velocity, the gun is strained to an extent far exceeding that demanded by the use of balls, on which the parabolic theory treats. The modern weapon and its projectiles are therefore a distinct departure from the ancient cannon, and in consequence of the increased strain to which it is subjected, it is built up in coils, so distributed as to equalise the action of the combined stresses, and at the same time with a due regard to its minimum weight and mobility. Guns of modern design are made entirely of steel in the form of ribbon-wire, the ultimate tensile strength of which = 100 tons per square inch. By winding on with varying tension, any desired state of initial stress may be given; and thus on firing, every part of the structure is made to take its due share of stress. For calculating the strength of a wire gun, the winding tensions must be known, as well as dimensions and strength of material employed.

693. The height reached is given approximately by the formula,

$$h = \tfrac{1}{2}gt(T - t),$$

where T = total time of flight in seconds,

t = time of flight in seconds to a point where height of trajectory is h feet.

Assuming the vertex to be reached at half-time, and putting $t = \frac{1}{2}T$, and $g = 32$, we get the greatest height. $H = 4T^2$ is a useful approximation for comparing the flatness of trajectories of different guns.

694. Velocity and Momentum of Recoil.

v = muzzle velocity of projectile,
W = weight of gun and carriage,
V = velocity of recoil of gun and carriage,
w = weight of projectile,
w_1 = " powder charge,
C = a constant deduced by experiment;

$$WV = v(w + Cw_1).$$

695. Gravimetric Density.

To find gravimetric density GD of a charge.

Let S = cubic space allotted per lb. of powder in the chamber.

$$GD = \frac{27 \cdot 73}{S}.$$

696. Work done by Exploding Powder

No. of Expansions	Work per Lb. burned—Foot-Tons	No. of Expansions	Work per Lb. burned—Foot-Tons	No. of Expansions	Work per Lb. burned—Foot-Tons
1·25	19·226	5·5	95·210	11	121·165
1·5	31·986	6	98·638	12	124·239
1·75	41·494	6·5	101·744	13	127·036
2	49·050	7	104·586	14	129·602
2·5	60·642	7·5	107·192	15	131·970
3	69·347	8	109·600	16	134·108
3·5	76·315	8·5	111·840	17	136·218
4	82·107	9	113·937	18	138·138
4·5	87·064	9·5	115·905	19	139·944
5	91·385	10	117·757	20	141·647

This Table is made out for charges of unit gravimetric density.

Divide cubic contents of bore by cubic content of cartridge-chamber, which will give number of expansions. Multiply the number found opposite this in the Table by number of lb. in the charge, and the result will be the work done.

If the charge be not of unit gravimetric density.

Suppose gravimetric density = ·8, and number of expansions = 5;

Work done per lb. of powder = work done in 5 expansions minus work done in $\frac{1 \cdot 0}{\cdot 8}$, or in 1·25 expansions = (91·385 − 19·226) foot-tons = 72·159 foot-tons.

In practice only a portion of this, called the factor of effect, varying from 0·7 to 0·9, is obtained. Thus, suppose factor of effect 0·8, the work realised is $= 72 \cdot 159 \times \cdot 8$ foot-tons $= 57 \cdot 727$ foot-tons per lb. of powder in the charge.

$\frac{wV^2}{2g \times 2240}$ foot-tons is also a measure of work contained in the projectile, in which w = weight of projectile in lb., V = the muzzle velocity in feet per second.

By equating the two expressions, the probable muzzle velocity can be estimated before actual trial has taken place.

697. Penetration of Armour.

T = thickness of wrought-iron that can be penetrated by direct fire (inches),
d = diameter of projectile (inches),
v = striking velocity, feet per second;

$$T = \frac{v}{608 \cdot 3}\sqrt{\frac{w}{d}} - \cdot 14d.$$

For steel and compound or steel-faced armour, penetration $= T \times \cdot 8$ (approximately). Captain Orde Brown's rough rule: $T = \cdot 001 v.d.$

Various causes will modify the above, and when striking at an angle—about 40° from the normal—projectiles will be deflected.

698. Penetration of Rifle-Bullets.—Thickness of various materials proof against magazine-rifle fire at all ranges:—

Earth parapet free from stones, not rammed,	24	inches.
Clay, ...	24	"
Fine loamy sand, ...	20	"
Wrought-iron or mild steel plate, ...	$\frac{3}{8}$	inch.
Fir, dry or green, ...	38	inches.
Elm, green, ...	36	"
Oak, "	24	"
Sand-bags, filled, header, ...	1	bag.
" " stretcher, ...	2	bags.

699. Use of Bashforth's Tables (A and B).—The two following Tables (Bashforth's) give the relations between (A) the time of flight of a projectile and its velocities at the beginning and end of that time, and (B) the distance of flight of a projectile and its velocities at the beginning and end of that distance. Thus, the initial velocity of a projectile being given, the velocity

TIME AND VELOCITY TABLE (A)

Tv = number in Table corresponding to velocity V, V = initial velocity, feet per second,
Tv = " " " " v, v = remaining " " "
t = time of flight of projectile in seconds, C = ballistic coefficient $= w \div nd^2$,
w = weight of projectile in lb., d = diameter of projectile in inches.
n = coefficient for head,

$$t = C(Tv - Tv).$$

Velocity—Feet per Second	0	10	20	30	40	50	60	70	80	90	Velocity—Feet per Second
800	225·768	226·021	226·267	226·504	226·734	226·955	227·169	227·375	227·575	227·768	800
900	227·954	228·135	228·309	228·478	228·641	228·799	228·953	229·101	229·245	229·385	900
1000	229·521	229·652	229·780	229·902	230·018	230·123	230·217	230·303	230·383	230·459	1000
1100	230·531	230·600	230·667	230·731	230·794	230·854	230·914	230·972	231·028	231·083	1100
1200	231·137	231·189	231·240	231·290	231·338	231·385	231·432	231·477	231·521	231·565	1200
1300	231·607	231·649	231·689	231·729	231·768	231·807	231·844	231·881	231·917	231·953	1300
1400	231·988	232·022	232·056	232·090	232·123	231·156	232·188	232·220	231·251	232·282	1400
1500	232·312	232·342	232·372	232·402	232·431	232·460	232·488	232·516	232·544	232·572	1500
1600	232·599	232·626	232·653	232·680	232·706	232·732	232·758	232·783	232·808	232·833	1600
1700	232·858	232·882	232·906	232·930	232·954	232·978	233·001	233·024	233·046	233·069	1700
1800	233·091	233·113	233·135	233·157	233·178	233·200	233·220	233·242	233·262	233·283	1800
1900	233·303	233·323	233·343	233·363	233·382	233·402	233·421	233·440	233·459	233·477	1900
2000	233·496	233·514	233·531	233·549	233·567	233·584	233·601	233·617	233·634	233·650	2000
2100	233·666	233·682	233·698	233·713	233·728	233·743	233·758	233·773	233·787	233·802	2100
2200	233·816	233·830	233·843	233·857	233·870	233·884	233·897	233·910	233·923	233·935	2200
2300	233·948	233·960	233·973	233·985	233·998	234·010	234·022	234·035	234·047	234·059	2300
2400	234·071	234·083	234·095	234·107	234·119	234·131	234·143	234·155	234·167	234·179	2400

DISTANCE AND VELOCITY TABLE (B)

Sv = number in Table corresponding to velocity V, V = initial velocity, feet per second, of projectile,
Sv = " " " " v, v = remaining " " " "
s = distance of flight (or range) in feet, C = ballistic coefficient = $w \div nd^2$,
w = weight of projectile in lb., d = diameter of projectile in inches.
n = coefficient for head,

$$s = C(Sv - Sv).$$

Velocity—Feet per Second	0	10	20	30	40	50	60	70	80	90	Velocity—Feet per Second
800	36512·6	36716·1	36916·0	37111·7	37303·1	37490·0	37672·4	37850·6	38024·8	38195·0	800
900	38361·5	38524·3	38683·5	38839·4	38991·9	39141·2	29287·4	39430·6	39570·8	39708·3	900
1000	39842·9	39975·0	40104·3	40230·1	40349·4	40459·2	40558·7	40650·5	40736·8	40819·0	1000
1100	40897·0	40974·2	41048·2	41120·5	41191·4	41261·0	41329·5	41396·8	41462·9	41528·0	1100
1200	41591·9	41654·8	41716·7	41777·5	41837·5	41896·5	41954·6	42011·8	42068·3	42123·9	1200
1300	42178·8	42233·0	42286·4	42339·2	42391·4	42443·0	42493·9	42544·4	42594·3	42643·7	1300
1400	42692·6	42741·2	42789·3	42837·1	42844·4	42931·4	42978·1	43024·5	43070·6	43116·4	1400
1500	43162·0	43207·2	43252·3	43297·2	43342·0	43386·5	43430·9	43475·1	43519·1	43563·0	1500
1600	43606·6	43650·0	43693·3	43736·3	43779·2	43831·9	33864·4	43906·8	43949·0	43990·9	1600
1700	44032·7	44074·3	44115·7	44157·0	44198·0	44238·9	44279·6	44320·2	44360·5	44400·7	1700
1800	44440·8	44480·6	44520·3	44559·8	44599·2	44638·4	44677·4	44716·3	44755·0	44793·7	1800
1900	44832·2	44870·5	44908·7	44946·7	44984·5	45022·2	45059·6	45096·9	45133·9	46170·6	1900
2000	45207·1	45213·3	45279·2	45314·9	45350·3	45385·4	45420·2	45454·7	45488·9	45522·8	2000
2100	45556·4	45589·7	45622·8	45655·5	45688·0	45720·2	45752·2	45783·9	45815·4	45846·6	2100
2200	45877·5	45908·3	45938·7	45969·0	45999·0	46028·7	46058·3	46087·6	46116·7	46145·7	2200
2300	46174·6	46203·5	46232·3	46261·2	46290·1	46319·0	46348·0	46377·0	46406·0	46435·1	2300
2400	46464·2	46493·4	46522·6	46551·9	46581·3	46610·6	46640·1	46669·5	46698·9	46728·4	2400

after any given interval of flight may be found, or *vice versâ;* or, the initial and remaining velocities being known, the time or distance of flight may be found.

The coefficient n in the formulæ depends on the shape of head of the projectile, steadiness of flight, and density of the air. For ogival-headed projectiles, the heads of which are struck with a radius of $1\frac{1}{2}$ diameter, $n=1$; for more modern projectiles with heads of 2 to $2\frac{1}{2}$ diameter, and under normal atmospheric conditions, n may be taken from about 0·88 up to unity—from the lightest to the heaviest. With the 0·303 magazine bullet, $n=0·64$.

For high velocities, at all ordinary low angles of elevation, these Tables and formulæ give results agreeing very closely with actual practice.

Example on the Use of the Tables

Suppose the muzzle velocity of a 10″ gun to be 2040 feet per second, and that it has a remaining velocity of 1879 feet per second. It is required to find the distance of flight, or range of the projectile in yards, given the weight of projectile =500 lb.

By referring to Table B we have the formula,

$$s=C(Sv-Sv),$$

where s=range in feet.

We first determine C; and by taking the value of n as $=·88$ (see notes preceding the Tables), we get the ballistic coefficient $C=w\div nd^2$.

$\therefore$ C=500 lb. $\div ·88\times 10'^2=5·68$.

The initial velocity Sv=2040, and the number in the Table corresponding to this velocity is 45350·3.

$\therefore$ Sv=45350·3.

The remaining velocity is 1879 feet per second, and the number in the Table corresponding to a velocity of 1870 feet per second (for 1870 is the nearest velocity in the Table to that given) is 44716·3.

$\therefore$ Sv=44716·3.

$$\begin{aligned}\therefore s=\text{range}&=C(Sv-Sv)\\&=5·68(45350·3-447163·3)\\&=5·68\times 634\\&=3601·12 \text{ feet, or } 1200·37 \text{ yards.}\end{aligned}$$

Further, let the time of flight of the above-named projectile be also required.

The given muzzle velocity=2040 feet per second, and the number in Table A corresponding to this velocity is Tv=233·567.

The given remaining velocity = 1879 feet per second, and as the nearest velocity in Table A is 1870, the number corresponding to this velocity is $Tv = 233 \cdot 242$.

The ballistic coefficient C is $w \div nd^2$, and this has already been found $= 5 \cdot 68$.

$$\begin{aligned}\therefore\ t &= \text{time of flight of projectile in seconds}\\ &= C(Tv - Tv)\\ &= 5 \cdot 68(233 \cdot 567 - 233 \cdot 242)\\ &= 5 \cdot 68 \times \cdot 325\\ &= 1 \cdot 846 \text{ seconds.}\end{aligned}$$

$\therefore$ the time of flight of a projectile whose weight = 500 lb., and whose muzzle and remaining velocities are respectively 2040 and 1879 feet per second, ranges over a distance of 1200·37 yards in 1·846 seconds of time.

It may be mentioned that the weight of powder in the charge of this gun, which affords the above results, is 252 lb., and that the projectile at this range will penetrate 20·5 inches of wrought-iron armour-plating.

It will therefore prove interesting to learn what measure of work is stored up in a projectile whose weight is 500 lb., and which travels at the rate of 2040 feet per second.

By the formula already afforded, we find that this measure of work $= \dfrac{wV^2}{2g \times 2240}$.

The symbol (g) is the acceleration due to gravity (see 'Parabolic Theory').

$$\begin{aligned}\therefore\ \frac{wV^2}{2g \times 2240} &= \frac{500 \times 2040^2}{2 \times 32 \times 2240}\\ &= 14515 \cdot 34 \text{ foot-tons.}\end{aligned}$$

Should the muzzle velocity and time of flight be given, the remaining velocity can easily be found ; for,

$$\begin{aligned}Tv &= \text{remaining velocity}\\ &= Tv - t/c.\end{aligned}$$

Let us suppose that it be required to find the remaining velocity of a shell whose weight is 500 lb. and diameter 10 inches; let the value of (n) be assumed $= \cdot 88$, and the muzzle velocity of the shell = 2040 feet per second, with time of flight = 1·846 seconds.

We first determine the value of C, the ballistic coefficient, *vide* formula $C = w \div nd^2$, which in this case $= 500 \div \cdot 88 \times 100$.

$\therefore\ C = 5 \cdot 68.$

The given time of flight $=1 \cdot 846$ seconds.

The muzzle velocity $=2040$ feet per second $=$ (*vide* Table) $233 \cdot 567$.

$$\begin{aligned} \therefore \text{T}v &= \text{remaining velocity} \\ &= \text{Tv} - t/c. \\ &= 233 \cdot 567 - \frac{1 \cdot 846}{5 \cdot 68} \\ &= 233 \cdot 567 - \cdot 325 \\ &= 233 \cdot 242\,; \end{aligned}$$

and the velocity corresponding to this number in the Tables $=1870$ feet per second.

$\therefore$ the shell has a remaining velocity of 1870 feet per second at the end of a time of flight $=1 \cdot 846$ seconds.

700. The reader's attention is directed to formula $v=1600\sqrt{\frac{3c}{b}}$, which appears in Art. 680. This formula can only be applied to guns which have more or less windage; but in almost all modern weapons the system of obturation adopted is such as to totally exclude this factor. The object aimed at is to utilise to the very fullest extent the force set up by the expansion of the gas in the powder-chamber. The prevention of gas waste or escape in a breech-loading gun is technically termed 'obturation,' and is derived from the Latin *obturo*, I stop or close up.

The 'windage' of a gun (a term already used) is the difference between the diameter of the bore of the gun and that of its projectile.

With muzzle-loading guns it was an inevitable necessity to have windage, otherwise it would have been impossible to load the gun. It has, however, its disadvantages—namely, that a large volume of the gas generated by the ignition of the charge passed both over and under the projectile whilst being propelled through the bore of the gun, and thus escaped without having fulfilled its allotted work in respect to the projectile; and consequently this loss of potential energy materially affected the muzzle or initial velocity of the projectile.

The formula can, however, be modified and applied to modern guns by increasing the value of the coefficient 3 to $3 \cdot 75$; thus—

$$\text{Initial velocity } v=1600\sqrt{\frac{3 \cdot 75\text{C}}{b}}.$$

It would be better, perhaps, to alter the symbol (b) to P where P = weight of projectile; this distinction would better characterise the formulæ, and at the same time assist the memory.

Thus, for flat trajectories the initial velocity $=1600\sqrt{\frac{3\cdot75C}{P}}$, where C = weight of charge in lb., and P = weight of projectile also in lb. The formula affords a close approximation to the initial velocity, but should not be preferred to that by which the Tables are calculated.

Exercises

1. A 5″ breech-loading gun, whose shell weighs 16 lb., has a muzzle velocity of 1800 feet per second, and a remaining velocity of 1200 feet per second; find the distance of flight, or range in yards, of the projectile, and state the measure of work contained in the shell; given $n=1$.

    Distance of flight = 1823·296 yards; measure of work = 361·6 foot-tons.

2. Taking the previous exercise, let it be required to find the time of flight of the projectile in seconds, with the velocities there mentioned. . . . . = 1·250 seconds.

3. Supposing the shell mentioned in the first exercise had a striking velocity of 1200 feet per second, determine its penetration by formula for direct fire, in both wrought-iron and steel-faced armour.

    Wrought-iron, 3·002 inches; steel-faced armour, 2·401 inches.

4. The muzzle velocity of a 3-pounder Hotchkiss quick-firing gun is 1873 feet per second; its remaining velocity is required, given time of flight of projectile = 3 seconds, weight of projectile = 3 lb., $n = \cdot88$, $d = 1\cdot85$ inches. . . . = 1060 feet per second.

5. With the data before you in the above question and answer, state the distance of flight, or range in yards, of the shell.

    = 4141 feet, or 1380·3 yards.

6. What would be the greatest height the projectile would attain to in a given time of flight = 3 seconds?

    = 36 feet, approximately.

701. **Strength of Guns.**—In calculating the circumferential strength of a gun built up by shrinking on successive layers of metal, the general formula employed is

$$P_{n-1}=\frac{r^2_n - r^2_{n-1}}{r^2_n + r^2_{n-1}}(T_{n-1}+P_n)+P_n,$$

where n is the number of layers. Thus, for a 6-inch breech-loading, steel gun, having over the powder-chamber a tube,

breech-piece, and jacket, commence from the exterior and put $n=3, 2, 1$ successively; thus—

$$P_2=\frac{r_3^2-r_2^2}{r_3^2+r_2^2}T_2,$$

$$P_1=\frac{r_2^2-r_1^2}{r_2^2+r_1^2}(T_1+P_2)+P_2,$$

$$P_0=\frac{r_1^2-r_0^2}{r_1^2+r_0^2}(T_0+P_1)+P_1,$$

in which r_0 is the radius of the powder-chamber; r_1, r_2, and r_3 the outer radii of the tube, breech-piece, and jacket respectively; P_0, P_1, P_2, P_3 are the radial pressures in tons per square inch at the surfaces, where the radii are r_0, r_1, and r_2 respectively.

Note.—In the case of the outside layer, or jacket, $P_3=0$, as the pressure of the atmosphere may be neglected.

T_0, T_1, and T_2 are the maximum allowable *hoop tensions* in tons per square inch at r_0, r_1, and r_2 respectively.

Practically, with modern gun-steel, the values for strength calculations may be taken at $T_0=15$, T_1 and T_2, &c. $=18$. A large margin of safety is thus provided, as 40 tons per square inch would be an average ultimate tensile strength.

For wrought-iron coils, T_0, T_1, &c. $=9$.

If F is taken as the circumferential factor of safety, usually about 1·5, and P the safe working pressure in the chamber,

$$PF=P_0.$$

For the longitudinal strength of the above gun (where the breech-screw gears into the breech-piece), with the same system of notation, p_0 denoting longitudinal pressure,

$$p_0=\frac{r_3^2-r_1^2}{r_0^2}\cdot T_1.$$

The longitudinal factor of safety f generally equals 6 or 7, and $fP=p_0$.

702. Concluding Remarks, and Momentum of Recoil.—From what has already been said, the reader will at once perceive that the science of gunnery aims at the further development of the last theory. We have but touched the fringe of this interesting science, as space will permit us to do no more; but before closing the subject we can safely predict that the high initial velocities of projectiles fired from breech-loading rifled guns, such as the 16·25-inch and 13·5-inch, with charges of 1800 lb. and 1250 lb. of powder respectively, will shortly be eclipsed by the introduction of electricity as an agent of propulsion.

Experiments have already been made in this direction with lighter projectiles, and the marvellous results obtained therefrom, as regards range, velocity, and time of flight, are such as will necessitate all future formulæ being expressed in terms intimately associated with this prime-motor.

In order to obtain a high initial velocity for any gun, various complex considerations present themselves, and these require to be regarded as the sum of so many positive and negative quantities.

The construction of any gun depends on the total stress the material will be subjected to in the different parts of the gun, modified, of course, by the particular circumstances connected with its manipulation, which may either be field, naval, mountain, siege, or position (that is, coast defence).

Having decided upon its calibre and determined its employment, we have to consider the questions of the charge and weight of the projectile; and in arriving at these we are governed by the all-important matter of initial velocity, which in turn regulates the momentum of the shell, and consequently its penetrative work.

The twist of rifling (expressed in so many turns or revolutions of the projectile in a certain number of calibres) produces frictional resistance.

For instance, let the diameter of the bore of a gun be 3 inches, and let the twist of rifling be expressed as 1 turn in 30 calibres. It will be readily understood that the projectile, in passing through the bore of the gun, makes one complete turn round its longer axis in a distance = 90 inches or 30 calibres.

Then, again, there is another factor, which cannot be overlooked —namely, the 'momentum of recoil.'

When a gun fires a projectile, the force of the explosion produces momentum in the gun equal in amount but opposite to that of the projectile, and causes recoil. The other effects produced in the gun and the projectile are not, however, numerically equal.

According to a well-known law in dynamics, we are told '*that when two bodies mutually act upon each other, the momenta developed in the same time are equal, but opposite in direction;*' or, every action is accompanied by an equal and opposite reaction.

Example.—The 5-inch B.L. gun whose weight is 2 tons fires a projectile weighing 50 lb. with an initial velocity of 1800 feet per second. Find the velocity of the gun's recoil and the mean force of the explosion, supposing the bore of the gun to = 25·1 calibres.

Let W, w = weight of gun and projectile respectively.
V, v = velocity " " "

By the above law, momentum of gun = momentum of projectile.

$$\therefore WV = wv;$$

that is,

$$2 \times 2240 \times V = 50 \times 1800.$$

$$\therefore V = \frac{50 \times 1800}{2 \times 2240} = 20{\cdot}089 \text{ feet per second.}$$

Now, to find the mean effort executed during the explosion of the powder, we must first ascertain the acceleration of the projectile along the bore of the gun. Since the bore is 25·1 calibres, or 10·458 feet, in length, and the initial velocity of the projectile as it leaves the gun = 1800 feet per second, we have

$$v^2 = 2as,$$

a formula deduced from uniformly accelerated motion, where

a = acceleration per unit time,
s = distance described during interval $(t_2 - t_1)$.

$$\therefore 1800^2 = 2 \times a \times 10{\cdot}458.$$

$$\therefore a = \frac{1800^2}{20{\cdot}916} = 154905{\cdot}335 \text{ feet per second per second.}$$

But

$$P = \frac{w}{g} \times a.$$

This equation expresses the force P in the same units as (w), and if w be stated in lb. weight, this will be in what is termed gravitation units.

$$\therefore P = \frac{50}{32} \times 154905{\cdot}335 = 242039{\cdot}595, \text{ \&c., lb.}$$

Large charges of powder alone will not produce a high velocity, although in a great measure they assist it. The object to which gunnery is rapidly trending is minimum charges and higher velocities.

Although the introduction of electricity will revolutionise this science to a very great extent, still, under circumstances where high angle fire appears necessary and advantageous, there can be but little doubt that the theory on which the general problems rest will still be found the pangenesis of formulæ connected with this science.

Its renaissance is dependent on a recognition of the theories already treated, for they embody certain fundamental laws of natural science inseparable from any speculation or experiment connected with gunnery.

These laws cannot therefore be affected in any way by a mere change of an agent representing force. The force, if electricity, is

still a force, and only differs from other forces by virtue of its highly subtile character and the magnitude of its power.

The word power is very frequently misapplied by writers and students, for they often call the mere pull, pressure, or force exercised on or by an agent the power.

It should never be employed in any other sense than as *expressing a rate of doing work, or activity.*

In electrical engineering the unit of power is called the watt, and it equals 10^7 ergs per second, or 746 watts = 1 horse-power.

PROJECTIONS

GENERAL DEFINITIONS

703. The representation on a plane of the important points and lines of an object as they appear to the eye when situated in a particular position is called the **projection** of the object.

704. The plane on which the delineation is made is called the **plane of projection.**

705. The point where the eye is situated is called the **point of sight,** or the **projecting point.**

706. The point on the plane of projection where a perpendicular to it from the point of sight meets the plane is called its **centre.**

707. The line joining the point of sight and the centre is called the **axis** of the plane of projection.

708. Any point, line, or other object to be projected is called the **original,** in reference to its projection.

709. A straight line drawn from the point of sight to any original point is called a **projecting line.**

710. The surface which contains the projecting lines of all the points of any original line is called a **projecting surface.** When the original line is straight, the projecting surface will be a **projecting plane.**

COR.—The projection of any point is the intersection of its projecting line with the primitive.

STEREOGRAPHIC PROJECTION OF THE SPHERE

DEFINITIONS

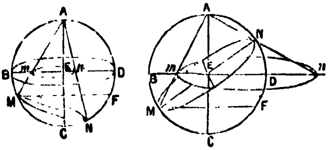
711. The **stereographic projection** of the sphere is that in which a great circle is assumed as the plane of projection, and one of its poles as the projecting point.

712. The great circle upon whose plane the projection is made is called the **primitive.**

713. By the **semi-tangent** of an arc is meant the tangent of half that arc.

714. By **the line of measures** of any circle of the sphere is meant that diameter of the primitive, produced indefinitely, which is perpendicular to the line of common section of the circle and the primitive.

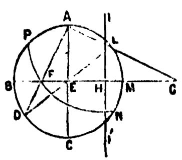
715. Let A be the pole of the primitive BD, and MN a circle to be projected; MN being in the first figure a small circle, and in the second a great circle; then the point M has for its projection the point m, and n is the projection of N, and the circle mn is the projection of the circle MN. The line AM is the projecting line of the point M, and the plane AMN is the projecting plane of the diameter MN, whose projection is the line mn.



In the stereographic projection, the projection of every circle of the sphere is a circle.

716. Problem I.—To find the locus of the centres of the projections of all the great circles that pass through a given point.

Let F be any given point within the primitive ABCM.



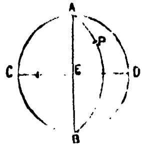
Through F draw the diameter BM and AC perpendicular to it; draw AF, and produce it to D; draw the diameter DL; draw AL, and produce it to meet BM in G; bisect FG perpendicularly by II′, and II′ is the required locus. Thus any circle, PFN, passing through F, and having its centre in any point as I in IHI′, is the projection of a great circle, and hence it cuts the primitive in two points, P, N, diametrically opposite.

717. **Problem II.—Through any two points in the plane of the primitive, to describe the projection of a great circle.**

1. When one of the points is in the centre of the primitive.

Draw a diameter passing through the other point, and it will be the required projection. For the great circle passes through the pole of the primitive.

2. When one of the points is in the circumference, and the other is neither in the circumference nor in the centre.

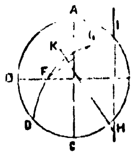


Let A and P be the two points, and ACBD the primitive.

Draw the diameter AB, and describe the circle APB through the three points A, P, B; and it is the required circle.

3. When neither of the points is in the centre or circumference.

Let F, G be the given points, and ABC the primitive.



Find IH the locus of the centres of all the projections of great circles passing through one of the points, as F (Art. 716); join F, G, and bisect FG perpendicularly by KH; and the centre of every circle through F and G is in KH; but the centre of the required circle is in IH; hence H is its centre; and a circle, DFG, through the two given points, described from the centre H, is the circle required.

718. **Problem III.—About some given point, as a pole, to describe the projection of a great circle.**

1. When the given point is the centre of the primitive.

The required projection is evidently the primitive itself.

2. When the given point is in the circumference of the primitive.

Draw a diameter through the given point, and another diameter perpendicular to the former; the latter diameter is the required projection.

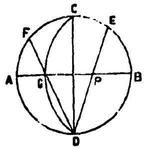
For, since the primitive passes through the pole of the required projection, its original circle must pass through the pole of the primitive, and its projection is a diameter.

3. When the given point is neither in the centre nor the circumference of the primitive.

Let P be the given point, and ADBC the primitive.

Through P draw the diameter AB, and another CD perpendicular

to it. Draw DP, and produce it to E; make the arc EF equal to a quadrant; draw DF, cutting AB in G; and the circle CGD, through the points C, G, D, is the required circle.



For, considering APB as the primitive, and D its pole, PG is evidently the projection of a quadrant EF. Now, if ADBC be the primitive, since APB passes through P, the pole of the required circle, it must pass through C, D, the poles of AB. Hence the required circle must pass through C, G, and D.

COR.—Hence the method of finding the pole of a projected great circle is evident.

1. When the projection is a diameter of the primitive. The extremities of the diameter perpendicular to it are evidently its poles.

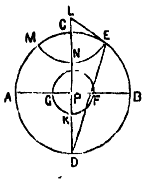
2. When the given projection is inclined to the primitive, as CGD.

Join C, D, and draw the diameter AB perpendicular to CD. Draw DG, and produce it to F; make the arc FE a quadrant; draw DE, cutting AB in P, and P is the pole of the given circle.

719. **Problem IV.—To describe the projection of a small circle about some given point as a pole.**

1. When the pole is in the centre of the primitive, or the original small circle parallel to the primitive.

Let AB, CD be two perpendicular diameters of the primitive. Make CE equal to the distance of the small circle from its pole—as, for example, 34°. Draw DE, cutting AB in F; from P as a centre, with the radius PF, describe the circle FGK, which will be the required projection.



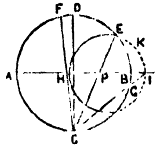
For PF is evidently the projection of CE, and the centre of the required circle is evidently in P.

2. When the given pole is in the circumference of the primitive, or the original circle is perpendicular to the primitive.

Let C be the given pole; AB, CD two perpendicular diameters. Make CE equal to the distance of the circle from its pole. Draw EL a tangent to the primitive at E, and let it meet DC produced in L. A circle described from the centre L, with the radius LE—namely, MNE—is the required circle.

3. When the pole is neither in the centre nor the circumference of the primitive.

Let P be the given point, and AB, CD two perpendicular diameters of the primitive. Draw CP, and produce it to E; lay off EF, EG, each equal to the distance of the circle from its pole—for instance, 62°; draw CF, CG, cutting AB in H and I, and on HI, as a diameter, describe the circle HKI, and it is the required projection. For if AB be the primitive, and C its pole; E the pole of a small circle, and F, G two points in its circumference, then HI is the diameter of its projection. Hence, if ACBD be the primitive, HI is evidently the diameter of the projected small circle, whose pole is P.



COR.—The method of finding the projected pole of a given projected small circle is manifest from this problem.

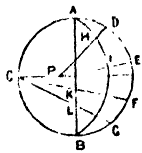
1. When the small circle is concentric with the primitive, the centre of the latter is the projected pole of the former.

2. When the small circle is perpendicular to the primitive, as MNE, its pole is in C, the middle of the arc MCE.

3. When the circle is inclined to the primitive, as HKI, draw a diameter AB through its centre, and CD perpendicular to it; draw CH, CI, cutting the primitive in F, G; bisect FEG in E; draw CE, and P is the required pole.

720. **Problem V.—To measure any given arc of a projected circle.**

1. If the given arc be a part of the primitive, it may be measured as the arc of any other circle (Art. 130 or 162).



2. When the given arc is a part of a circle projected into a straight line.

Let KL be any given arc of the projected circle AKB; find C its pole, and draw CK, CL, cutting the primitive in F and G, and FG is the measure of KL, and is in the present instance 32°.

3. When the given circle is inclined to the primitive.

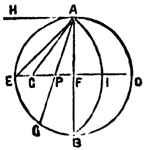
Let HI be the given arc of the projected circle AIB. Find P its pole; draw PH, PI, cutting the primitive in D, E, and DE is the measure of HI, which is therefore, in the present example, 45°.

721. Problem VI. – To measure the projection of a spherical angle.

1. When the circles containing the given angle are the primitive and a diameter of it.

The angle is a right angle.

2. When one of the circles is the primitive, and the other is a circle inclined to it.



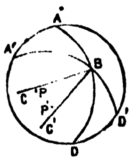
Let AEB be the primitive, and AIB the other circle, and IAD the angle. Find F and C their centres; draw AC, AF, and the angle CAF measures the given angle. Or, find F and P their poles; draw AP, AF, cutting the primitive in G and B, and GB measures the given angle, which is in the present instance 40°.

3. When one of the circles is a diameter of the primitive, and the other is inclined to the latter.

Let AFB and AIB be the two circles, and FAI the given angle.

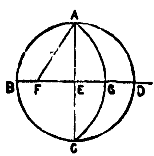
Draw the radius AC of the circle AIB, and AH perpendicular to AFB, and the angle HAC measures the given angle. Or, find P and E the poles of the circles; draw AE, AP; then GE measures the given angle, which is 50°.

4. When both the circles are inclined to the primitive.



Let ABD, A'BD' be the two circles, and ABA' the given angle. Find C, C', the centres of the circles, then the two radii drawn from these to B will contain an angle CBC' equal to that at B. Or, find P, P', the poles of the circles, and lines drawn from B through these points will intercept on the primitive an arc which measures the given angle. The angle in this instance is 32°.

722. Problem VII.—Through a given point in a given projected great circle, to describe the projection of another great circle cutting the former at a given angle.



Let ABCD be the primitive, and Z the given angle.

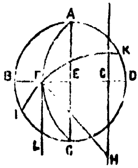
1. When the given circle is the primitive.

Let A be the given point; draw the perpendicular diameters AC, BD; make angle EAF $=Z=32°$, suppose; and from F as a centre, with a radius FA, describe the circle AGC; it is the required projection, and angle GAD$=32°$.

When the angle is a right angle, the diameter AC is evidently the required projection.

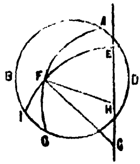
2. When the given projected circle is a diameter of the primitive.

Let BD be the given projection, and F the given point. Find GH the locus of all the great circles passing through F; draw FL perpendicular to BD, and FH, making an angle LFH $= Z = 46°$, for instance; from the centre H, with the radius HF, describe the circle IFK; it is the required projection, and angle DFK $= 46°$.



If the angle be a right angle, G is the centre, and AFC the required projection, for angle LFG $=$ a right angle. Or, since the required circle is in this case perpendicular to BFD, it must pass through its poles A and C. Hence the circle AFC, passing through the three points A, F, C, is the required projection.

3. When the given circle is inclined to the primitive.



Let AFC be the given circle, and F the given point in it. Find EG the locus of the centres of all the great circles passing through F. Draw FH a radius of the given circle, and draw FG, making the angle GFH $= Z = 23°$, suppose; from the centre G, with the radius GF, describe IFE; and it is the required projection, and angle IFC $= 23°$.

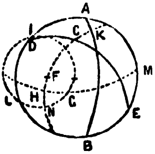
When the angle Z is a right angle, draw from F a line perpendicular to FH, and it will cut EG in the centre of the required circle. Or since in this case the required projection must pass through the pole of AFC, find its pole, and describe the projection of a great circle passing through this pole and the point F (Art. 717), and it will be the required circle.

723. Problem VIII.—Through a given point in the plane of the primitive, to describe the projection of a great circle cutting that of another great circle at a given angle.

Let AKB be the given circle, Z the given angle, and C the given point in the plane of the primitive AMB.

Find F the pole of AKB, and about it describe a small circle IGN, at a distance from its pole equal to the measure of angle $Z = 44°$, for example. About the given point C, as a pole, describe a great circle LHM, intersecting the small circle in L and G. About either of these points, as G, for a pole, describe a great

circle DCE, and it is the required projection. For the circle DCE must pass through C, since C is at the distance of a quadrant from G, a point of the circle LGM. Also, the distance between F and G, the poles of AKB and DCE, is the measure of the given angle, and hence the inclination of the circles is equal to that angle = 44°.



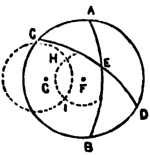
SCHOL. 1.—Let an arc of a great circle FCK be described through F and C; then, FK and CH being quadrants, FH = CK. Now, FH must not exceed FN, the measure of the angle, otherwise the circle LHM would not meet IGN, and the problem would be impossible. But CK = FH; therefore the distance of the given point from the given circle must not exceed the measure of the angle.

SCHOL. 2.—If the point C were in the centre of the primitive, the circle LGM would coincide with the primitive. If C were in the circumference of the primitive, the circle LGM would be a diameter perpendicular to that passing through C.

724. **Problem IX.—To describe the projection of a great circle that shall cut the primitive and a given great circle at given angles.**

Let ADB be the primitive, AEB the given circle, and X, Y the given angles which the required circle makes respectively with these circles, and let these angles be respectively 47° and 45°.

About F, the pole of the primitive, describe a small circle at a distance of 47°, the measure of angle X, and about G, the pole of AEB, describe another small circle at a distance of 45°, the measure of angle Y. Then from either of the points of intersection H, I, as I for a pole, describe the great circle CED, and it is the required circle. For the distances of its pole I from F and G, the poles of the given circles, are equal to the measures of the angles X and Y; and therefore the inclinations of CED to the given circles are equal to these angles—that is, angle ACE = 47°, and AEC = 45°.



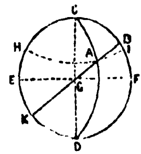
SCHOL.—When any of the angles exceeds a right angle, the distance of the small circle from its pole is greater than a quadrant. The same small circle will be determined by finding the more remote pole—that is, the projection of the pole nearest to the projecting point—and then describing a small circle about it at a distance equal to the supplement of the measure of the angle.

STEREOGRAPHIC PROJECTION OF THE CASES OF TRIGONOMETRY

PROJECTION OF THE CASES OF RIGHT-ANGLED TRIGONOMETRY

725. Case 1.—Given the hypotenuse AC = 64°, and the angle C = 46°, to construct the triangle, and to measure its other parts.

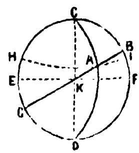
Let ECFD be the primitive; draw the circle CAD, making angle C = 46° (Art. 722); about C, as a pole, describe the small circle IAH at a distance = 64° from C (Art. 719); then through A draw the diameter BK; and ABC is the given triangle.



Measure the sides AB, BC, and angle A (Arts. 720 and 721); and it will be found that AB = 40° 17′, BC = 54° 55′, and A = 65° 35′.

726. Case 2.—Given the hypotenuse AC = 70° 24′, and the side BC = 65° 10′, to construct the triangle.

Make the arc BC = 65° 10′, and describe the small circle IAH at a distance from its pole C equal to 70° 24′ (Art. 719); draw the diameter BAG, and then through A and C describe the great circle CAD; and ABC is the required triangle.

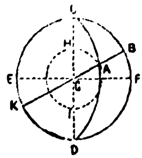


Measure the side AB, and angles A and C, as in the preceding problem.

Angle C = 39° 42′, A = 74° 26′, and AB = 37°.

727. Case 3.—Given the side AB = 37°, and BC = 65° 10′, to construct the triangle.

Make BC = 65° 10′; draw the diameter BAK; and about G, as a pole, describe the small circle AIH at a distance from G = the complement of AB = 53° (Art. 719), then is AB = 37°; through A and C describe the great circle CAD (Art. 717); and ABC is the required triangle.



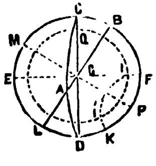
Measure AC, and angle A and C as before.

AC = 70° 24′, A = 74° 26′, and C = 39° 42′.

728. Case 4.—Given angle A = 32° 30′, and C = 106° 24′, to construct the triangle.

Draw a diameter BL, and find its pole P (Art. 718, Cor.); about the pole P describe the small circle KI′I at a distance from P of

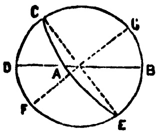
32° 30′; and about G, the pole of the primitive, describe a small circle I′IQ at a distance from it = 73° 36′, the supplement of angle C (Art. 719); and about I, the intersection of these small circles, describe the great circle CAD (Art. 718); and ABC is the required triangle.



Measure AB, BC, AC as before.

AC = 117° 31′, AB = 126° 42′, and BC = 28° 28′.

729. Case 5.—Given the side BC = 140° 53′, and angle C = 105° 53′, to construct the triangle.



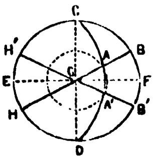
Make BGC = 140° 53′; draw the diameter BAD, and through C describe the circle CAE, making angle FCE = 74° 7′, the supplement of 105° 53′; and ABC is the required triangle.

Measure AB, AC, and angle A.

AC = 70° 24′, AB = 114° 17′, and A = 138° 16′.

730. Case 6.—Given AB = 40° 25′, and angle C = 44° 56′, to construct the triangle.

Describe the circle CAD, making angle ACB = 44° 56′; and about G, as a pole, describe the small circle AA′ at a distance from G = 49° 35′, the complement of AB; then through A and A′ draw the diameters BH, B′H′, and ABC, A′B′C are two triangles, constructed from the same data—that is, having their sides AB, A′B′ of the given magnitude, and the angle C common.

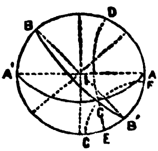


Measure AC, BC, and angle A; also A′C and B′C, and angle A′.

AC = 66° 38′, BC = 58° 36′, and A = 68° 25′; and A′C = 113° 22′, B′C = 121° 24′, and A′ = 111° 35′; the three latter parts are the supplements of the three former.

PROJECTION OF CASES OF OBLIQUE-ANGLED SPHERICAL TRIGONOMETRY

731. Case 1.—Given the side AB = 132° 11′, BC = 143° 46′, and AC = 67° 24′, to construct the triangle.

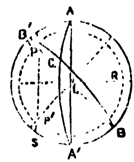


Make ADB = 132° 11′; about A, as a pole, describe the small circle DCE at a distance AD of 67° 24′; and about B′, the small circle FCG at a distance B′F = 36° 14′, the supplement of BC; then through A, C, and B, C, describe the great circles ACA′, BCB′; and ABC is the required triangle.

By measurement, angle A = 143° 18′, B = 111° 4′, and C = 131° 30′.

732. Case 2.—Given the angle A = 114° 30′, B = 83° 12′, and C = 123° 20′, to construct the triangle.

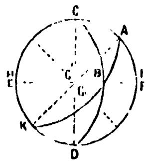
Describe the great circle ACA′, making angle BAC = 114° 30′; then about G, as a pole, describe a small circle PP′R at a distance from it = 83° 12′ (Art. 719); and about the remote pole of ACA′ describe the small circle P′PS at a distance from it = 56° 40′ = the supplement of 123° 20′; then about either of the points of intersection P, P′, as P, describe the great circle B′CB; and ABC is the required triangle.



It will be found by measurement that the side BC = 125° 24′.

733. Case 3.—Given the side AC = 44° 14′, BC = 84° 14′, and angle C = 36° 45′, to construct the triangle.

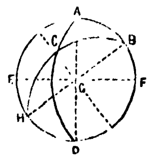
Make AC = 44° 14′; make angle ACB = 36° 45′ (Art. 722); draw the small circle IBH about C, as a pole, at a distance = 84° 14′; and through the points A, B describe the circle ABK (Art. 717); and ABC is the required triangle.



By measurement, AB = 51° 6′, angle A = 130° 5′, and B = 30° 26′.

734. Case 4.—Given angle A = 130° 5′, B = 30° 26′, and the side AB = 51° 6′, to construct the triangle.

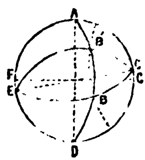
Make AB = 51° 6′, angle BAC = 130° 5′, or EAC = 49° 55′ (Art. 722), and ABC = 30° 26′; and ABC is the required triangle.



By measurement, AC = 44° 14′, BC = 84° 14′, and angle C = 36° 45′.

735. Case 5.—Given the side AC = 80° 19′, BC = 63° 50′, and angle A = 51° 30′, to construct the triangle.

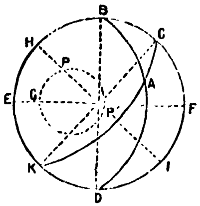
Make AC = 80° 19′, and angle BAC = 51° 30′ (Art. 722); about C, as a pole, describe B′B at a distance = 63° 50′; and through B and C describe the circle EBC; or through B′ and C describe EB′C; and either ABC or AB′C is the required triangle.



By measurement, in the triangle ABC, AB = 120° 46′, angle B = 59° 16′, and angle C = 131° 32′. In the triangle AB′C, angle B′ is the supplement of B = 180° − 59° 16′ = 120° 44′; but AB′ is not the supplement of AB, nor angle ACB′ of ACB. It is found that AB′ = 28° 34′, and ACB′ = 24° 36′.

736. CASE 6.—Given angle $A = 31° 34'$, $B = 30° 28'$, and the side $BC = 40°$, to construct the triangle.

Make $BC = 40°$, and angle $ABC = 30° 28'$ (Art. 722); about the pole of BAD, and at a distance $= 31° 34'$, describe a small circle PP′G, cutting the diameter PP′, which is perpendicular to CK in P and P′; about P, as a pole, describe the great circle CAK, and ABC is the required triangle.



The great circle described about P′ as a pole would cut the circle BAD at the given angle; but it would be an exterior angle of the triangle, to which the side BC belongs. But if A were < B, there would then be two triangles; in this case the two poles, P and P′, would lie on the same side of CK.

By measurement, $AC = 38° 30'$, $AB = 70°$, and $C = 130° 3'$.

SPHERICAL TRIGONOMETRY

737. Spherical Trigonometry treats of the methods of computing the sides and angles of **spherical triangles.**

DEFINITIONS

738. A **sphere** is a solid every point in whose surface is equidistant from a certain point within it.

This point is called the **centre.** A sphere may be conceived to be formed by the revolution of a semicircle about its diameter as an axis.

739. A line drawn from the centre to the surface of a sphere is called its **radius**; and a line passing through the centre of the sphere, and terminated at both extremities by its surface, is called a **diameter.**

740. Circles whose planes pass through the centre of the sphere are called **great** circles; and all others, **small** circles.

741. A line limited by the spherical surface, perpendicular to the plane of a circle of the sphere, and passing through the centre of the circle, is called the **axis** of that circle; and the extremities of the axis are the **poles** of the circle.

742. The **distance** of two points on the surface of the sphere means the arc of a great circle intercepted between them.

743. **A spherical angle** is an angle at a point on the surface of the sphere, formed by arcs of two great circles passing through the point, and is measured by the inclination of the planes of the circles, or by the inclination of their tangents at the angular point.

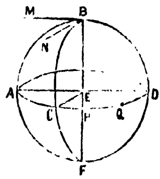
744. A **spherical** triangle is a triangular figure formed on the spherical surface by arcs of three great circles, each of which is less than a semicircle.

When one of the sides of a spherical triangle is a quadrant, it is called a **quadrantal** triangle.

745. The sides of a spherical triangle being arcs of great circles of the same sphere, their lengths are proportional to the number of degrees contained in them; and hence the sides of spherical triangles are usually estimated by the number of degrees they contain.

The definitions of trigonometrical ratios given in 'Plane Trigonometry' are employed in reference to the sides and angles of spherical triangles.

746. A spherical angle is measured by that arc of a great circle whose pole is the angular point which is intercepted by the sides of the angle.



Thus, the spherical angle ABC, which is the same as the angle contained by the planes ABF, CBF of the two arcs AB, BC that contain the angle, is measured by the arc AC of a great circle ACD, whose pole is the angular point B; or by the angle MBN contained by the tangents MB, NB to the arcs BA, BC. For angle MBN = angle AEC, and AEC is measured by AC.

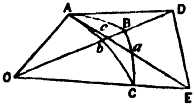
747. Two arcs are said to be of the same **species, affection,** or **kind** when both are less or both greater than a quadrant; and consequently the same term is applied to angles in reference to a right angle.

The species of the sides and angles of spherical triangles can generally be easily determined by means of the algebraical signs of their cosines, cotangents, &c.

748. **To find the relations between the trigonometrical functions of the three sides and the three angles of any spherical triangle.**

Let ABC be a spherical triangle, and let O be the centre of the sphere on which it is described; then OA = OB = OC, and let

AD be a tangent to the arc AB, produced to meet OB in D, and AE a tangent to the arc AC, produced to meet OC in E and join DE. Then, if A, B, and C represent the three angles, and a, b, and c the sides opposite them; since AD and AE are tangents to the arc AB and AC, the angle DAE is the measure of the spherical angle BAC; also, c is the measure of the angle AOD, and b is the measure of the angle AOE, and a is the measure of the angle BOC or DOE; hence



$$\frac{AO}{OD}=\cos c,\ \frac{AD}{OD}=\sin c,\ \frac{OA}{OE}=\cos b, \text{ and } \frac{AE}{OE}=\sin b.$$

Therefore in the triangle DAE,

$$DE^2=AD^2+AE^2-2AD.AE\cos A; \quad . \quad . \quad [a];$$

and from triangle DOE,

$$DE^2=OD^2+OE^2-2OD.OE\cos a. \quad . \quad . \quad . \quad [b].$$

Subtracting [a] from [b], and observing that OD^2-AD^2 and OE^2-AE^2 are each equal to OA^2 (Eucl. I. 47), since the angles OAD and OAE are right angles, we obtain

$$0=2OA^2+2AD.AE\cos A-2OD.OE\cos a;$$

transposing and dividing by 2OD . OE,

$$\cos a=\frac{OA}{OD}\times\frac{OA}{OE}+\frac{AD}{OD}\times\frac{AE}{OE}\cos A,$$

or $\cos a=\cos c.\cos b+\sin c.\sin b.\cos A.$

Similarly, $\cos b=\cos a.\cos c+\sin a.\sin c.\cos B,$. [c].

and $\cos c=\cos a.\cos b+\sin a.\sin b.\cos C.$

Again, transposing and dividing by the coefficients of the cosines of the angles,

$$\left.\begin{aligned}\cos A&=\frac{\cos a-\cos b\cos c}{\sin b\sin c},\\ \cos B&=\frac{\cos b-\cos a\cos c}{\sin a\sin c},\\ \text{and}\quad \cos C&=\frac{\cos c-\cos a\cos b}{\sin a\sin b},\end{aligned}\right\} \quad . \quad . \quad . \quad [d];$$

whence, $1-\cos^2A$, or $\sin^2A=1-\dfrac{(\cos a-\cos b\cos c)^2}{\sin^2b\sin^2c}$;

and reducing to a common denominator, and putting $1-\cos^2b$ for $\sin^2b$, and $1-\cos^2c$ for $\sin^2c$ in the numerator, there results

$$\sin^2A=\frac{1-\cos^2b-\cos^2c-\cos^2a+2\cos a\cos b\cos c}{\sin^2b\sin^2c}.$$

Taking the root of this, and dividing the two sides by sin a, the second side will be a **symmetrical** function of a, b, c, which we shall call M—namely,

$$\frac{\sin A}{\sin a}=\frac{\sqrt{1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c}}{\sin a\sin b\sin c}=\mathrm{M}.$$

But if A and a be now changed into B and b, or into C and c, the second side will remain the same; hence the **first** side must continue constant, and we shall have

$$\mathrm{M}=\frac{\sin A}{\sin a}=\frac{\sin B}{\sin b}=\frac{\sin C}{\sin c};\ \text{hence} \quad . \quad . \quad . \quad [e].$$

749. In every spherical triangle the sines of the angles are proportional to the sines of the opposite sides.

According to the property of the supplemental triangle,* change, in [c], a into 180° − A, &c., and we shall have

$$\left.\begin{array}{ll} & -\cos A=\cos B\cos C-\sin B\sin C\cos a.\\ \text{Similarly,} & -\cos B=\cos A\cos C-\sin A\sin C\cos b,\\ \text{and} & -\cos C=\cos A\cos B-\sin A\sin B\cos c.\end{array}\right\} \quad . \quad . \quad [f].$$

$$\text{Cor.—Whence,}\quad\left.\begin{array}{l}\cos a=\dfrac{\cos A+\cos B\,.\cos C}{\sin B\,.\sin C}.\\[2ex] \cos b=\dfrac{\cos B+\cos A\,.\cos C}{\sin A\,.\sin C},\\[2ex] \cos c=\dfrac{\cos C+\cos A\,.\cos B}{\sin A\,.\sin B}.\end{array}\right\} \quad . \quad . \quad . \quad [g].$$

To eliminate b from equation 1 of [c], put $\sin b=\dfrac{\sin B\sin a}{\sin A}$ [e], and $\cos b=\cos a\cos c+\sin a\sin c\cos B$ [c]; substituting in the result $1-\sin^2 c$ for $\cos^2 c$, and dividing the whole by the common factor $\sin a\sin c$, we have

$$\left.\begin{array}{ll} & \sin c\cot a=\cos c\cos B+\sin B\cot A.\\ \text{Similarly,} & \sin c\cot b=\cos c\cos A+\sin A\cot B,\\ & \sin a\cot c=\cos a\cos B+\sin B\cot C.\end{array}\right\} \quad . \quad . \quad [h].$$

$$\left\{\begin{array}{l}\sin a\cot b=\cos a\cos C+\sin C\cot B,\\ \sin b\cot a=\cos b\cos C+\sin C\cot A,\\ \text{and}\ \sin b\cot c=\cos b\cos A+\sin A\cot C.\end{array}\right\} \quad . \quad . \quad [h].$$

* If the angular points of a spherical triangle be made the poles of three great circles, these three circles by their intersections will form a triangle which is said to be supplemental to the former; and the two triangles are such that the sides of the one are the supplements of the arcs which measure the angles of the other.

The equations [c], [e], [f], [h] are the foundation of the whole of Spherical Trigonometry, and serve for the solution of all triangles; but as they are not suited to logarithmic calculation, we proceed to deduce from them more convenient formulæ.

SOLUTION OF RIGHT-ANGLED SPHERICAL TRIANGLES

750. In the preceding formulæ, if one of the angles, as B, be a right angle, then sin B = 1, and cos B = 0, and we at once have, by making these substitutions in the above formulæ, [c], [e], [f], and [h].

From [c], . . . $\cos b = \cos a \,.\, \cos c$ . . . . . (l).

" [e], . . . $\begin{cases} \sin a = \sin A \,.\, \sin b & (m). \\ \sin c = \sin C \,.\, \sin b & (n). \end{cases}$

" [f], . . . $\begin{cases} \cos b = \cot A \,.\, \cot C & (o). \\ \cos A = \cos a \,.\, \sin C & (p). \\ \cos C = \cos c \,.\, \sin A & (q). \end{cases}$

" [h], . . . $\begin{cases} \sin c = \tan a \,.\, \cot A & (r). \\ \sin a = \tan c \,.\, \cot C & (s). \\ \cos A = \tan c \,.\, \cot b & (t). \\ \cos C = \tan a \,.\, \cot b & (u). \end{cases}$

Collecting the values of each quantity into one line, and multiplying the first side by R, to make it true for any radius, we have

$R \,.\, \cos b = \cot A \,.\, \cot C = \cos a \,.\, \cos c$ . . (o) and (l),
$R \,.\, \sin a = \tan c \,.\, \cot C = \sin b \,.\, \sin A$ . . (s) " (m),
$R \,.\, \sin c = \tan a \,.\, \cot A = \sin b \,.\, \sin C$ . . (r) " (n),
$R \,.\, \cos A = \tan c \,.\, \cot b = \cos a \,.\, \sin C$ . . (t) " (p),
$R \,.\, \cos C = \tan a \,.\, \cot b = \cos c \,.\, \sin A$ . . (u) " (q).

The above ten equations are all included in two rules, called Napier's Rules, for the circular parts; they are the following:—

751. If in a right-angled spherical triangle the right angle be omitted, there remain other **five** parts. Napier observed that if the two sides which contain the right angle, the **complements** of the other two angles, and the **complement** of the side opposite the right angle be called the **five circular parts**, then any three of these being taken, they will either be adjacent, or one of them will be separated from each of the other two by another of the circular parts. Let now that part which lies between the other two, or which is separated from the other two, be called the **middle** part; and the remaining two, when they all lie together, the **adjacent** parts; and when they are separated from it, the **opposite** parts; then,

752. R. × sine of the middle part = product of the $\begin{cases}\text{tangents of the adjacent parts,} \\ \text{cosines " opposite "}\end{cases}$

It will, in fact, easily be seen that these two conditions contain all the ten preceding equations, which are true as first given, for radius = 1; and in the second form and in the rule, are true for any radius.

These equations, taken in connection with the signs of the trigonometrical ratios, demonstrate various general properties which it will be of use to observe in all right-angled spherical triangles.

1st. From equation (l) we conclude that any one of the three sides is $<$ or $>$ 90°, according as the other two sides are of the same or different species.

2nd. Equation (o) shows that if the hypotenuse be compared with the two adjacent angles A and C, any one of these three arcs is $<$ or $>$ 90°, according as the two others are of the same or different species.

3rd. The equations (p, q) prove that each of the angles A and C is always of the same species as the opposite sides a and c; and conversely.

4th. Equations (t, u) prove that the hypotenuse and a side are of the same species when the included angle is acute, and of different species when the included angle is obtuse.

5th. Equation (l) proves that if $\cos a = 0$, or $a = 90°$, $\cos b = 0$, and $\therefore b = 90°$; hence $\cot b = 0$, and from (t) $\cos A = 0$, or $A = 90°$; so that the sides a and b are both quadrants, and perpendicular to the third side c; and C is the pole of the arc c, consequently c is the measure of the angle C; the triangle is then isosceles, and has two right angles, and the third side and third angle contain the same number of degrees.

The above five theorems will be found useful when any triangle is divided into two right-angled triangles by an arc drawn from one of its angles perpendicular to the opposite side.

753. To find expressions suited to logarithmic calculation for the three angles of a spherical triangle, in terms of the three sides.

By Art. 748 (d), $\cos A = \dfrac{\cos a - \cos b \cdot \cos c}{\sin b \cdot \sin c}$; hence

$$1 + \cos A = 1 + \frac{\cos a - \cos b \cdot \cos c}{\sin b \cdot \sin c}$$

$$= \frac{\cos a - (\cos b \cdot \cos c - \sin b \cdot \sin c)}{\sin b \cdot \sin c};$$

$$\therefore\ 2\cos^2\tfrac{1}{2}A = \frac{\cos a - \cos(b+c)}{\sin b \cdot \sin c}$$

$$= \frac{2\sin\tfrac{1}{2}(a+b+c)\sin\tfrac{1}{2}(b+c-a)}{\sin b \sin c};$$

and hence, if $s = \tfrac{1}{2}(a+b+c)$,

$$\left.\begin{aligned} \cos\tfrac{1}{2}A &= \sqrt{\frac{\sin s \cdot \sin(s-a)}{\sin b \cdot \sin c}}, \\ \text{Similarly,} \quad \cos\tfrac{1}{2}B &= \sqrt{\frac{\sin s \cdot \sin(s-b)}{\sin a \cdot \sin c}}, \\ \text{and} \quad \cos\tfrac{1}{2}C &= \sqrt{\frac{\sin s \cdot \sin(s-c)}{\sin a \cdot \sin b}}. \end{aligned}\right\} \quad . \quad . \quad . \quad [i].$$

Again, $$1 - \cos A = 1 - \frac{\cos a - \cos b \cdot \cos c}{\sin b \cdot \sin c}$$

$$= \frac{\cos b \cdot \cos c + \sin b \cdot \sin c - \cos a}{\sin b \cdot \sin c};$$

$$\therefore\ 2\sin^2\tfrac{1}{2}A = \frac{\cos(b-c) - \cos a}{\sin b \cdot \sin c}$$

$$= \frac{2\sin\tfrac{1}{2}(a+b-c)\sin\tfrac{1}{2}(a+c-b)}{\sin b \cdot \sin c};$$

and hence $$\sin\tfrac{1}{2}A = \sqrt{\frac{\sin\tfrac{1}{2}(a+b-c)\sin\tfrac{1}{2}(a+c-b)}{\sin b \cdot \sin c}};$$

$$\left.\begin{aligned} \therefore\ \sin\tfrac{1}{2}A &= \sqrt{\frac{\sin(s-b) \cdot \sin(s-c)}{\sin b \cdot \sin c}}. \\ \text{Similarly,} \quad \sin\tfrac{1}{2}B &= \sqrt{\frac{\sin(s-a) \cdot \sin(s-c)}{\sin a \cdot \sin c}}, \\ \text{and} \quad \sin\tfrac{1}{2}C &= \sqrt{\frac{\sin(s-a) \cdot \sin(s-b)}{\sin a \cdot \sin b}}. \end{aligned}\right\} \quad [j].$$

Also, $$\tan\tfrac{1}{2}A = \frac{\sin\tfrac{1}{2}A}{\cos\tfrac{1}{2}A}$$

$$= \sqrt{\frac{\sin(s-b) \cdot \sin(s-c)}{\sin b \cdot \sin c} \times \frac{\sin b \cdot \sin c}{\sin s \cdot \sin(s-a)}}$$

$$= \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \cdot \sin(s-a)}}$$

$$= \sqrt{\frac{\sin(s-a) \cdot \sin(s-b) \cdot \sin(s-c)}{\sin s \cdot \sin^2(s-a)}};$$

$$\left.\begin{aligned}
&\text{and therefore } \tan \tfrac{1}{2}A \\
&\quad = \frac{1}{\sin (s-a)}\sqrt{\frac{1}{\sin s} \times \sin (s-a) \,.\, \sin (s-b) \,.\, \sin (s-c)}. \\
&\text{Similarly, } \tan \tfrac{1}{2}B \\
&\quad = \frac{1}{\sin (s-b)}\sqrt{\frac{1}{\sin s} \times \sin (s-a) \,.\, \sin (s-b) \,.\, \sin (s-c)}, \\
&\text{and } \tan \tfrac{1}{2}C \\
&\quad = \frac{1}{\sin (s-c)}\sqrt{\frac{1}{\sin s} \times \sin (s-a) \,.\, \sin (s-b) \,.\, \sin (s-c)}.
\end{aligned}\right\} \quad [k].$$

The three angles of a spherical triangle can be calculated logarithmically from either of the three sets of formulæ given above; but the last, which gives the tangent of the semi-angle, will be found the most convenient in practice, as all the angles can be found in terms of four arcs, while those formulæ which give the semi-angles in terms of the cosine or sine require the use of seven arcs; besides, the angles can be found with greater accuracy from the tangents than from the sine or cosine, as the tangent varies more rapidly than either the sine or cosine.

754. When the three angles are given to find the sides, **the supplements of the given angles** may be taken for the **sides** of a new triangle, and the **angles** of this triangle, found from the formulæ of last article, will be **the supplements of the sides** of the given triangle, from which the sides can easily be found.

Formulæ similar to the above, expressing the sides in terms of the angles, may be deduced from [g] in the same manner as those in the last article; they are the following: $s = \frac{1}{2}(A + B + C)$.

$$\text{Sin}\,\frac{a}{2} = \sqrt{\frac{-\cos s \,.\, \cos (s-A)}{\sin B \,.\, \sin c}},$$

and $$\cos\frac{a}{2} = \sqrt{\frac{\cos (s-B) \,.\, \cos (s-C)}{\sin B \,.\, \sin C}}.$$

$$\text{Sin}\,\frac{b}{2} = \sqrt{\frac{-\cos s \,.\, \cos (s-B)}{\sin A \,.\, \sin C}},$$

and $$\cos\frac{b}{2} = \sqrt{\frac{\cos (s-A) \,.\, \cos (s-C)}{\sin A \,.\, \sin C}}.$$

$$\text{Sin}\,\frac{c}{2} = \sqrt{\frac{-\cos s \,.\, \cos (s-C)}{\sin A \,.\, \sin B}},$$

and
$$\cos\frac{c}{2}=\sqrt{\frac{\cos(s-A)\,.\,\cos(s-B)}{\sin A\,.\,\sin B}};$$

$$\cot\frac{a}{2}=\frac{1}{\cos(s-A)}\sqrt{-\frac{1}{\cos s}\,.\,\cos(s-A)\,.\,\cos(s-B)\,.\,\cos(s-C)},$$

$$\cot\frac{b}{2}=\frac{1}{\cos(s-B)}\sqrt{-\frac{1}{\cos s}\,.\,\cos(s-A)\,.\,\cos(s-B)\,.\,\cos(s-C)},$$

$$\cot\frac{c}{2}=\frac{1}{\cos(s-C)}\sqrt{-\frac{1}{\cos s}\,.\,\cos(s-A)\,.\,\cos(s-B)\,.\,\cos(s-C)}.$$

755. When the parts given are either two sides and the contained angle, or two angles and the side lying between, the other parts are most conveniently found by a set of formulæ called Napier's Analogies, which may be established as follows :—

Let $M=\sqrt{\frac{1}{\sin s}\,.\,\sin(s-a)\,.\,\sin(s-b)\sin(s-c)}$, then

$$\tan\frac{A}{2}=\frac{M}{\sin(s-a)},\ \tan\frac{B}{2}=\frac{M}{\sin(s-b)},$$

and
$$\tan\frac{C}{2}=\frac{M}{\sin(s-c)};\ \text{hence}$$

$$\tan\frac{A+B}{2}=\frac{\dfrac{M}{\sin(s-a)}+\dfrac{M}{\sin(s-b)}}{1-\dfrac{M^2}{\sin(s-a)\,.\,\sin(s-b)}}$$

$$=\frac{\sqrt{\sin s\,.\,\sin(s-a)\,.\,\sin(s-b)\,.\,\sin(s-c)}\{\sin(s-b)+\sin(s-a)\}}{\sin(s-a)\,.\,\sin(s-b)\{\sin s-\sin(s-c)\}}$$

$$=\sqrt{\frac{\sin s\,.\,\sin(s-c)}{\sin(s-a)\,.\,\sin(s-b)}}\times\frac{\sin(s-b)+\sin(s-a)}{\sin s-\sin(s-c)}$$

$$=\cot\frac{C}{2}\cdot\frac{2\sin\dfrac{c}{2}\,.\,\cos\dfrac{(a-b)}{2}}{2\cos\left(\dfrac{a+b}{2}\right).\sin\dfrac{c}{2}}=\cot\frac{C}{2}\cdot\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}.$$

In a similar manner, after some reduction, we find

$$\tan\frac{A-B}{2}=\sqrt{\frac{\sin s\,.\,\sin(s-c)}{\sin(s-a)\,.\,\sin(s-b)}}\times\frac{\sin(s-b)-\sin(s-a)}{\sin s+\sin(s-c)}$$

$$=\cot\frac{C}{2}\cdot\frac{2\cos\dfrac{c}{2}\sin\frac{1}{2}(a-b)}{2\sin\frac{1}{2}(a+b)\,.\,\cos\dfrac{c}{2}}=\cot\frac{C}{2}\cdot\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}.$$

Therefore, $\tan \frac{1}{2}(A+B) = \cot \frac{1}{2}C \cdot \dfrac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)},$

and $\tan \frac{1}{2}(A-B) = \cot \frac{1}{2}C \cdot \dfrac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)},$ . . [v]

In a similar manner, it may be shown from Art. 753 (k) that

$\tan \frac{1}{2}(a+b) = \tan \frac{1}{2}c \cdot \dfrac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)},$

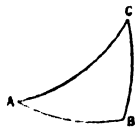
and $\tan \frac{1}{2}(a-b) = \tan \frac{1}{2}c \cdot \dfrac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)},$ . . [w]

The above four equations, which can easily be converted into proportions, are called Napier's Analogies.

756. Rule for determining the sign of the answer in a proportion. If the fourth term is a cosine, tangent, or cotangent, and of the arcs whose cosines, tangents, or cotangents enter in the first three terms, if one or three are greater than a quadrant, so is the fourth term.

757. Case 1.—Given the hypotenuse and one of the angles of a right-angled triangle, to find the other parts.

Example.—In the spherical triangle ABC right-angled at B, the hypotenuse AC is 64°, and the angle C 46′; what are the remaining parts?



1. To find BC

When angle C is the middle part, BC and the complement of AC are the adjacent parts; therefore R . cos C = cot AC . tan BC; and as BC is wanted, the proportion must be (Art. 750)

Cot AC : R = cos C : tan BC.

+ Cot AC 64°	=	9·6881818
+ Radius	=	10·
+ Cos C 46°	=	9·8417713
+ Tan BC 54° 55′ 35·8″ . . .	=	10·1535895

Since the signs of the first three terms are +, for radius is always positive, that of the fourth must be so, and + tan B shows that B is < 90.

2. To find AB

AB being the middle part, AC and C are opposite parts; therefore (Art. 750, n) R . sin AB = sin C . sin AC; or, since AB is required,

R : sin AC = sin C : sin AB.

Radius	=	10·
Sin AC 64°	=	9·9536602
Sin C 46°	=	9·8569341
Sin AB 40° 16′ 52″	=	9·8105943

The sine for 9·8105943 may be either that of 40° 16′ 52″, or its supplement 139° 43′ 8″; but in the given triangle, the angle C opposite to the side AB is *acute*; hence AB is <90 (Art. 742).

3. To find angle A

When AC is the middle part, angles A and C are the adjacent parts, and (Art. 750, *o*) R . cos AC = cot A . cot C; hence

Cot C : R = cos AC : cot A.

Cot C 46°	=	9·9848372
Radius	=	10·
Cos AC 64°	=	9·6418420
Cot A 65° 35′ 4″	=	9·6570048

EXERCISES

1. The hypotenuse is = 75° 20′, and one of the oblique angles = 57° 16′; what are the other parts?

The two sides = 64° 10′ 20″ and 54° 28′ 3″, and the other angle = 68° 30′ 4″.

2. The hypotenuse is = 64° 40′, and an angle = 64° 38′ 11″; find the other parts.

The other angle = 47° 55′ 50″, its opposite side = 42° 8′ 24·5″, and the other side = 54° 45′ 25″.

758. CASE 2.—Given the hypotenuse and a side.

EXAMPLE.—Let the hypotenuse AC and the side BC of the triangle ABC be given equal to 70° 24′ and 65° 10′ respectively, to find the other parts.

1. To find angle C

By Art. 750 (*n*), R . cos C = cot AC . tan BC;

hence R : cot AC = tan BC : cos C.

Radius	=	10·
Cot AC 70° 24′	=	9·5515524
Tan BC 65° 10′	=	10·3346338
Cos C 39° 41′ 40″	=	9·8861862

2. To find AB

By Art. 750 (*l*), R . cos AC = cos BC . cos AB;

hence Cos BC : R = cos AC : cos AB.

Cos BC 65° 10′	=	9·6232287
Radius	=	10·
Cos AC 70° 24′	=	9·5256298
Cos AB 36° 59′ 27″	=	9·9024011

3. To find angle A

By Art. 750 (m), R . sin BC = sin AC . sin A;
hence Sin AC : R = sin BC : sin A.

Sin AC 70° 24′	=	9·9740774
Radius	=	10·
Sin BC 65° 10′	=	9·9578626
Sin A 74° 26′ 26″	=	9·9837852

EXERCISES

1. The hypotenuse is = 75° 20′, and a side is = 64° 10′; required the other parts.

The other side = 54° 28′ 32″, its opposite angle = 57° 16′ 32″, and the other angle = 68° 29′ 40″.

2. The hypotenuse AC is = 50°, and the side BC = 44° 18′ 39″; what are the other parts?

AB = 26° 3′ 53″, angle A = 65° 46′ 6″, and angle C = 35°.

759. CASE 3.—Given the two sides.

EXAMPLE.—The side AB is = 37°, and BC is = 65° 10′; find the other parts.

1. To find AC

By Art. 750 (l), R . cos AC = cos AB . cos BC;
hence R : cos AB = cos BC : cos AC.

Radius	=	10·
Cos AB 37°	=	9·9023486
Cos BC 65° 10′	=	9·6232287
Cos AC 70° 24′ 9″	=	9·5255773

2. To find angle A

By Art. 750 (r), R . sin AB = cot A . tan BC;
hence Tan BC : R = sin AB : cot A.

Tan BC 65° 10′	=	10·3346338
Radius	=	10·
Sin AB 37°	=	9·7794630
Cot A 74° 26′ 14·5″	=	9·4448292

3. To find angle C

By Art. 750 (s), R . sin BC = cot C . tan AB;
hence Tan AB : R = sin BC : cot C.

Tan AB 37°	=	9·8771144
Radius	=	10·
Sin BC 65° 10′	=	9·9578626
Cot C 39° 42′ 14″	=	10·0807482

EXERCISES

1. The two sides are $=54° 28'$ and $64° 10'$; find the other parts.
   The angles are $=57° 16' 1{\cdot}4''$ and $68° 29' 48''$, and the hypotenuse $=75° 19' 48''$.

2. The two sides are $=42° 12'$ and $54° 41' 28''$; what are the other parts?
   The angles are $=48° 0' 49''$ and $64° 33' 24''$, and the hypotenuse $=64° 38' 54''$.

760. CASE 4.—Given the two oblique angles.

EXAMPLE.—The angle C is $=106° 24'$, and angle $A=32° 30'$; required the other parts.

1. To find AC

By Art. 750 (*o*), $R \,.\, \cos AC = \cot A \,.\, \cot C$;
hence $R : \cot A = \cot C : \cos AC$.

+ Radius	=	10·
+ Cot A 32° 30′	=	10·1958127
− Cot C 106° 24′ (73° 36′) . .	=	9·4688139
− Cos AC 117° 30′ 55″	=	9·6646266

In the Tables the cosine here belongs to an arc of 62° 29′ 5″; but since the sign of cot C, one of the terms, is negative, that of the fourth term, cos AC, must also be negative (Art. 756); and hence $AC > 90$.

2. To find AB

Angle C being *M*, A and comp. AB are *O* and *o*.

By Art. 750 (*q*), $R \,.\, \cos C = \sin A \,.\, \cos AB$;
hence $\sin A : R = \cos C : \cos AB$.

+ Sin A 32° 30′	=	9·7302165
+ Radius	=	10·
− Cos C 106° 24′	=	9·4507747
− Cos AB 121° 42′ 3″	=	9·7205582

AB is also $<90°$, for angle C is so (Art. 756).

3. To find BC

By Art. 750 (*p*), $R \,.\, \cos A = \sin C \,.\, \cos BC$;
hence $\sin C : R = \cos A : \cos BC$.

+ Sin C 106° 24′	=	9·9819608
+ Radius	=	10·
+ Cos A 32° 30′	=	9·9260292
+ Cos BC 28° 27′ 31″	=	9·9440684

BC is $<90°$, for angle A is so.

EXERCISES

1. The two angles are $=39° 42'$ and $74° 26'$; find the other parts.
   The sides are $=36° 59' 39''$ and $65° 9' 28''$, and the hypotenuse $=70° 23' 39''$.

2. The angles A and C are respectively $=138° 15' 45''$ and $105° 52' 39''$; what are the other parts?
   The sides AB and BC are $=114° 15' 54{\cdot}2''$ and $140° 52' 39{\cdot}6''$, and $AC=71° 24' 30{\cdot}3''$.

761. CASE 5.—Given a side about the right angle and its adjacent angle.

EXAMPLE.—The side BC is $=140° 53'$, and angle C is $=105° 53'$; find the other parts.

1. To find AC

By Art. 750 (*u*), R . cos C = cot AC . tan BC;
hence Tan BC : R = cos C : cot AC.

− Tan BC 140° 53′ . . .	=	9·9101766
+ Radius	=	10·
− Cos C 105° 53′	=	9·4372422
+ Cot AC 71° 23′ 55·3″ . .	=	9·5270656

Angle A is of the same species with BC, and hence A and C are of the same species; therefore (Art. 752) $AC<90$.

2. To find AB

By Art. 750 (*s*), R . sin BC = cot C . tan AB;
hence Cot C : R = sin BC : tan AB.

− Cot C 105° 53′	=	9·4541479
+ Radius	=	10·
+ Sin BC 140° 53′ . . .	=	9·7999616
− Tan AB 114° 16′ 33″ . . .	=	10·3458137

The side AB and angle C are of the same species (Art. 752).

3. To find angle A

By Art. 750 (*p*), R . cos A = sin C . cos BC;
hence R : sin C = cos BC : cos A.

+ Radius	=	10·
+ Sin C 105° 53′	=	9·9830942
− Cos BC 140° 53′ . . .	=	9·8897850
− Cos A 138° 15′ 57″ . . .	=	9·8728792

Angle A is of the same species as BC.

Exercises

1. A side and its adjacent angle are respectively = 119° 11′ and 126° 54′; find the other parts.

The hypotenuse = 71° 27′ 43″, the other side = 130 41′ 42″, and the other angle = 112° 57′ 0·7″.

2. The side AB is = 54° 28′ 10″, and angle A = 68° 29′ 48″; what are the other parts?

AC = 75° 19′ 54·3″, BC = 64° 10′ 3·2″, and C = 57° 16′ 10·3″.

762. Case 6.—When a side about the right angle and its opposite angle are given.

Example.—Given AB = 40° 25′, and angle C = 44° 56′, to find the other parts.

1. To find AC

By Art. 750 (n), R . sin AB = sin AC . sin C;
hence Sin C : R = sin AB : sin AC.

Sin C 44° 56′	=	9·8489791
Radius	=	10·
Sin AB 40° 25′	=	9·8118038
Sin AC 66° 37′ 48″	=	9·9628247

Or (Art. 752), AC is also 180° − 66° 37′ 48″ = 113° 22′ 12″.

2. To find angle A

By Art. 750 (q), R . cos C = sin A . cos AB;
hence Cos AB : R = cos C : sin A.

Cos AB 40° 25′	=	9·8815842
Radius	=	10·
Cos C 44° 56′	=	9·8499897
Sin A 68° 24′ 30″	=	9·9684055

Or, A is also 180° − 68° 24′ 30″ = 111° 35′ 30″.

3. To find BC

By Art. 750 (s), R . sin BC = cot C . tan AB;
hence R : cot C = tan AB : sin BC.

Radius	=	10·
Cot C 44° 56′	=	10·0010107
Tan AB 40° 25′	=	9·9302195
Sin BC 58° 36′ 0·6″	=	9·9312302

Or, BC is also 180° − 58° 36′ 0·6″ = 121° 23′ 59·4″.

As either of the triangles ABC, A′B′C (fig. to Art. 730), fulfils the conditions given in this case, it is hence called the **ambiguous** case. In practical applications of this subject it is generally easily known which of the two triangles is to be taken. If, for example, it were known that the side BC or the angle A is less than 90°, the triangle ABC alone would satisfy the given conditions, and the triangle A′B′C would be excluded.

Exercises

1. Given AB or A′B′ (fig. to Art. 730) = 26° 4′, and the opposite angle C = 35°, to find the other parts.

   AC = 50° 0′ 18″, or AC′ = 129° 59′ 42″; angle A = 65° 46′ 13″, or angle A′ = 114° 13′ 47″; and BC = 44° 18′ 57″, or B′C = 135° 41′ 3″.

2. Given (fig. to Art. 730) AH or A′H′ 115°, and angle C = 114° 14′, to find the other parts.

   AC = 83° 39′ 43″, or A′C = 96° 20′ 17″; angle A = 103° 46′ 50″, or A′ = 76° 13′ 10″; also, CH = 105° 8′ 33″, or CH′ = 74° 51′ 27″.

3. Proof triangle in which all the parts are given, and in which, if any two of the five parts be taken as the given parts, the other three will be found by the previous rules; it will therefore afford an exercise in each of the ten cases of right-angled trigonometry; the right angle is A.

Elements	Log Sine	Log. Cosine	Log. Tangent
a = 71° 24′ 30″	9·9767235	9·5035475 +	10·4731759 +
b = 140 52 40	9·8000134	9·8897507 −	9·9102626 −
c = 114 15 54	9·9598303	9·6137969 −	10·3460333 −
B = 138 15 45	9·8232909	9·8728568 −	9·9504341 −
C = 105 52 39	9·9831068	9·4370867 −	10·5460201 −

QUADRANTAL SPHERICAL TRIANGLES

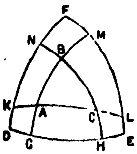
763. If the supplements of the sides and angles of a quadrantal triangle be taken, they will be the angles and sides respectively of a right-angled spherical triangle; the supplement of the quadrantal side being the right angle.

This is evident from the properties of the **polar** triangle, which are the following:—

764. If the angular points of a spherical triangle are made the poles of three great circles, another spherical triangle will be

formed, such that the sides of each triangle are the supplements of the angles opposite to them in the other triangle.

Thus, if the angular points A, B, C of the triangle ABC are respectively the poles of the sides EF, DE, DF opposite to them in the triangle DEF, then EF is the supplement of angle A, DE of B, DF of C, BC of D, AB of E, and AC of F.



Hence, if ABC were a quadrantal triangle, AB being the quadrantal side, then DEF would be a right-angled triangle, E being the right angle.

Quadrantal triangles can therefore be solved by the rules for right-angled triangles. It will also be sufficient to know two parts of such a triangle besides its quadrantal side; for then two parts of the supplemental right-angled triangle are known; and if its other parts are then calculated, the supplements of its sides and angles will be respectively the angles and sides of the given triangle.

Exercise

A side and the angle opposite to the quadrantal side are $=136° \ 8'$ and $61° \ 37'$; find the other parts.

The other side $=65° \ 26' \ 35''$, and the other two angles $=53° \ 9'$ and $142° \ 26' \ 2''$.

OBLIQUE-ANGLED SPHERICAL TRIGONOMETRY

765. The number of cases in oblique-angled trigonometry formed in reference to the given parts is six, as in the former section.

These cases, except when the three sides or three angles are given, can be solved by the method used in the preceding section, as explained afterwards under the next head. The solutions may, however, frequently be more conveniently effected by means of other methods, which are here employed for that purpose.

The rules used in the first four cases to determine the species of the part sought are:—

766. The half of a side or angle of a spherical triangle is less than a quadrant.

For a side or an angle of a spherical triangle is less than two right angles.

Other two rules are given under the fifth and sixth cases, to be employed in their solution.

The rule in Art. 759 is also applicable to oblique-angled spherical triangles.

767. Half the difference of any two parts of a triangle is less than a quadrant.

For each part is less than 180°.

768. It is to be observed in forming examples in spherical trigonometry that the sum of the three sides of a spherical triangle is less than the circumference of a circle, and the sum of any two sides is greater than the third; also, the greater angle is opposite to the greater side, and conversely.

769. CASE I.—When the three sides are given.

This case can be conveniently solved by any of the three following rules:—

RULE I.—From half the sum of the three sides subtract the side opposite to the required angle; then add together the logarithmic sines of the half-sum and of this difference and the logarithmic cosecants of the other two sides; and half the sum, diminished by 10 in the index, will be the logarithmic cosine of half the required angle.

Let the three sides be denoted by a, b, c; the angles respectively opposite to them being A, B, and C; and half the sum of the sides by s;

then (Art. 753, i) $\cos \frac{1}{2}A = \left(\frac{\sin s \, . \, \sin (s-a)}{\sin b \, . \, \sin c}\right)^{\frac{1}{2}}$.

And for B and C the formulæ are exactly analogous; that for B, for instance, being formed from the above by changing A into B, a into b, and b into a.

RULE II.—From half the sum of the three sides subtract separately the sides containing the angle; add together the logarithmic sines of the two remainders and the logarithmic cosecants of these two sides; and half the sum, diminished by 10 in the index, will be the logarithmic sine of half the required angle.

If A is the required angle,

then (Art. 753, j) $\sin \frac{1}{2}A = \left(\frac{\sin (s-b) \sin (s-c)}{\sin b \, . \, \sin c}\right)^{\frac{1}{2}}$.

RULE III.—From half the sum of the three sides subtract the side opposite to the given angle, and also each of the sides containing it; then add together the logarithmic cosecant of the half-sum and the logarithmic sines of the three remainders; and half the sum will be a constant, which, being diminished by the sine of

the half-sum, minus the side opposite the angle sought, will be the logarithmic tangent of half the required angle.

Let A be required, then (Art. 753, k)

$$\tan \tfrac{1}{2}A = \frac{1}{\sin (s-a)}\left(\operatorname{cosec} s \,.\, \sin (s-a) \,.\, \sin (s-b) \,.\, \sin (s-c)\right)^{\frac{1}{2}}.$$

770. The third rule is given in a new form, and is both more accurate in particular cases and more easy in practice than either of the other two when all the three angles are sought.

EXAMPLE.—The sides of a spherical triangle are $=143° 46'$, $67° 24'$, and $132° 11'$; find the angles (see fig. in Art. 731).

Let $a = 143° 46'$
$b = 67\ 24$
$c = 132\ 11$
$2)\overline{343\ 21}$
then $s = 171\ 40\ 30''$
$s-a = 27\ 54\ 30$
$s-b = 104\ 16\ 30$
$s-c = 39\ 29\ 30$

By Rule III

L cosec s .	=	10·8392676
L sin $(s-a)$.	=	9·6703000
L sin $(s-b)$.	=	9·9863791
L sin $(s-c)$.	=	9·8034339
		2)40·2993806
		20·1496903 = C
C − L sin $(s-a)$ = L tan $\frac{1}{2}$A	=	10·4793903
C − L sin $(s-b)$ = L tan $\frac{1}{2}$B	=	10·1633112
− L sin $(s-c)$ = L tan $\frac{1}{2}$C	=	10·3462564

Hence angle $A = 143° 18' 34''$
$B = 111\ \ 3\ 18$
and $C = 131\ 29\ 32$

EXERCISES

1. Find the three angles of the spherical triangle whose three sides are $a = 33° 4'$, $b = 74° 16'$, and $c = 94° 18'$.

$A = 26° 34' 54{\cdot}6''$, $B = 52° 7' 47{\cdot}6''$, and $C = 125° 7' 57{\cdot}2''$.

2. The sides of a spherical triangle are $=62° 54' 4''$, $125° 20'$, and $131° 30'$; what are its angles?

$=83° 12' 10''$, $114° 30'$, and $123° 20' 32''$.

771. CASE 2.—When the three angles of a spherical triangle are given.

RULE I.—From half the sum of the three angles subtract the angle opposite to the required side; then add together the logarithmic cosines of the half-sum and of this remainder and the logarithmic cosecants of the other two angles; and half the sum, diminished by 10 in the index, will be the logarithmic sine of half the required side.

Let a be the required side, and S half the sum of the angles;

then (Art. 754) $\sin \frac{1}{2}a = \left(\frac{-\cos S \cdot \cos (S-a)}{\sin B \cdot \sin C} \right)^{\frac{1}{2}}$.

RULE II.—From half the sum of the three angles subtract separately the angles adjacent to the required side; then add together the logarithmic cosines of the two remainders and the logarithmic cosecants of the other two angles; and half the sum, diminished by 10 in the index, will be the logarithmic cosine of half the required side.

Let a be the required side;

then (Art. 754) $\cos \frac{1}{2}a = \left(\frac{\cos (S-B) \cos (S-C)}{\sin B \cdot \sin C} \right)^{\frac{1}{2}}$.

RULE III.—From half the sum of the three angles subtract each angle separately; then add together the logarithmic secant of the half-sum and the logarithmic cosines of the three remainders; and half the sum will be a constant, from which, if the logarithmic cosine of the half-sum, diminished by any angle, be subtracted, the remainder will be the logarithmic cotangent of half the side opposite to that side. (See Art. 754.)

EXAMPLE

The angles of a spherical triangle are = 114° 30′, 83° 12′, and 123° 20′; find the sides.

To find the side a by Rule III

Here
$$\begin{aligned}
A &= 114° \ 30' \\
B &= \ \ 83 \ \ 12 \\
C &= 123 \ \ 20 \\
&2)\overline{321 \ \ \ 2}
\end{aligned}$$

S	= 160 31	L sec = 10·0256087
S − A	= 46 1	L cos = 9·8416404
S − B	= 77 19	L cos = 9·3415580
S − C	= 37 11	L cos = 9·9012980
		2)39·1101051
		C = 19·5550525

Log. cot $\frac{1}{2}a$ = C − L cos (S − A) = 9·7134121; ∴ $\frac{1}{2}a$ = 62° 39′ 55″
2

∴ a = 125 19 50

In the same manner, the other sides may be found to be b = 62° 54′ 16″, and c = 131° 23′ 32″.

EXERCISES

1. The three angles are $=111^\circ\ 4'$, $143^\circ\ 18'$, and $31^\circ\ 30'$; find the sides. The sides are $=67^\circ\ 25'\ 35''$, $143^\circ\ 44'\ 46''$, and $132^\circ\ 10'\ 26''$.

2. The three angles A, B, C of a spherical triangle are respectively $=70^\circ\ 39'$, $48^\circ\ 36'$, and $119^\circ\ 15'$; what are the sides?

The side $a=89^\circ\ 16'\ 53{\cdot}4''$, $b=52^\circ\ 39'\ 4{\cdot}5''$, and $c=112^\circ\ 22'\ 58{\cdot}6''$.

772. The two following cases can be solved by means of the analogies of the circular parts,* which are expressed in the following manner :—

Let one of the six parts of a triangle be omitted, and let the part opposite to it, or its supplement when it happens to be an angle, be called the **middle** part (M); the two parts next it, the **adjacent** parts (A, a); and the two remaining parts, the **opposite** parts (O, o); then

$$\sin \tfrac{1}{2}(A+a) : \sin \tfrac{1}{2}(A-a) = \tan \tfrac{1}{2}M : \tan \tfrac{1}{2}(O-o),$$

and $$\cos \tfrac{1}{2}(A+a) : \cos \tfrac{1}{2}(A-a) = \tan \tfrac{1}{2}M : \tan \tfrac{1}{2}(O+o).$$

By means of these two analogies, half the sum and half the difference of O and o are found, and each of them is then easily found.

773. When A, a, O, and o are given, M can be found from the first of these analogies by placing it for the last term, and $\sin \frac{1}{2}(A-a)$ for the first, and the other two indifferently for the second and third; thus,

$$\sin \tfrac{1}{2}(A-a) : \sin \tfrac{1}{2}(A+a) = \tan \tfrac{1}{2}(O-o) : \tan \tfrac{1}{2}M.$$

Or, M can be similarly found from the second analogy.

774. CASE 3.—When two sides and the contained angle are given, as a, b, and C.

Omit the side c, and make the supplement (Art. 772) of C the middle part M; then the sides a, b are the adjacent parts A, a; and the angles A, B the opposite parts O, o; hence (Art. 755, v)

$$\sin \tfrac{1}{2}(a+b) : \sin \tfrac{1}{2}(a\sim b) = \cot \tfrac{1}{2}\mathrm{C} : \tan \tfrac{1}{2}(\mathrm{A}\sim \mathrm{B}),$$

$$\cos \tfrac{1}{2}(a+b) : \cos \tfrac{1}{2}(a\sim b) = \cot \tfrac{1}{2}\mathrm{C} : \tan \tfrac{1}{2}(\mathrm{A}+\mathrm{B}).$$

The half-sum and half-difference of A and B being found by these two analogies, each of them is then easily found.

To find the side c

Reject C, and make c the middle part; then c is M; angles A, B are adjacent parts; and the sides a, b are opposite parts; hence (Art. 773),

$$\sin \tfrac{1}{2}(\mathrm{A}\sim \mathrm{B}) : \sin \tfrac{1}{2}(\mathrm{A}+\mathrm{B}) = \tan \tfrac{1}{2}(a\sim b) : \tan \tfrac{1}{2}c.$$

* These are called Napier's Analogies, as they were discovered by him.

EXAMPLE.—In a spherical triangle two sides are $=84° 14' 29''$ and $44° 13' 45''$, and the contained angle $=36° 45' 28''$; required the remaining parts.

Here
$$a=84° 14' 29''$$
$$b=44\ 13\ 45$$
$$C=36\ 45\ 28$$
$$\tfrac{1}{2}(a+b)=64° 14' 7'',\ \tfrac{1}{2}(a-b)=20° 0' 22'',$$
and
$$\tfrac{1}{2}C=18° 22' 44''.$$

1. To find $\frac{1}{2}(A+B)$

Sec $\frac{1}{2}(a+b)$	=	10·3618336
Cos $\frac{1}{2}(a-b)$	=	9·9729690
Cot $\frac{1}{2}C$	=	10·4785395
Tan $\frac{1}{2}(A+B)$	=	10·8133421

Hence $\frac{1}{2}(A+B)=81° 15' 44{\cdot}4''$.

2. To find $\frac{1}{2}(A \sim B)$

Cosec $\frac{1}{2}(a+b)$	=	10·0454745
Sin $\frac{1}{2}(a-b)$	=	9·5341789
Cot $\frac{1}{2}C$	=	10·4785395
Tan $\frac{1}{2}(A \sim B)$	=	10·0581929

Hence $\frac{1}{2}(A \sim B)=48° 49' 38''$.

Since $a>b$, therefore $A>B$; hence
$$A=81° 15' 44{\cdot}4''+48° 49' 38''=130° 5' 22{\cdot}4'',$$
and
$$B=81° 15' 44{\cdot}4''-48° 49' 38''=32° 26' 6{\cdot}4''.$$

3. To find the side c

Cosec $\frac{1}{2}(A \sim B)$ 48° 49′ 38″ . .	=	10·1233621
Sin $\frac{1}{2}(A+B)$ 81° 15′ 44·4″, . .	=	9·9949302
Tan $\frac{1}{2}(a \sim b)$ 20° 0′ 22″ . .	=	9·5612100
Tan $\frac{1}{2}c$ 25° 33′ 5·8″ . . .	=	9·6795023

And $c=51° 6' 11{\cdot}6''$.

EXERCISES

1. Given two sides $=89° 17'$, $52° 39'$, and the contained angle $=119° 15'$, to find the other parts.

   The other side is $=112° 23' 2''$, and the other angles $=70° 39' 3''$ and $48° 35' 58{\cdot}5''$.

2. The sides a and b are $=109° 21'$ and $60° 45'$, and angle C is $=127° 20' 55{\cdot}5''$; find the other parts.

   Angles A and B are $=90° 43' 6{\cdot}6''$ and $67° 37' 1{\cdot}4''$, and the side c is $=131° 24'$.

775. CASE 4.—When two angles and the interjacent side are given.

Let the angles A and B and the interjacent side c be given.

1. To find the sides a and b

Omit C, and let c be the middle part; then A and B are the adjacent parts, and a and b the opposite parts; hence

$$\sin \tfrac{1}{2}(A+B) : \sin \tfrac{1}{2}(A \sim B) = \tan \tfrac{1}{2}c : \tan \tfrac{1}{2}(a \sim b),$$
$$\cos \tfrac{1}{2}(A+B) : \cos \tfrac{1}{2}(A \sim B) = \tan \tfrac{1}{2}c : \tan \tfrac{1}{2}(a+b).$$

2. To find angle C

Omit c, and make the supplement of C the middle part; then the sides a, b are adjacent parts; and the angles A, B are opposite parts; hence (Art. 772), since $\tan \tfrac{1}{2}M = \cot \tfrac{1}{2}C$,

$$\sin \tfrac{1}{2}(a \sim b) : \sin \tfrac{1}{2}(a+b) = \tan \tfrac{1}{2}(A \sim B) : \cot \tfrac{1}{2}C.$$

EXAMPLE.—The angles A and B are $=130° \, 5' \, 22{\cdot}4''$ and $32° \, 26' \, 6{\cdot}4''$, and the side c is $=51° \, 6' \, 11{\cdot}6''$; required the other parts.

1. To find a and b

$$\tfrac{1}{2}(A+B) = 81° \, 15' \, 44{\cdot}4'',$$
$$\tfrac{1}{2}(A \sim B) = 48° \, 49' \, 38'', \text{ and } \tfrac{1}{2}c = 25° \, 33' \, 5{\cdot}8''.$$

Sin $\tfrac{1}{2}(A+B)$.	9·9949302	Cos $\tfrac{1}{2}(A+B)$.	9·1815881
Sin $\tfrac{1}{2}(A \sim B)$.	9·8766379	Cos $\tfrac{1}{2}(A \sim B)$.	9·8184449
Tan $\tfrac{1}{2}c$ . .	9·6795022	Tan $\tfrac{1}{2}c$ . .	9·6795022
	19·5561401		19·4979471
Tan $\tfrac{1}{2}(a \sim b)$.	9·5612099	Tan $\tfrac{1}{2}(a+b)$.	10·3163590
And $\tfrac{1}{2}(a \sim b) = 20° \, 0' \, 22''$.		And $\tfrac{1}{2}(a+b) = 64° \, 14' \, 7''$.	

And since A > B, therefore $a > b$; whence

$$a = 84° \, 14' \, 29'', \; b = 44° \, 13' \, 45''.$$

2. To find angle C

Cosec	$\tfrac{1}{2}(a \sim b)$ 20° 0′ 22″ . .	=	10·4658211
Sin	$\tfrac{1}{2}(a + b)$ 64° 14′ 7″ . .	=	9·9545255
Tan	$\tfrac{1}{2}(A \sim B)$ 48° 49′ 38″ . .	=	10·0581929
Cot	$\tfrac{1}{2}C$ 18° 22′ 44″ . . .	=	10·4785395

And $C = 36° \, 45' \, 28''$.

EXERCISES

1. The angles A and B are $=82° \, 27'$ and $57° \, 30'$, and side c is $=126° \, 37'$; what are the other parts?

The angle C is $=124° \, 42'$, and the sides a and $b = 104° \, 34' \, 28''$ and $55° \, 25' \, 32''$.

2. Given $A=66° \ 57' \ 3·6''$, $B=97° \ 20' \ 31·6''$, and the side $c=41° \ 9' \ 46''$, to find the third angle and other two sides.

$C=42° \ 30' \ 55''$, $a=63° \ 39' \ 58''$, and $b=75° \ 0' \ 51·6''$.

776. CASE 5.—When two sides and the angle opposite to one of them are given.

Let a, b, and A be given; then B can be found by the analogy,

$$\sin a : \sin b = \sin A : \sin B.$$

When B is found, there are then two sides and their opposite angles known; and hence c and C can be found as in the third and fourth cases; thus—

$$\sin \tfrac{1}{2}(A \sim B) : \sin \tfrac{1}{2}(A+B) = \tan \tfrac{1}{2}(a \sim b) : \tan \tfrac{1}{2}c,$$

$$\sin \tfrac{1}{2}(a \sim b) : \sin \tfrac{1}{2}(a+b) = \tan \tfrac{1}{2}(A \sim B) : \cot \tfrac{1}{2}C.$$

There will, however, be sometimes two values of B, as in the analogous case of plane trigonometry, and consequently two triangles can be formed from the same data; hence this is an *ambiguous* case, as is also the next case, for a similar reason (see fig. to Art. 735).

When B has two values, so have c and C. The values of B are supplementary; and by using first one of its values, the corresponding values of c and C will be found by the last two analogies, and then all the parts of one of the triangles are known. When the other value of B is taken, and the corresponding values of c and C are computed in the same manner, all the parts of the other triangle will then be known.

Whenever a differs from 90° in excess or defect less than b does, there will be only *one* triangle, and therefore only *one* value of B, which will be of the *same* species as b; in other cases B has two values that are supplementary.

When the difference of a from 90° is less than that of b, then it is evident that $\sin a > \sin b$; that is, if $(\frac{1}{2}\pi \sim a) < (\frac{1}{2}\pi \sim b)$, then $\sin a > \sin b$.

EXAMPLE.—The sides a, b are $=38° \ 30'$ and $40°$, and angle $A=30° \ 28'$; required the other parts.

To find angle B

$$\text{Sin } a : \sin b = \sin A : \sin B.$$

Cosec	a 38° 30′	=	10·2058504
Sin	b 40°	=	9·8080675
Sin	A 30° 28′	=	9·7050397
Sin	B 31° 34′ 14″,	=	9·7189576
Or,	B = 148° 25′ 46″.		

B has two values, for $(\frac{1}{2}\pi \sim a) > (\frac{1}{2}\pi \sim b)$, since

$$90° - 38° \ 30' = 51° \ 30', \text{ and } 90° - 40 = 50°, \text{ or } \sin a < \sin b.$$

Taking the triangle that has B acute, then $B = 31° \ 34' \ 14''$. There are now known a, b, A, and B, to find c and C, which are calculated exactly as in the third and fourth cases; and when this is done all the parts of this triangle are known.

Taking next the triangle that has B obtuse, then $B = 148° \ 25' \ 46''$; hence in this triangle are known a, b, A, and B; and consequently c and C in it are found also as in the preceding triangle.

It will be found in the triangle in which B is acute that $C = 130° \ 3' \ 50''$, and $c = 70° \ 0' \ 29''$.

Exercises

1. Given $a = 24° \ 4'$, $b = 30°$, and $A = 36° \ 8'$, to find the other parts.
   $B = 46° \ 18' \ 6''$, or $133° \ 41' \ 54''$; $C = 103° \ 59' \ 50''$, or $11° \ 23' \ 33''$; and $c = 42° \ 8' \ 49''$, or $7° \ 51' \ 5{\cdot}4''$.

2. Given $a = 76° \ 35' \ 36''$, $b = 50° \ 10' \ 30''$, and $A = 121° \ 36' \ 19{\cdot}8''$, to find the other parts.
   $B = 42° \ 15' \ 13{\cdot}5''$, $C = 34° \ 15' \ 2{\cdot}8''$, and $c = 40° \ 0' \ 10''$.

777. Case 6.—When two angles and a side opposite to one of them are given.

Let A, B, and a be given; then b will be found by the analogy,

$$\sin A : \sin B = \sin a : \sin b.$$

When b is found, there are then two sides and their opposite angles known; and hence c and C can be found as in the preceding case; thus—

$$\sin \tfrac{1}{2}(A \sim B) : \sin \tfrac{1}{2}(A + B) = \tan \tfrac{1}{2}(a \sim b) : \tan \tfrac{1}{2}c,$$

$$\sin \tfrac{1}{2}(a \sim b) : \sin \tfrac{1}{2}(a + b) = \tan \tfrac{1}{2}(A \sim B) : \cot \tfrac{1}{2}C.$$

There will sometimes be two values of b admissible, as there were of B in the preceding case, and consequently also two triangles (see fig. in Art. 736).

When b has two values, so have c and C. The values of b are supplementary, and by taking one of them there will then be known in one of the triangles the parts A, B, a, b; hence c and C can now be found, and all the parts of this triangle will be known. Taking then the second value of b, the remaining parts c, C of the other triangle can similarly be found.

Whenever A differs from 90° in excess or defect by less than B does, there will be only *one* triangle, and therefore only *one* value of b, which will be of the *same* species as B; in other cases b has two values that are supplementary.

When $(\frac{1}{2}\pi \sim A) < (\frac{1}{2}\pi \sim B)$, then $\sin A > \sin B$.

Example.—The angles A and B are $=31° 34'$ and $30° 28'$, and the side a is $=40°$; required the other parts.

To find the side b

$$\sin A : \sin B = \sin a : \sin b.$$

Cosec	A $31° 34'$	$=$	10·2810914
Sin	B $30° 28'$	$=$	9·7050397
Sin	a $40°$	$=$	9·8080675
Sin	b $38° 30' 18·5''$	$=$	9·7941986

The side b has only one value, for $\sin A > \sin B$.

In the triangle are now known the parts A, B, a, b, and the remaining parts c and C may be computed in the same manner as in the third and fourth cases.

Exercises

1. Given $A=51° 30'$, $B=59° 16'$, and $a=63° 50'$, to find the other parts.

$b=80° 19' 9''$, or $99° 40' 51''$; $C=131° 29' 53''$, or $155° 22' 19''$; and $c=120° 48' 5''$, or $151° 27' 3''$.

2. Given $A=97° 20' 31·6''$, $B=66° 57' 3·6''$, and $a=75° 0' 51·3''$, to find the other parts.

$C=42° 30' 54·7''$, $b=63° 39' 57·8''$, and $c=41° 9' 45·6''$.

Besides determining the species of the parts of oblique spherical triangles by means of the algebraical sines of the required parts, they can also be ascertained by certain theorems in spherical geometry.

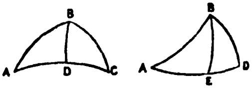
OTHER SOLUTIONS

The preceding methods of solution are generally the most convenient when all the parts of a spherical triangle are required; but when only one part is required, it will be more concise and simple to use some of the following methods:—

778. The third, fourth, fifth and sixth cases can be solved by dividing the given triangle into two right-angled triangles by means of a *perpendicular* from one of the angles upon the opposite side, so that one of the right-angled triangles shall contain two of the given parts.

By the method, however, of right-angled trigonometry alone, it would be necessary always to calculate the perpendicular; but this unnecessary calculation is avoided by eliminating the perpendicular from two equations.

779. THE THIRD CASE.—Let the given parts be a, b, and C; and let a perpendicular BD be drawn from angle B upon the side b; let θ be the segment of b that is nearest to C, reckoning from C towards A when C is acute, but from C in AC produced when C is obtuse; then $AD=b-\theta$ when C is acute, and $AD=b+\theta$ when C is obtuse.



1. To find θ and A

From the triangle BCD, by making C, or its supplement when it is obtuse, the middle part, BC and CD are the adjacent parts; therefore,

$$\cos C=\tan\theta\cot a, \text{ or } \tan\theta=\tan a\,.\,\cos C \quad . \quad . \quad [1].$$

Again, from the triangles ABD and BDC, by making AD and DC the middle parts, we have

$$\sin\theta=\cot C\tan BD, \text{ and } \sin(b\mp\theta)=\cot A\tan BD;$$

hence, by division,

$$\frac{\sin\theta}{\sin(b\mp\theta)}=\frac{\cot C\,.\,\tan BD}{\cot A\,.\,\tan BD}=\frac{\cot C}{\cot A}=\frac{\tan A}{\tan C};$$

or

$$\sin(b\mp\theta):\sin\theta=\tan C:\tan A \quad . \quad . \quad . \quad [2].$$

2. To find the side c

In the triangles ABD and CBD, we have by right-angled trigonometry,

$$\cos c=\cos(b\mp\theta)\cos BD, \text{ and } \cos a=\cos\theta\cos BD;$$

$$\therefore\ \frac{\cos c}{\cos a}=\frac{\cos(b\mp\theta)\,.\,\cos BD}{\cos\theta\,.\,\cos BD}=\frac{\cos(b\mp\theta)}{\cos\theta};$$

hence

$$\cos c=\frac{\cos a\,.\,\cos(b\mp\theta)}{\cos\theta};$$

or

$$\cos\theta:\cos(b\mp\theta)=\cos a:\cos c \quad . \quad . \quad . \quad [3].$$

The angle B can be found exactly in the same manner as A by supposing the perpendicular to be drawn from A upon the side a. The formulæ for this purpose are easily obtained from those for A by merely changing A into B, a into b, and b into a.

EXERCISE

Given $a=89°\ 17'$, $b=52°\ 39'$, and $C=119°\ 15'$, to find A, B, and c.
$A=70°\ 39'\ 3''$, $B=48°\ 35'\ 57''$, and $c=112°\ 22'\ 50''$.

780. THE FOURTH CASE.—Let the given parts be A, B, c, and let a perpendicular be drawn from B upon b; let angle ABD $=\phi$, then,

To find the side a and the angle C

From the triangle ABD, $\cos c = \cot\phi \,.\, \cot A$;

or $\cos c \,.\, \tan A = \cot\phi$; $\therefore$ R : $\cos c = \tan A : \cot\phi$. [4].

From the triangles ABD and CBD we have

$$\cos\phi = \cot c \,.\, \tan BD, \text{ and } \cos(B\sim\phi) = \cot a \,.\, \tan BD;$$

$$\therefore \frac{\cos(B\sim\phi)}{\cos\phi} = \frac{\cot a \,.\, \tan BD}{\cot c \,.\, \tan BD} = \frac{\cot a}{\cot c} = \frac{\tan c}{\tan a};$$

hence $\cos(B\sim\phi) : \cos\phi = \tan c : \tan a$ . . . [5].

Again, from the same triangles we have

$$\cos A = \sin\phi \,.\, \cos BD, \text{ and } \cos C = \sin(B\sim\phi) \,.\, \cos BD;$$

$$\therefore \frac{\cos A}{\cos C} = \frac{\sin\phi \cos BD}{\sin(B\sim\phi)\cos BD} = \frac{\sin\phi}{\sin(B\sim\phi)};$$

and hence $\sin\phi : \sin(B\sim\phi) = \cos A : \cos C$ . . . [6].

EXERCISE

Given $A = 82^\circ\ 27'$, $B = 57^\circ\ 30'$, and $c = 126^\circ\ 37'$, to find a, b, and C.

$a = 104^\circ\ 34'\ 30''$, $b = 55^\circ\ 25'\ 32''$, and $C = 124^\circ\ 42'\ 7''$.

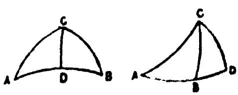
781. THE FIFTH CASE.—Let a, b, and A be given, and let a perpendicular CD be drawn from C upon c; let the segment of c next to A be denoted by θ, and the opposite angle ACD by ϕ; then,

1. To find the angle B

$$\text{Sin B} = \frac{\sin b}{\sin a} \,.\, \sin A, \text{ or } \sin a : \sin b = \sin A : \sin B.$$

When a is nearer to 90° than b, B has only *one* value, which is of the *same* species as b. When a differs more from 90° than b, then B has two supplementary values (Art. 777).

2. To find the side c



Let $AD = \theta$, then $BD = (c\sim\theta)$; and

$\cos A = \cot b \,.\, \tan\theta$, or

$\tan\theta = \tan b \,.\, \cos A$ . . [7].

Also, $\cos b = \cos\theta \cos CD$,

and $\cos a = \cos(c\sim\theta)\cos CD$;

$$\therefore \frac{\cos b}{\cos a} = \frac{\cos\theta \,.\, \cos CD}{\cos(c\sim\theta)\cos CD} = \frac{\cos\theta}{\cos(c\sim\theta)};$$

hence $\cos b : \cos a = \cos\theta : \cos(c\sim\theta)$ . . . [7].

782. The value of θ found above does not show whether the perpendicular is within or without the triangle, as c is not known; but the species of A and B determine this circumstance, for the perpendicular falls within or without the triangle according as angles A and B are of the same or of different affection.

3. To find the angle C

Let $ACD=\phi$, then $BCD=C\sim\phi$; and from the triangle ACD we have $\cos b=\cot A . \cot \phi$, or $\cot\phi=\cos b \tan A$ . . [8].

Also, from the triangles ACD and BCD,

$$\cos\phi=\cot b \tan CD, \text{ and } \cos(C\sim\phi)=\cot a \tan CD.$$

$$\therefore \frac{\cos\phi}{\cos(C\sim\phi)}=\frac{\cot b . \tan CD}{\cot a \tan CD}=\frac{\cot b}{\cot a}=\frac{\tan a}{\tan b};$$

hence $\tan a : \tan b=\cos\phi : \cos(C\sim\phi)$ . . . [9].

783. When B has two values, one of them—for instance, its acute value—should first be taken, and the side c and angle C of the triangle to which it belongs are then to be calculated; then, its other value being taken, the side c and angle C of the triangle to which it belongs are to be calculated.

Exercise

Given $a=76° 35' 30''$, $b=50° 10' 30''$, and $A=121° 36'$, to find B, C, and c. $B=42° 15' 26''$, $C=34° 15' 15''$, and $c=40° 0' 14''$.

784. The Sixth Case.—Let A, B, and a be given, and let a perpendicular be drawn, as in the preceding case; then,

1. To find the side b

$$\text{Sin } A : \sin B=\sin a : \sin b \quad . \quad . \quad . \quad [10].$$

When A is nearer to 90° than B, b has only *one* value, which is of the *same* species as B; in any other case b has two supplementary values (Art. 777).

2. To find the angle C

By last case [8], $\cot\phi=\cos b . \tan A$, and from the triangles ACD and BCD we have

$$\cos A=\sin\phi . \cos CD,$$

and

$$\cos B=\sin(C\sim\phi)\cos CD$$

$$\therefore \frac{\cos A}{\cos B}=\frac{\sin\phi . \cos CD}{\sin(C\sim\phi)\cos CD}=\frac{\sin\phi}{\sin(C\sim\phi)};$$

hence $\cos A : \cos B=\sin\phi : \sin(C\sim\phi)$ . . [11].

It is known, as in Art. 782, whether the perpendicular falls within or without the triangle.

The species of $C-\phi$ is not thus determined, as its sine is the fourth term; but those of B and a being known in triangle BCD, that of $DCB=C-\phi$ is known, if the formula (Art. 750, o) $\cos a = \cot B . \cot (C \sim \phi)$ be used; or, $R : \cos a = \tan B : \cot (C-\phi)$. [12].

3. To find the side c

By Art. 781, $\tan \theta = \tan b . \cos A$, and from the triangles ACD and BCD we have

$$\sin \theta = \cot A . \tan CD, \text{ and } \sin (c \sim \theta) = \cot B \tan CD ;$$

$$\therefore \frac{\sin \theta}{\sin (c \sim \theta)} = \frac{\cot A . \tan CD}{\cot B . \tan CD} = \frac{\cot A}{\cot B} = \frac{\tan B}{\tan A};$$

hence $\tan B : \tan A = \sin \theta : \sin (c \sim \theta)$ . . [13].

The species of $c-\theta$ is not determined, as its sine is the last term; but in triangle BCD, the species of $(c-\theta)$ is known from those of a and B.

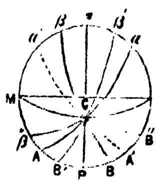
785. When b has two values, one of them—for instance, its acute value—can first be taken, and the angle C and side c of the triangle to which it belongs are then to be calculated; its other value being next taken, the angle C and side c of the triangle to which it belongs are to be computed.

EXERCISE

Given $A = 97° 20' 30''$, $B = 66° 57'$, and $a = 75° 1'$, to find C, b, and c.
$C = 42° 31' 16''$, $b = 63° 39' 59''$, and $c = 41° 10' 8''$.

786. All the forms of a triangle that can exist when two sides and an angle opposite to one of them are given are contained in the following diagram:—

Let MM′, Pπ be two perpendicular diameters of the circle MPM′, which is the base of a hemisphere; and let C be any point in its surface, and the arcs passing through C be all the halves of great circles, except CB″, Cβ″, which are portions of great circles. Let all these be symmetrically situated in the semicircles PMπ, PM′π, so that every two corresponding arcs —as, for instance, AC, A′C—are equally inclined to PCπ.



Hence every two corresponding arcs, reckoning from C, as CB and CB′ or Cβ and Cβ′, are equal. Also, of all these arcs, Cπ is the greatest, CP the least; and any arc, as Ca, nearer to the greatest is greater than any other, as CA′—that is, more remote

(Solid Geom., p. 54). Also, the arcs CM, CM′ are quadrants; and therefore all the arcs, reckoning from C to the semicircle MPM′, are less than quadrants, and those terminating in the semicircle MπM′ are greater than quadrants.

Let PC$=h$, and denote the parts of the triangle ABC as usual; then

$$\text{R} \, . \sin h = \sin \text{B} \sin a,$$

and

$$\sin \text{B} = \frac{\text{R} \, . \sin h}{\sin a}.$$

It is evident that when h is constant, sin B is least when $\sin a$ is greatest—that is, when a is a quadrant and equal to CM′ or CM. Hence, of all the angles subtended by CP at the points B, A′, B″, M′, α,...that at M′ is the least, and that at a point nearer to M′, either in the quadrant PM′ or πM′, is less than one more remote; and these angles, therefore, are all less than right angles, for $h<a$; and their adjacent angles are greater than right angles. Also, when two arcs CA′ and Cα are supplementary, the acute angles at A′ and α are equal.

When each of the sides a and b is less than a quadrant, and angle A is acute, and $\sin a > \sin b$, only one triangle can be formed, and the unknown angle B is of the same affection as b. This appears from the triangle ACB″, where B″C$=a$, AC$=b$, and angle CAP$=$A; for B″C$>$AC, and $<\alpha$C.

When each of the sides a and b is greater than a quadrant, and $\sin a > \sin b$, and A is obtuse, there can be only one triangle, and angle B will be of the same affection as b. This appears from the triangle αCβ'', where β''C$=a$, αC$=b$, and angle C$\alpha\pi=$A; where β''C is intermediate between αC and its supplement AC, and therefore $\sin a > \sin b$.

When $\sin a < \sin b$ there will be two triangles. This appears when A is acute from the triangles ACB, ACB′, in which CB $=$CB′; and when A is obtuse, from the triangles αCβ, αCβ'. And in this case the two values of B are supplementary.

By examining all the possible cases, it will be found that they are comprehended in the rules for the signs of the trigonometrical ratios (Art. 195).

Let A, B, and a be given. When each of the given parts is less than a quadrant, and sin A$>$sin B, there can be only one triangle, and the unknown side b will be of the same species as B. This appears from triangle ACB″, where B″C$=a$. Were sin A$<$sin B, there could then be two triangles—as BCA, BCα'. And by examining in the same way all the possible cases, the theorem stated in Art. 776 is easily established.

ASTRONOMICAL PROBLEMS

CIRCLES AND OTHER PARTS OF THE CELESTIAL SPHERE

787. To an observer placed on the surface of the earth the heavenly bodies appear to be situated on the surface of a concave sphere, of which the place of the observer is the centre; for the magnitude of the earth is a mere point in reference to the distance of all celestial bodies, except those belonging to the solar system, and it becomes sensible in regard to the distances of the latter only when accurate observations are taken with proper instruments.

The apparent diurnal revolution of these bodies from east to west, caused by the real daily rotation of the earth on its axis in the opposite direction from west to east, and the apparent annual motion of the sun in the heavens, arising from the earth's annual revolution in its orbit in an opposite direction, are, for convenience, in the practice of astronomy and navigation, considered as real motions; and the positions of these bodies are determined accordingly, for any given time, with the aid of the principles of spherical trigonometry.

DEFINITIONS

788. The **celestial sphere** is the apparent concave sphere on the surface of which the heavenly bodies appear to be situated.

789. The **axis of the celestial sphere** is a straight line passing through the earth's centre, terminated at both extremities by the celestial sphere. About this axis the heavenly bodies appear to revolve.

790. The **poles of the celestial sphere** are the extremities of its axis, one of them being called the **north**, the other the **south pole.**

The poles appear as fixed points in the heavens, without any diurnal rotation, the bodies near them appearing to revolve round them as centres.

791. The **equinoctial** or **celestial equator** is a great circle in the heavens equidistant from the poles; and it divides the celestial sphere into the northern and southern hemispheres.*

This circle referred to the earth is the equator; also the axis of the earth is a portion of that of the celestial sphere.

792. The **ecliptic** is a great circle that intersects the equator obliquely, and is that in which the sun appears to perform its annual motion round the earth.

793. The two points in which the ecliptic and equinoctial intersect are called the **equinoxes**, or **equinoctial points**; that at which the sun crosses the equator towards the north is called the **vernal** equinox, and the other the **autumnal** equinox.

794. The **zodiac** is a zone extending about 8° on each side of the ecliptic; and it is divided into twelve equal parts, called the **signs of the zodiac.**

The names and characters of these signs are:—

♈ Aries,	♋ Cancer,	♎ Libra,	♑ Capricornus,
♉ Taurus,	♌ Leo,	♏ Scorpio,	♒ Aquarius,
♊ Gemini,	♍ Virgo,	♐ Sagittarius,	♓ Pisces.

The first six lie on the north of the equinoctial, and are called **northern** signs; the other six, on the south of that circle, are called **southern** signs. Each sign contains 30°.

The signs are reckoned from west to east according to the apparent annual motion of the sun. The first, Aries, is near the vernal equinox; and Libra, near the autumnal equinox.†

795. The **solstitial points** are the middle points of the northern and southern halves of the ecliptic; the northern is called the **summer**, and the southern the **winter, solstice.**‡

796. The **horizon** is the name of three circles: one, the **true** or **rational**; another, the **sensible**; and a third, the **visible** or **apparent** horizon.

The first is the intersection of a horizontal plane passing through

* This circle is called the equinoctial because when the sun is in it the nights, and consequently the days, are equal everywhere on the earth's surface.

† About three thousand years ago the western part of Aries nearly coincided with the vernal equinox; but from the slow westerly recession of this point, called the precession of the equinoxes, it is nearly a whole sign to the west of Aries.

‡ These points are so named because when the sun (*sol*) has arrived at either of them it appears to *stop* (*sto*), in reference to its motion north and south, and then to *return*; and hence, also, the origin of the term tropics, from a Greek word (τρόπος) which means a turn.

the earth's centre with the celestial sphere; the second is the intersection with this sphere of a plane parallel to the former touching the earth's surface at the place of the observer; and the third is the intersection with the same sphere of the conic surface, of which the vertex is at the eye of the observer, and the surface of which touches on every side the surface of the earth, considered as a sphere.

797. A vertical line passing through the earth's centre and the place of the observer may be called the **axis** of the horizon; and the extremities of this axis, where it meets the celestial sphere, the **poles** of the horizon; the upper pole being called the **zenith**, and the lower the **nadir.**

798. The north, east, south, and west points of the horizon are called the **cardinal points.**

799. **Meridians** are great circles passing through the poles of the celestial sphere; they are also called **hour** circles.

These circles correspond to meridians on the earth.

800. A meridian passing through the equinoctial points is called the **equinoctial colure**; and that passing through the solstitial points, the **solstitial colure.**

801. Circles passing through both poles of the ecliptic are called **circles of celestial longitude.**

802. **Vertical circles** are great circles passing through both the poles of the horizon.

803. A vertical circle passing through the east and west points of the horizon is called the **prime vertical.**

804. Small circles parallel to the equinoctial are called **parallels of declination.**

805. Small circles parallel to the ecliptic are called **parallels of celestial latitude.**

806. Small circles parallel to the horizon are called **parallels of altitude.**

807. The **right ascension** of a heavenly body is an arc of the equinoctial intercepted between the vernal equinox and a meridian passing through the body.

808. The **longitude** of a heavenly body is an arc of the ecliptic intercepted between the vernal equinox and a circle of longitude passing through the body.

809. The **azimuth** of a body is an arc of the horizon intercepted between the north or south point and a vertical circle passing through the body.*

810. The **amplitude** of a body is an arc of the horizon intercepted between the east or west point and a vertical circle passing through the body.†

811. The **declination** of a body is its distance from the equinoctial, measured by the arc of the meridian passing through it which is intercepted between the body and the equinoctial.

812. The **latitude** of a body is the arc of a circle of longitude intercepted between the body and the ecliptic.

813. The **altitude** of a body is the arc of a vertical circle passing through the body, intercepted between it and the true horizon.

814. The **dip** or **depression of the horizon** is the angle of depression of the visible horizon below the sensible, in consequence of the eye of the observer being situated above the surface of the earth.

815. The **observed** altitude is the altitude indicated by the instrument, the **apparent** altitude is the result after correcting the observed altitude for the error of the instrument and the dip, and the **true** altitude is the result after correcting the apparent altitude for refraction and parallax. The **meridian** altitude of a body is its altitude when on the meridian. When a body is on the meridian it is said to **culminate**; and its **culmination** is said to be **upper** or **lower** according as it is then in its highest or lowest position.

816. The **polar distance** or **codeclination** of a body is its distance from the pole of the equinoctial, measured by the arc of a meridian intercepted between the body and the pole.

817. The **zenith distance** or **coaltitude** of a body is its distance from the zenith, measured by the arc of a vertical circle intercepted between the body and the zenith.

818. The **obliquity of the ecliptic** is the inclination of the ecliptic to the equator. This inclination is nearly 23° 27½′.

819. The **horary angle** of a body at any instant is an angle at the pole of the equator, contained by the meridian passing through

* The azimuths may be named according to the quadrants in which they lie; thus, N.E. when between the north and east points, S.W. between the south and west points, and so on.

† Amplitudes may be named, also, according to the quadrants in which they lie, as the azimuths are named.

the body and the meridian of the place of observation. It measures the time between the instant of observation and the instant of the body's passage over the meridian of the observer.

820. The **rising** or **setting** of a body is the time when its centre is apparently in the horizon when rising or setting.

821. The **diurnal arc** of any body is that portion of its parallel of declination which is situated above the horizon, and its **nocturnal arc** that portion of the same parallel which is below the horizon.

The diurnal arc, reckoned at the rate of 15° to 1 hour, will express the interval of continuance of a body above the horizon, for a star in **sidereal** time, for the sun in **solar** time, for the moon in **lunar** time, and for a planet in **planetary** time (Art. 831).

822. The **precession of the equinoxes** is a small motion of the equinoxes towards the west.*

823. A **tropical year** is the time in which the sun moves from the vernal equinox to that point again.

824. A **sidereal year** is the time in which the sun moves from a fixed star to the same star again, or the time in which it performs an absolute revolution.†

825. **Apparent time** is that which depends on the position of the sun, and is also called **solar time.** This is the time shown by a sun-dial, the days of which are unequal.

826. **Mean time** is the time shown by a well-regulated clock, the days of which are equal.

827. An **apparent solar day** is the time between two successive transits of the sun's centre over the meridian, and is of variable length.‡

828. A **mean solar day** is a constant interval of time, and is the mean of all the apparent solar days in a year; or it is what an apparent solar day would be were the sun's motion in the equinoctial uniform.

* This retrograde motion is about 1° in 72 years, or 50·2″ annually; and in consequence of it the sun returns to the vernal equinox sooner than it would do were this point at rest; hence the origin of the term.

† The tropical year, in consequence of the precession of the equinoxes, is shorter than the sidereal year by the time the sun takes to move over 50·2″, or 20 m. 19·9 s. The length of the former is 365 d. 5 h. 48 m. 51·6 s., and that of the latter 365 d. 6 h. 9 m. 11·5 s.

‡ In consequence of the obliquity of the ecliptic and the sun's unequal motion in its orbit, its motion referred to the equinoctial is not uniform, and consequently the intervals between its successive transits are variable.

829. The **equation of time** is the difference between mean and apparent time.

It is just the difference between the time shown by a regulated timepiece and a sun-dial; at mean noon it is the difference between twelve o'clock mean time and the mean time of the sun's passing the meridian.

830. A **sidereal day** is the interval between two successive transits of the same star over the meridian. It is the time in which the earth performs an absolute rotation on its axis; and as this motion is uniform, the sidereal day is always of the same length.

The sidereal day **begins** when the vernal equinox—that is, the first point of Aries—arrives at the meridian; and its length is 23 h. 56 m. 4·056 s. in mean solar time, or 24 sidereal hours.

A meridian of the earth returns to the same star in a shorter interval than it does to the sun; the difference, expressed in mean time, is called the **retardation** of mean on sidereal time; and when expressed in sidereal time, it is called the **acceleration** of sidereal on mean time.*

831. Generally, the interval of time between the departure of a given meridian from a celestial body and its return to that body is called a **day** in reference to the body. If the body is a star, the interval is called a **sidereal** day; if the sun, a **solar** day; if the moon, a **lunar** day; each day consisting of 24 hours, the hours for these days being of course of different magnitudes.

The **astronomical** day begins at noon, and is reckoned till next noon; and it is thus twelve hours later than the **civil** day.

832. The refraction of the atmosphere causes the altitude of a celestial body to appear greater than it would be were there no atmosphere; the **increase** of altitude from this cause is the **refraction** of the body (see Art. 849).

When the body is in the horizon its refraction is greatest, and when in the zenith it is nothing; at other altitudes the refraction is intermediate.

* The retardation for 24 hours of mean time is = 3 m. 55·9094 s., or 24 sidereal hours = 23 h. 56 m. 4·0906 s. of mean time. The acceleration for 24 hours of sidereal time is = 3 m. 56·5554 s., or 24 hours of mean solar time are = 24 h. 3 m. 56·5554 s. of sidereal time. These equivalents are obtained from the fact that the sun's mean increase of right ascension in a mean solar day is 59′ 8·3″, or 3 m. 55·9 s. of mean time; so that a meridian of the earth moves over 360° in a sidereal day, and 360° 59′ 8·3″ in a mean solar day; the former motion, which is just the time of the earth's rotation on its axis, is performed in 24 sidereal hours, and the latter in 24 h. 3 m. 56·5554 s. of sidereal time; or the former is performed in 23 h. 56 m. 4·09 s. mean time, and the latter in 24 hours of mean time.

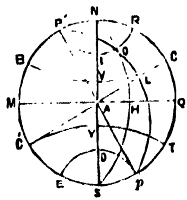
833. The **parallax** of a celestial body is the quantity by which its altitude, when seen from the surface of the earth, is **diminished,** compared with its altitude seen from the earth's centre.

The parallax, like the refraction, is greatest when the body is in the horizon, and is nothing when it is in the zenith.

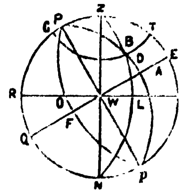
When the body is in the horizon its parallax is called **horizontal** parallax; its parallax at any altitude is called its parallax in **altitude**; and its parallax supposed to be subtended by the greatest or equatorial radius of the earth is called its **equatorial** parallax.

834. Most of these definitions will be readily understood from the two following diagrams:—

Let PQM be a meridian passing through the pole of the ecliptic; MQ the equator, N and S its north and south poles; CC′ the ecliptic, P and p its north and south poles. Then A is the first point of Aries, C that of Cancer, and C′ that of Capricornus; and angle CAQ, measured by CQ, is the obliquity of the ecliptic. The parallels of declination CB, C′T are the tropical circles, the former being the tropic of Cancer, and the latter that of Capricorn; and PR, Ep are the polar circles, the former being the arctic, and the latter the antarctic, circle. Also, if POp and NOS are respectively a circle of celestial longitude and a meridian passing through any celestial body O, then AL is its longitude, OL its latitude, AH its right ascension, and OH its declination. The meridian NQSM is the solstitial colure, and NAS the equinoctial colure.



Again, let RH be the horizon, Z and N its poles, the former being the zenith and H the south point; EQ the equator, P and p its north and south poles; also, let B be any celestial body, and ZBN a vertical circle through it; then BL is its altitude, HL its azimuth; and if O is its position when rising, OW is its amplitude. Let A be the first point of Aries, and POp a meridian through O; then the distance between A and F is the right ascension of O, the distance between A and W its oblique ascension, and WF its ascensional difference. The small circle GBT, parallel to RH, is a parallel of altitude, and ZWN is the prime vertical.



835. **Problem I.—To convert degrees of right ascension or of terrestrial longitude into the corresponding time; and conversely.**

RULE.—To convert degrees into time, multiply by 4, and consider the product of the degrees by 4 as minutes of time, the product of the minutes of space as seconds of time, and so on.

To convert time into degrees, reduce the hours to minutes, and consider the number of minutes of time as degrees, the seconds of time as minutes of space, and so on; divide by 4, and the quotient will be the required number of degrees.

EXAMPLES.—1. Convert 36° 12′ 40″ to time.

(36° 12′ 40″) × 4 = 144 m. 50 s. 40 t. = 2 h. 24 m. 50 s. 40 t.

2. Convert 2 h. 24 m. 50 s. 40 t. to degrees.

$\frac{1}{4}$(2 h. 24 m. 50 s. 40 t.) = $\frac{1}{4}$(144 m. 50 s. 40 t.) = 36° 12′ 40″.

If the sun moved uniformly it would pass over 360° of the equator or equinoctial in 24 hours of mean time—that is, 15° in 1 hour; hence, if d = the number of degrees, and h = the number of hours corresponding,

$$1 \text{ h.} : h = 15^\circ : d, \text{ and therefore}$$

$$h = \frac{d}{15} = \frac{4}{60}d, \text{ also } d = 15h = \frac{60h}{4};$$

and from these expressions the rules are easily obtained.

EXERCISES

1.	80°	32′	40″		are equivalent to	5 h.	22 m.	10 s.	40 t.
2.	161	5	20		" "	10	44	21	20
3.	98	14	48		" "	6	32	59	12
4.	5 h.	22 m.	10 s.	40 t.	are equivalent to	80°	32′	40″	
5.	0	28	6		" "	7	1	30	
6.	14	1	12		" "	210	18	0	

836. **Problem II.—To express civil time in astronomical time; and conversely.**

RULE.—When the given time is P.M., the civil time and astronomical time are the same; and when the civil time is A.M. add 12 hours to it, and the sum will be the astronomical time, reckoning from the noon of the preceding day. The rule for the converse problem is evident.

EXAMPLES.—1. April 6 at 3 h. 12 m. P.M. civil time is in astronomical time also 6th April 3 h. 12 m.

2. June 1 at 10 h. 15 m. A.M. of civil time is 31st May 22 h. 15 m. of astronomical time.

Exercises

	Civil Time					Astronomical Time		
1.	Feb. 10 d.	4 h.	20 m.	P.M.	is	Feb. 10 d.	4 h.	20 m.
2.	July 13	2	12	A.M.	"	July 12	14	12
3.	Aug. 1	11	40	A.M.	"	July 31	23	40
4.	Oct. 9	10	1	P.M.	"	Oct. 9	10	1

837. **Problem III.—To reduce the time under any given meridian to the corresponding astronomical time at that instant at Greenwich; and conversely.**

RULE.—To find the time at Greenwich corresponding to that at another place: to the given time expressed astronomically apply the longitude in time by addition when it is W., and by subtraction when it is E.

To find the time at a given place corresponding to a given time at Greenwich: to the given time expressed astronomically apply the longitude in time by addition when it is E., and by subtraction when it is W.

EXAMPLES.—1. Find the time at Greenwich corresponding to 20th June, at 9 h. 12 m. A.M., at a place in longitude $-14° 2' 30''$ W.

Given time June,	.	19 d.	21 h.	12 m.	0 s.
Longitude in time W.,	+	0	56	10	
The *reduced* time at Greenwich,	.	19	22	8	10

2. Find the time at Greenwich corresponding to 30th August, at 2 h. 40 m. 10 s. P.M., at a place in longitude $=75° 34' 45''$ E.

Given time August,	.	30 d.	2 h.	40 m.	10 s.
Longitude in time,	−		5	2	19
Astronomical time at Greenwich,	.	29	21	37	51

From these examples the converse problem is evident.

Exercises

1. Find the time at Greenwich corresponding to 18th July, at 5 h. 24 m. A.M., at a place in longitude $=40° 20'$ W.

July = 17 d. 20 h. 5 m. 20 s.

2. Find the time at Greenwich corresponding to 19th June, at 1 h. 12 m. 40 s. P.M., in longitude $=90° 37' 30''$ E.

June = 18 d. 19 h. 10 m. 10 s.

3. Find the time at a place in longitude $=40° 20'$ W. corresponding to 18th July, at 8 h. 5 m. 20 s. A.M., at Greenwich.

July = 17 d. 17 h. 24 m.

4. Find the time in longitude = 90° 37′ 30″ E. corresponding to 19th June, at 7 h. 10 m. 10 s. A.M., at Greenwich.

June = 19 d. 1 h. 12 m. 40 s.

838. Problem IV.—To reduce the registered* declination or right ascension of the sun to any given meridian and to any time of the day.

RULE.—As 24 hours is to the given time, so is the change of declination for 24 hours to its change for the given time. When the declination is increasing add this proportional part to it, and when diminishing subtract it, and the result will be the declination required.

By the same method, the sun's right ascension can be found for any time at a given place, only it is always increasing.

Or, if t = reduced time at Greenwich past the previous noon,
v' = variation of declination in 24 hours,
v = " " given time,
D' = declination at noon at Greenwich,
D = " required ;

then $24 : t = v' : v$, and $v = \frac{1}{24}v't$, $D = D' \pm v$.

Or, if P.L. denote proportional logarithms,

P.L., v = P.L., t + P.L., v'.

EXAMPLE.—Find the sun's declination in 1854, 30th August, at 2 h. 40 m. 10 s. P.M., at a place in longitude = 75° 34′ 45″ E.

Time at Greenwich (Art. 837), 29th August	= 21 h.	37 m.	51 s.	= t
Sun's declination at noon on 29th . .	= 9°	24′	1″	= D
" " " 30th . .	= 9	2	35	
Variation of declination in 24 h. . .	= 0	21	26	= v'
And 24 h. : 21 h. 37 m. 51 s. = 21′ 26″ : v, and v . .	= 0°	19′	19″	
Hence the required declination $D' - v = D$ . . .	= 9	4	42	

Or, by proportional logarithms—

	P.L., t 21 h.	37 m.	51 s.	=	4515	
	P.L., v' 0°	21′	26″	=	4912	
hence	P.L., v 0	19	19	=	9427	
and	D' = 9	24	1			
therefore	D = 9	4	42			= required declination.

By changing declination to right ascension in the preceding rule, it will be adapted to the finding of the sun's right ascension.

* These elements of the sun's place are registered in the *Nautical Almanac* for noon of every day at Greenwich.

Exercises

1. Find the sun's declination in 1854, 12th October, at noon, at a place in longitude 4° 15′ W., the declination at Greenwich at noon, 12th October, being = 7° 22′ 52″ S. and increasing, and its variation in 24 hours = 22′ 32″. . . . . . . = 7° 23′ 8″.

2. Find the sun's declination in 1854, 20th June, at 9 h. 12 m. P.M., at a place in longitude = 14° 2′ 30″ W.; having given D′ = 23° 27′ 13″ N., and v' = 20″, and the declination increasing. = 23° 27′ 21″.

3. Find the sun's declination in 1854, 29th May, at 2 h. 37 m. 20 s. A.M., in longitude = 32° 40′ W.; having given D′ = 21° 27′ 33″ N., and v' = 9′ 31″. . . . . = 21° 34′ 13″.

4. What is the sun's right ascension in 1854, 4th September, at 4 h. 45 m. 39 s. P.M., at a place in longitude = 72° 35′ 15″ W., when D′ = 10 h. 52 m. 4·65 s., and v' = 3 m. 37 s.? = 10 h. 53 m. 31·45 s.

PROPORTIONAL LOGARITHMS

839. These logarithms, which are useful in calculating small quantities, such as minutes of time or space, as they generally require to be carried out only to four decimal places, are obtained in this manner:—

Let a, b, c,...be any quantities; assume another quantity q, such that it exceeds any of the quantities a, b, c,...then Lq - La is the proportional logarithm of a, Lq - Lb that of b, and so on; also, P. Lq = Lq - Lq = o. The quantity q so assumed is 3 hours or 3 degrees, and sometimes 24 hours.

840. Problem V.—To reduce the registered declination and right ascension of the moon to any given meridian and to any time of the day.*

Rule.—Find the reduced time, and the declination for the preceding hour; then, as 10 minutes is to the time past that hour, so is the variation in 10 minutes to the variation in the past time, which being applied to the given declination by addition or subtraction, according as the declination is increasing or diminishing, will give the declination required.

The same rule applies for finding the right ascension, only 1 hour or 60 minutes must be used for 10 minutes; and as the right ascension is always increasing, the variation is always to be added.

* The moon's declination and right ascension are given for every hour in the *Nautical Almanac*, and the variation of the former for every 10 minutes.

Let D′, D = the earlier given and required declination,
R′, R = " " " right ascension,
t = time past the hour preceding the reduced time,
v' = the variation for 60 m. of right ascension, or 10 m. of declination,
v = the correction sought; then

for the declination, 10 m. : t m. $= v' : v$,
and $v = \frac{1}{10}tv'$; hence $D = D' \pm v$;
and for the right ascension, 60 m. : t m. $= v' : v$,
and $v = \frac{1}{60}tv'$; hence $R = R' + v$.

EXAMPLE.—What will be the declination and right ascension of the moon on 15th November 1841, at 0 h. 30 m. A.M., at a place in longitude = 36° 45′ W.?

Given time on 14th November	=	12 h.	30 m.	
Longitude in time	= +	2	36	
Reduced time (Art. 837)	=	15	6	and $t = 6$ m.

	Right Ascension	Declination
On 14th, at 15 h.,	R′ = 16 h. 53 m. 37·35 s.,	D′ = 26° 18′ 47·5″
" " 16 h.,	16 55 57·69	26 20 25·9
Change in 60 m.,	v' = 0 2 20·34	0 1 38·4

Diff. dec. in 10 m. 16·4″.

For declination, v' in 10 m. = 16·4″;
hence $v = \frac{1}{10}tv' = \frac{6}{10} \times 16{\cdot}4 = 9{\cdot}8''$,
and $D = D' + v = 26^\circ\ 18'\ 57{\cdot}3''$.

For the right ascension,

$$v = \frac{1}{60}tv' = \frac{6}{60} \times 140{\cdot}34 \text{ s.} = 14{\cdot}03 \text{ s.},$$

and $R = R' + v = 16$ h. 53 m. 51·38 s.

From the variation of declination 1′ 38·4″ in 60 m., its change in the past time 6 m. could be found in the same manner as that for right ascension.

EXERCISES

1. What was the moon's right ascension and declination at Greenwich on the 18th of October 1841, at 10 h. 40 m. P.M., from these data?

October 1841	Right Ascension	Declination
On 18th, at 10 h.,	R′ = 17 h. 2 m. 18·88 s.,	D′ = 26° 34′ 19·7″
" " 11 h.,	17 4 38·13	26 35 31·9
	R = 17 h. 3 m. 51·71 s., and	D = 26° 35′ 7·8″

2. Required the moon's right ascension and declination on the 27th of October 1854, at 10 h. 43 m. A.M., at a place in longitude $=40° 15'$ E., from these data :—

October 1854	Right Ascension	Declination
On 26th, at 20 h.,	R′=19 h. 4 m. 5·50 s.,	D′=26° 49′ 5·1″
" " 21 h.,	19 6 44·40	26 47 4·8

Reduced time $=26$ d. 20 h. 2 m., R $=19$ h. 4 m. 10·79 s., and D $=26° 49' 1{\cdot}09''$.

841. **Problem VI.—To reduce the registered semi-diameter or horizontal parallax of the moon to any meridian and any time of the day.***

RULE.—Find the civil time at Greenwich; then, as 12 hours is to the reduced time, so is the change in either of these elements in 12 hours to its change for the intermediate time, which is to be applied by addition or subtraction to the earlier given element according as it is increasing or decreasing.

Let s', s = the earlier given and required semi-diameter,
p', p = " " " horizontal parallax,
t = " time in hours past noon or midnight,
v' = " change in either of these elements for 12 h.,
and v = " " for the reduced time; then
for the semi-diameter,

$$12 \text{ h.} : t \text{ h.} = v' : v, \quad v = \tfrac{1}{12} t v', \text{ and } s = s' \pm v;$$

and for the horizontal parallax,

$$12 \text{ h.} : t \text{ h.} = v' : v, \quad v = \tfrac{1}{12} t v', \text{ and } p = p' \pm v.$$

EXAMPLE.—Find the semi-diameter and horizontal parallax of the moon at a place in longitude $=4° 20' 15''$ W., on 20th March 1854, at 7 h. 42 m. 39 s. P.M.; having given the registered elements for the preceding noon and midnight.

The reduced time is 20th, 8 h. P.M.

20th March 1854	Semi-diameter	Horizontal Parallax
At noon, ...	$s'=16'$ 9·3″	$p'=59'$ 10·3″
" midnight, . . .	16 10·2	59 13·5
Change in 12 h., . .	0·9	3·2
Computed change in 8 h.,	0·6	2·1
Required elements, . .	16 9·9	59 12·4

For to find s, $v = \frac{1}{12} t v' = \frac{8}{12} \times {\cdot}9'' = 0{\cdot}6''$, and $s = s' + v$;
and " p, $v = \frac{1}{12} t v' = \frac{8}{12} \times 3{\cdot}2'' = 2{\cdot}1''$, " $p = p' + v$.

* These elements are registered for every noon and midnight.

Exercises

1. Find the semi-diameter and horizontal parallax of the moon on 17th November 1854, at 10 h. 30 m. P.M., at Greenwich from these elements :—

17th Nov. 1854	Semi-diameter	Horizontal Parallax
At noon, ...	$s=15'\ 39{\cdot}2''$	$p=57'\ 19{\cdot}8''$
„ midnight, . . .	15 46	57 44·7

$s=15'\ 45{\cdot}2''$, and $p=57'\ 41{\cdot}6''$

2. Find the semi-diameter and horizontal parallax of the moon on the 10th of November 1854, at 4 h. 7 m. 10 s. P.M., at St Helena, in longitude$=5°\ 42'\ 30''$ W., from these elements :—

10th Nov. 1854	Semi-diameter	Horizontal Parallax
At noon, ...	$s'=14'\ 48{\cdot}9''$	$p'=54'\ 15{\cdot}8''$
„ midnight, . . .	14 48·3	54 13·5

Reduced time$=$4 h. 30 m., $s=14'\ 48{\cdot}7''$, and $p=54'\ 14{\cdot}9''$

AUGMENTATION OF THE MOON'S SEMI-DIAMETER

842. Since when the moon is in the zenith it is nearer to the observer than when in the horizon by the radius of the earth, its apparent magnitude is consequently increased, and at intermediate altitudes its augmentation will be intermediate. The amount of this augmentation for any given altitude is sensibly constant for the same diameter, and is given in a Table, and can be easily applied.

For the altitude of 15°, and semi-diameter $14'\ 43{\cdot}7''$, this augmentation is $4''$, so that the semi-diameter found above must be augmented by this quantity for this altitude, and would then be

$$=14'\ 43{\cdot}7''+4''=14'\ 47{\cdot}7''.$$

The semi-diameter of the moon given in the *Nautical Almanac* is that which it would appear to have when seen from the centre of the earth. If this semi-diameter be denoted by s, and its apparent semi-diameter at the given place by s', and a its altitude, then s' can be calculated from the equation,

$$s'=s+ms^2\sin a.$$

Where $m=k\sin 1''$, $k=3{\cdot}6697$, and $k=\frac{h}{s}$, where h and s are the registered horizontal parallax and semi-diameter. The value of m is ·00001779, for the ratio of h to s is constant. When

$$a=0,\ \text{then } s'=s.$$

CONTRACTION OF THE MOON'S SEMI-DIAMETER

843. The lower limb of the moon is apparently more elevated by refraction than its upper limb, as its altitude is less; and consequently every diameter of the moon except the horizontal one is contracted, and the vertical one is subject to the greatest contraction. This contraction is greater the less the altitude, and is sensibly constant for the same diameter and for a given altitude; and is therefore conveniently applied by means of a Table.

The contraction for an altitude of 15° is 4″ for the vertical diameter; so that the semi-diameter 14′ 48″, previously found, now becomes $= 14' \, 47{\cdot}7'' - 4'' = 14' \, 43{\cdot}7''$.

This semi-diameter, neglecting these corrections, was found to be 14′ 43·7″; so that in this instance these two corrections exactly compensate each other.

THE SUN'S SEMI-DIAMETER

844. The sun's daily change of distance from the earth is so small compared with its distance that its semi-diameter does not sensibly change in apparent magnitude in the course of a day; so that its registered semi-diameter may be considered as constant for at least one day. Its distance also is so great compared with the earth's radius that its semi-diameter is not subject to an apparent augmentation dependent upon altitude.

The sun's diameter, however, like that of the moon, is subject to an apparent contraction by the unequal refraction of its upper and lower limbs, and its amount is sensibly the same as for the moon.

845. Problem VII.—Given the horizontal parallax of a celestial body, and its altitude, to find its parallax in altitude.

RULE.—Radius is to the cosine of the apparent altitude as the sine of the horizontal parallax to the sine of the parallax in altitude.

Let p' = the horizontal parallax,
p = " parallax in altitude,
a = " apparent altitude;

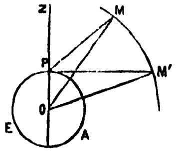
then
$$R : \cos a = \sin p' : \sin p,$$
or
$$\sin p = \sin p' \cos a \text{ when rad.} = 1.$$
Or, by proportional logarithms,
$$\text{P.L, } p = \text{P.L, } p' + \text{L sec } a - 10.$$

EXAMPLE.—When the horizontal parallax of the moon is $=54'$ $20''$, and its altitude $=36° 45'$, what is its parallax in altitude?

Here $a=36° 45'$, and $p'=54' 20''$.

By Logarithms		By Proportional Logarithms	
L, radius . . =	10·	L, radius . . =	10·
L, cos a . . =	9·9037701	L, sec a . . =	10·0962
L, sin p' . . =	8·1987581	P.L, p' . . =	·5202
L, sin p . . =	8·1025282	P.L, p . . =	·6164
Hence p . . =	43′ 32″.	Hence p . . =	43′ 32″

846. The principle on which the rule is founded is very simple.



Let PEA be the earth, O its centre, M′ the moon in the horizon, M its position at any altitude, OZ a vertical line; and let

angle OM′P $=p'$ the horizontal parallax,
" OMP $=p$ " parallax in altitude,
" MPM′ $=a$ " apparent altitude
OP $=r$ " earth's semi-diameter; and
OM or OM′ $=d$ " moon's distance;

then in triangle OPM, angle $P=90+a$, and $\sin P=\cos a$;
and " OPM, $\sin P$ or $\cos a : \sin p=d : r$;
" " OPM′, $R : \sin p'=d : r$;
hence $R : \cos a=\sin p' : \sin p$.

847. The sun's horizontal parallax varies only about $\frac{1}{8}$ of a second, and may in practice generally be considered as invariable. The **parallaxes in altitude** for the sun at any given time may therefore be considered the same for any other time; and thus, being **constant**, they are given in a Table.

EXERCISES

1. When the horizontal parallax is $=54' 16''$, and altitude $=$ $24° 29' 30''$, what is the parallax in altitude? . . $=49' 23''$.

2. When the horizontal parallax is $=57' 32''$, and the altitude $=50° 40'$, what is the parallax in altitude? . . . $=36' 28''$.

REDUCTION OF THE EQUATORIAL PARALLAX

848. The horizontal parallax given in the *Nautical Almanac* is calculated for the equatorial radius of the earth, and is the true horizontal parallax only at the equator; for, the earth's radius being less the greater the latitude, the horizontal parallax will be less at any other place. If l denote the latitude, and e the ellipticity of the earth, the value of which is nearly $\frac{1}{300}$, and if p' and

p'' denote the horizontal parallax at the given place and at the equator, then is $p' = p''\ (1 - e \sin^2 l)$.

For if a, r are the radii of the earth at the equator and the given place, it is proved in the theory of the figure of the earth that $r = a\ (1 - e \sin^2 l)$. Also (Art. 846), $r = d \sin p'$, and $a = d \sin p''$; therefore, since $\frac{p'}{p''} = \frac{\sin p'}{\sin p''}$ very nearly, $\frac{p'}{p''} = \frac{r}{a} = (1 - e \sin^2 l)$; hence $p' = p''\ (1 - e \sin^2 l)$.

A Table contains the corrections, calculated by this formula, that must be deducted from the equatorial horizontal parallax in order to reduce it to the horizontal parallax for any given latitude.

Thus, for the equatorial horizontal parallax in the preceding example, the reduction in the Table under 54′, and opposite to latitude 36°, is 3·7″; and the correct horizontal parallax for this latitude is $= 54'\ 20'' - 3{\cdot}7'' = 54'\ 16{\cdot}3''$.

849. **Problem VIII.—Given the observed altitude of a heavenly body, to find the altitude when corrected for refraction.**

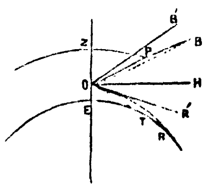
The refraction for the observed altitude is given in a Table, and is always to be subtracted from the observed altitude.

EXAMPLE.—Let the apparent altitude be 32° 10′, to find the true altitude.

In the Table the refraction for this altitude is 1′ 30″,

and the apparent altitude	=	32°	10′	0″
The refraction	= −	0	1	30
Hence the true altitude	=	32	8	30

Let ER be a part of the earth's surface, and ZP a portion of the upper limit of the atmosphere; B the real place of a heavenly body, B′ its apparent place; O the eye of the observer; OH a horizontal and OZ a vertical line. When a ray of light BO from the body B enters the atmosphere at P, which increases in density downwards, the direction of the ray approaches always nearer to that of the vertical line OZ, and thus



it moves in a curved path BPO; but the direction of the body is referred to the direction of the ray when entering the eye at O—that is, to the direction OB′ of the tangent to the curved path at O—and the body thus appears at B′ higher than its real position.

The greater the altitude of the body the less is its refraction, and in the zenith it vanishes.

850. The **mean** refraction of a body is its **true** refraction when the barometer stands at 29·6 inches, and Fahrenheit's thermometer at 50°. Bradley's formula for calculating the mean refraction is

$$r' = 57'' \tan (z - 3r'),$$

where r' = the mean refraction, and z = the zenith distance.

A Table of mean refractions can thus be calculated; and to find the true refraction r when the pressure of the atmosphere is h, and the temperature t, multiply the mean refraction r' by $\frac{400}{350+t} \cdot \frac{h}{29 \cdot 6}$; that is, $r = \frac{400h}{29 \cdot 6(350+t)} r'$. But a Table is also calculated containing the corrections that must be applied to the mean refractions when the pressure and temperature differ from 29·6 and 50.

Thus, the refraction in the preceding example—namely, 1′ 30″—is the mean refraction; but if the temperature and pressure were 69° and 30·35, then the correction

For altitude 32° 10′, and temperature 69°, is .	= − 4″
And " 32° 10′, " pressure 30·35, is . .	= + 2
Hence the correction for both is	= − 2
And the mean refraction was found . . .	= 1′ 30″
Therefore the true refraction	= 1 28
Apparent altitude	= 32° 10′ 0″
And the true altitude	= 32 8 32

851. Unless when great accuracy is required, or when the altitude is small, the corrections for change of pressure and temperature are unnecessary.

852. Problem IX.—Given the height of the eye of the observer above the surface of the earth, to find the depression of the visible horizon.

The depression of the horizon HOR (fig. to preceding problem) can be calculated when the height OE of the eye and the diameter of the earth are known; for it is just the angle at the earth's centre, subtended by the arc ER, because (fig. to Art. 585) angle RBH = BCH, and the latter angle can be calculated in the same manner as angle E (fig. to Art. 587).

The real depression, however, will be this angle diminished by $\frac{1}{11}$ of itself on account of refraction.

EXAMPLE.—Find the depression of the horizon when the height of the eye is = 30 feet.

Opposite to 30 in the Table is 5′ 18″, the dip.

853. Problem X.—Given the observed altitude of a fixed star, to find its true altitude.

RULE.—Correct the observed altitude by applying to it the index error of the instrument with its proper sign, and subtract the dip from the result, and the remainder will be the apparent altitude.

From the apparent altitude subtract the refraction, and the remainder will be the true altitude.

When the observed altitude is taken by a back observation, the dip must be added to it. When great accuracy is required, the corrections for the temperature and pressure of the atmosphere must be applied to the refraction.

EXAMPLE.—The observed altitude of a star was = 40° 20′ 34″, the height of the eye = 12 feet, and the index error 2′ 25″ in excess; find the true altitude.

Observed altitude	=	40° 20′ 34″
Index error	=	− 2 25
		40 18 9
Dip	=	− 3 21
Apparent altitude	=	40 14 48
Refraction	=	− 1 8
True altitude	=	40 13 40

EXERCISES

1. When the observed altitude of a star is = 25° 36 40″, the index error = 1′ 54″ in defect, and the height of the eye = 20 feet, what is the true altitude? . . . . = 25° 32′ 15″.

2. What is the true altitude of a star when its observed altitude is = 38° 2′ 20″, the height of the eye = 18 feet, and the temperature and pressure = 45° and 30·6? . . . . . . = 37° 57′ 4″.

854. Problem XI.—Given the observed altitude of the upper or lower limb of the sun, to find the true altitude of its centre.

RULE.—Apply the sun's registered semi-diameter to the observed altitude by addition or subtraction, according as the lower or upper limb was observed, and subtract the dip, and the result will be the

apparent altitude of its centre; from which subtract the refraction corresponding to it, and add the parallax in altitude, and the sum will be the required altitude.

EXAMPLE.—If the observed altitude of the sun's upper limb on the 6th of November 1854 should be = 28° 21′ 24″, and the height of the eye = 13, what would be the true altitude?

Observed altitude of sun's upper limb	=	28° 21′ 24″
Semi-diameter	=	− 16 11
		28 5 13
Dip	=	− 3 29
Apparent altitude of centre . . .	=	28 1 44
Refraction	=	− 1 47
		27 59 57
Parallax in altitude	=	+ 8
		28 0 5

EXERCISES

1. If the observed altitude of the sun's lower limb on the 19th of April 1854 was = 42° 10′ 15″, the height of the eye being = 25 feet, what was the true altitude of its centre, its semi-diameter being = 15′ 57″? . . . . = 42° 20′ 25″.

2. If the observed altitude of the sun's upper limb on the 11th of June 1854 was = 20° 40′ 15″, and the height of the eye = 16 feet, what was the true altitude of its centre, its semi-diameter being = 15′ 47″? . . . . = 20° 18′ 12″.

855. **Problem XII.—Given the observed altitude of the upper or lower limb of the moon, to find the true altitude of its centre.**

RULE.—To the observed altitude apply the semi-diameter by addition or subtraction according as the altitude of the lower or upper limb is given; from this result subtract the depression, and the remainder is the apparent altitude of the moon's centre; and to this altitude apply the refraction and parallax in altitude, as in the preceding problem.

EXAMPLE.—If on the 12th of July 1841, in latitude 56° 40′, the observed altitude of the moon's upper limb was = 57° 14′ 20″, the height of the eye = 22 feet, the semi-diameter = 15′ 35″, and the horizontal parallax = 57′ 14″; required the true altitude of the moon's centre.

Ob. alt. moon's upper limb	= 57° 14′ 20″	Hor. par. .	= 57′ 14″
Moon's semi-diameter .	= 15 35	Reduction	= − 8
Augmentation . . .	= 13	Tr. hor. par.	= 57 6
Aug. semi-diameter .	= − 15 48		
Depression . . .	= − 4 32	P.L., 57′ 6″,	= ·4986
Ap. alt. of centre a' .	= 56 54 0	Secant a' .	= ·2627
Moon's par. in alt. p .	= + 31 11	P.L., p, .	= ·7613
Ref. to ap. altitude . .	= − 0 37		
True altitude a . .	= 57 24 34		

856. The longitude of the place of observation and the time of observation must be known in order to determine the reduced time (Art. 837), and then the semi-diameter and horizontal parallax are found for the reduced time according to the rule in Art. 841.

Exercises

1. At a place in latitude = 36° 50′ the observed altitude of the moon's lower limb was = 24° 18′ 40″, the height of the eye = 17·3 feet, the moon's semi-diameter at the time of observation = 15′, and its horizontal equatorial parallax = 55′ 2″; what was the true altitude, the corrected semi-diameter, and parallax in altitude?

The true altitude of moon's centre = 25° 17′ 41″, semi-diameter = 15′ 6″, and parallax = 50′ 1″.

2. If, at a place in latitude = 24° 30′, and longitude = 23° E., on 31st May 1796, at 5 h. 36 m. P.M., the observed altitude of the moon's lower limb was = 23° 48′ 15″, the height of the eye = 17·3 feet, the moon's semi-diameter and horizontal parallax at the preceding noon and following midnight being = 15′ 49″, 15′ 56″, and 58′ 1″, 58′ 29″; required the true altitude of the moon's centre.

The reduced time = 4 h. 4 m. P.M.; corrected semi-diameter = 15′ 57″, parallax in altitude = 53′ 6″, and altitude = 24° 51′ 9″.

3. On 10th September 1841, in latitude = 28° 40′ N., longitude 24° 45′ W., at 5 h. 51 m. P.M., the observed altitude of the moon's lower limb was = 32° 40′ 15″, height of eye = 16 feet; required the true altitude of the moon's centre, having also given the moon's

	Semi-diameter	Horizontal Parallax
At noon preceding, ...	16′ 15″	59′ 37″
" midnight following, . .	16 19	59 51

Reduced time = 7 h. 30 m. P.M., corrected semi-diameter = 16′ 26″, parallax in altitude = 50′ 8″, and altitude = 33° 41′ 29″.

857. **Problem XIII.—To find the polar distance of a celestial object.**

RULE.—When the declination and the latitude of the place are of the same name, subtract the declination from 90°; and when of different names, add the declination to 90°; then the difference in the former or the sum in the latter case is the polar distance.

Let D = the declination of the body, P = the polar distance; then

$$P = 90 \mp D.$$

EXAMPLE.—What will be the moon's north polar distance on the 12th of November 1854 at noon at Greenwich, its declination being then = 21° 18′ 4·4″ N.?

$$P = 90° - D = 90° - 21° \ 18' \ 4{\cdot}4'' = 68° \ 41' \ 55{\cdot}6''.$$

EXERCISES

1. Find the moon's north polar distance on the 11th of September 1854 at 11 h. P.M. at Greenwich, its declination being then = 18° 21′ 15″ N. . . . . = 71° 38′ 45″.

2. Find the north polar distance of Mars on the 10th of December 1841 when on the meridian of Greenwich, its declination being at that time = 19° 15′ 16″ S. . . . . . . = 109° 15′ 16″.

858. **Problem XIV.—To convert intervals of mean solar time to intervals of sidereal time; and conversely.**

RULE.—As 1 h. is to 1 h. 0 m. 9·8565 s., so is the given interval of mean solar time to the required interval of sidereal time; and 1 h. is to 0 h. 59 m. 50·1704 s. as the given interval of sidereal time to the required interval of mean time.

859. Or find the equivalents by means of a Table of time equivalents, or by means of a Table of accelerations and retardations.

Let m = the mean time,
s = " equivalent sidereal interval,
a = " acceleration,
r = " retardation;

then $s = m + a$, $m = s - r$.

P′.L. a . . . . . . = P.L. m + 1·65949,
P′.L. r . . . . . . = P.L. s + 1·66068,

where P′.L. stands for 3 h. proportional logarithms, and P.L. for 24 h. ones.

EXAMPLE.—Convert 10 h. 20 m. 40 s. of sidereal time to mean time.

By the First Method

1 h. : 0 h. 59 m. 50·1704 s. = 10 h. 20 m. 40 s. : 10 h. 18 m. 58·32 s.

By the Second Method

P.L., 10 h. 20 m. 40 s.			= ·36550
Constant			= 1·66068
P′.L.,		1 m. 41·68 s.	= 2·02618
Sidereal time, 10 h.	20	40	
Required " 10	18	58·32	

The rules depend on the facts that a meridian describes 360° in 24 sidereal hours, and 360° 59′ 8·3″ in 24 hours of mean solar time. Hence it describes 59′ 8·3″ in 3 m. 55·91 s. mean time, and in 3 m. 56·56 s. sidereal time.

Hence	24 h.	mean time	= 24 h.	3 m.	56·56	s.	sidereal time,
and	24	sidereal "	= 23	56	4·09		mean "
Or,	1	mean "	= 1	0	9·856		sidereal "
and	1	sidereal "	= 0	59	50·170		mean "

The acceleration of sidereal on mean time in 24 sidereal hours is 3 m. 56·56 s., and the retardation of mean on sidereal time in 24 mean hours is 3 m. 55·91 s. A Table of accelerations and retardations for any number of hours, of minutes, &c. can easily be calculated.

The rule by proportional logarithms is obtained thus:—The first two terms of the proportion are either 3600 s. and 3609·85 s., or 3600 s. and 3590·17 s., and the difference of their logarithms is ·00119; then proportional logarithms may be taken for the other two terms; for if a, b, c, d are the terms of a proportion, then (Art. 839),

$$\mathrm{L}a \sim \mathrm{L}b = \mathrm{P.L.}\ c \sim \mathrm{P.L.}\ d,$$
$$\mathrm{L}b - \mathrm{L}a = \mathrm{P.L.}\ c - \mathrm{P.L.}\ d,$$
and
$$\mathrm{P.L.}\ d = \mathrm{P.L.}\ c + (\mathrm{L}a - \mathrm{L}b).$$

EXERCISES

1. Convert 7 h. 40 m. 15 s. of sidereal time to mean time.
= 7 h. 38 m. 59·6 s.

2. Convert 7 h. 38 m. 59·6 s. of mean time to sidereal time.
= 7 h. 40 m. 15 s.

860. **Problem XV.—Given the sun's registered mean right ascension at mean noon, to find its mean right ascension at any place and at any time of the day.***

RULE.—Find the reduced time; then, as 24 hours is to the reduced time, so is 3 m. 56·555 s. to a proportional part, which,

* In the *Nautical Almanac* the mean right ascension is called the sidereal time.

when added to the given right ascension at the preceding mean noon, will give that required.

Let A′ = the registered mean right ascension at mean noon,
A = " required mean right ascension,
d' = " increase of A′ in 24 mean hours = 3 m. 56·555 s.,
d = " proportional part for t,
t = " reduced time;

then $$24 : t = d' : d, \text{ and } d = \tfrac{1}{24} d't,$$
and $$A = A' + d.$$

EXAMPLES.—1. Find the sun's mean right ascension at mean noon on the 11th of April 1854 at a place in longitude = 36° 15′ W.

t = longitude in time = 2 h. 25 m.

Sun's given mean right ascension, or	A′ =	1 h.	17 m.	29·86 s.
Increase in time t, or . . .	d =	0	0	23·82
Right ascension required, or . .	A =	1	17	53·68

2. What is the sun's mean right ascension at 2 h. 40 m. P.M. on the 30th of April 1841 at a place in longitude = 50° 20′ 30″ W.?

Given time . .	=	2 h.	40 m.	0 s.
Longitude in time	= +	3	21	22
Reduced time t .	=	6	1	22

Sun's given right ascension, or .	A′ =	2 h.	33 m.	1·27 s.
Increase in time t, or . . .	d =	0	0	59·36
Right ascension required, or .	A =	2	34	0·63

861. The principle of the rule is evident, for 3 m. 56·555 s. is the increase of the sun's mean right ascension in 24 hours mean time, and terrestrial longitude reduced to time by the usual rule is mean time.

EXERCISES

1. Find the sun's mean right ascension at mean noon at a place in longitude = 45° 25′ W. on the 25th of June 1841, its registered mean R.A. at mean noon being = 6 h. 13 m. 48·5 s.
= 6 h. 14 m. 18·34 s.

2. Required the sun's mean R.A. on the 20th of July 1841, at 3 h. 20 m. P.M., at a place in longitude = 56° 15′ W.; its registered mean R.A. at mean noon on the same day being = 7 h. 52 m. 22·45 s.
= 7 h. 53 m. 32·26 s.

3. What will be the sun's mean R.A. on the 14th of November 1854, at 10 h. 40 m. A.M., at a place in longitude = 36° 24′ 15″ E.; its registered mean R.A. at mean noon on the 13th being = 15 h. 29 m. 5·9 s.? . . . . 15 h. 32 m. 25·4 s.

862. Problem XVI.—To convert any given mean time on any given day to the corresponding sidereal time; and conversely.

RULE.—When mean time is given, express it astronomically and convert it into the equivalent sidereal time; then to this result add the sidereal time at the preceding mean noon, and the sum will be the required sidereal time.

When sidereal time is given, subtract from it the sidereal time at the preceding noon, and convert the remainder into its equivalent mean time, and it will be the required time.

The sidereal time at the preceding noon—that is, the sun's mean right ascension—is found by the preceding problem.

Let m = the mean astronomical time,
s = " equivalent interval of sidereal time,
a = " acceleration for m,
r = " retardation for s,
S = " sidereal time,
S' = " registered sidereal time or sun's mean R.A. at preceding mean noon at the given place.

When m is given, $s=m+a$, and $S=S'+s$;
and when S is given, $s=S-S'$, and $m=s-r$.

EXAMPLES.—1. Find the sidereal time corresponding to 2 h. 22 m. 25·62 s. mean time at Greenwich, 2nd January 1854.

Here	$m=$	2 h.	22 m.	25·62 s.	
	$a=$	0	0	23·39	
	$s=$	2	22	49·01	
	$S'=$	18	47	10·92	at noon, 2nd Jan.
	$S=$	21	9	59·93	required time.

2. Find the mean time corresponding to 21 h. 9 m. 59·93 s. sidereal time at Greenwich, 2nd January 1854.

Here	$S=$	21 h.	9 m.	59·93 s.	
	$S'=$	18	47	10·92	at noon, 2nd Jan.
	$s=$	2	22	49·01	
	$r=$	0	0	23·40	
	$m=$	2	22	25·61	2nd Jan., required time.

3. Find the sidereal time corresponding to 3 h. 40 m. P.M. mean time on the 11th of April 1854 at a place in longitude =36° 15′ W.

The sun's mean R.A. at mean noon—that is, the sidereal time at the preceding noon at the given place, according to the first example of the preceding problem—is = 1 h. 17 m. 53·68 s.

	m = 3 h.	40 m.	0 s.
	a = 0	0	36·14
	s = 3	40	36·14
and	S′ = 1	17	53·68
hence	S = 4	58	29·82

863. The sidereal time at mean noon at any place is just the right ascension of its meridian at that time—that is, the sidereal interval since the transit of the first point of Aries—and this is just the right ascension of the mean sun at the mean noon. This time is given in the *Nautical Almanac* for Greenwich, and is easily found from the sun's right ascension at mean noon, by applying to it the equation of time; it could also be deduced from the sun's right ascension at apparent noon.

Exercises

1. Convert 2 h. 21 m. 13·08 s. mean solar time on the 2nd of January 1854 at Greenwich into the corresponding sidereal time, the sidereal time at mean noon being = 18 h. 47 m. 10·92 s.
=21 h. 8 m. 47·2 s.

2. Convert 21 h. 8 m. 47·2 s. of sidereal time on the 2nd of January 1854 at Greenwich into the corresponding mean solar time, the sidereal time at mean noon being = 18 h. 47 m. 10·92 s.
=2 h. 21 m. 13·08 s.

3. Find the sidereal time corresponding to 8 h. 20 m. A.M. mean time on the 26th of June 1841 at a place in longitude = 45° 25′ W., the registered sidereal time at the preceding mean noon being = 6 h. 13 m. 48·5 s. . . . . . . =2 h. 37 m. 38·75 s.

864. Problem XVII.—To find the mean time of the sun's transit over the meridian of any place.

RULE.—Find the equation of time for the reduced time corresponding to the longitude, and it will be the time from mean noon, either before or after, at which the transit of the centre happens.

To the time of the meridian passage of the centre apply the time of the semi-diameter's passing the meridian, by subtraction or addition, according as the time of transit of the first or second limb is required.

EXAMPLES.—1. Find the mean time of the transit of the sun's centre, and that of its first limb, over the meridian of Greenwich on the 10th of January 1854.

Equation of time at apparent noon to be added to apparent time . .	=0 h.	7 m.	50·43 s.
Time of semi-diameter's passage .	=	1	10·52
	0	6	39·91

Hence the time of transit of the centre is=0 h. 7 m. 50·43 s., and of the first limb=0 h. 6. m. 39·91 s.

2. Find the mean time of the transit of the sun's centre, and of its second limb, at a place in longitude=54° 30′ E. on the 28th of April 1854.

Longitude in time=3 h. 38 m. 0 s.

Reg. equation of time on the 28th .	= +	2 m.	36·28 s.
" " " 27th .	=0	2	26·87
Increase of equation of time in 24 h. .	=0	0	9·41
" " " 3 h. 38 m.	= −	0	1·43
Equation of time for reduced time .	=0	2	34·85
Time of passage of semi-diameter .	=0	1	5·80
	0	3	40·65

Hence the time of the passage of the centre is=0 h. 2 m. 34·85 s., and of the second limb=3 m. 40·65 s.

EXERCISES

1. Find the time of the meridian passage of the sun's centre, and of its second limb, at Greenwich on the 30th of April 1854, the equation of time at apparent noon being=2 m. 53·58 s., to be subtracted from apparent time, and the mean time of the semi-diameter's passing the meridian=1 m. 5·96 s.

For centre=29 d. 23 h. 57 m. 6·42 s., and for the second limb =29 d. 23 h. 58 m. 12·38 s.

2. Required the time of the meridian passage of the sun's centre, and that of its first limb, at Edinburgh, in longitude=12 m. 44 s. W., on the 16th of November 1854, the equation of time at apparent noon (to be subtracted from apparent time) on the 15th and 16th being=15 m. 15·56 s., and 15 m. 4·77 s., and the mean time of the semi-diameter's passage being 1 m. 8·71 s.

For centre=15 d. 23 h. 44 m. 55·32 s., and for the first limb =15 d. 23 h. 43 m. 46·61 s.

865. **Problem XVIII.—To find the mean time of a star's culmination at any given meridian.**

RULE.—Find the sun's mean right ascension at mean noon at the given place, and subtract it from the star's right ascension, increased if necessary by 24 hours; and the remainder, which is a sidereal interval, being converted into mean time, will be the mean time of culmination.

Let A' = the star's apparent right ascension at given time,
A = " sun's mean right ascension at mean noon preceding the transit at given place (Art. 860),
T' = " sidereal time of transit after mean noon,
T = " mean " " " "
then $T' = A' - A$, and $T = T' - r$, by Art. 859.

EXAMPLES.—1. When will Arcturus culminate at Greenwich on the 1st of April 1854?

R.A. of Arcturus, or . . .	$A' = +$	14 h.	9 m.	0·82 s.
R.A. of sun at mean noon, .	$A = -$	0	42	4·21
Sidereal time of cul. after noon,	$T' =$	13	26	56·61
Retardation, ...	$r = -$	0	2	12·20
Mean time of transit, . .	$T =$	13	24	44·41

2. Find the time of the passage of Arcturus over the meridian of a place in longitude = 62° 15′ W. on the 7th of December 1854.

Longitude in time 62° 15′ W. = 4 h. 9 m.

Sun's registered mean R.A. on 7th	=	16 h.	55 m.	21·88 s.
Increase or accel. for 4 h. 9 m., or a	= +	0	0	40·90
Hence	$A = -$	16	56	2·78
And	$A' = +$	14	9	2·69
Therefore	$T' =$	21	12	59·91
Retardation, or	$r = -$	0	3	28·55
Mean time of transit, . .	$T =$	21	9	31·36

Or, in civil time, on the 8th, at 9 h. 9 m. 31·36 s. A.M.

EXERCISES

1. At what time will Sirius culminate at Greenwich on the 17th of December 1854, its right ascension being = 6 h. 38 m. 45·46 s., and sun's mean right ascension, or the sidereal time, at mean noon being = 17 h. 39 m. 27·73 s.? . . . . At 12 h. 57 m. 10·06 s.

2. When did Aldebaran culminate at New York, in longitude = 73° 59′ W., on the 17th of November 1841, its right ascension

being 4 h. 26 m. 51·19 s., and the registered sidereal time at mean noon being = 15 h. 45 m. 29·03 s.? . ; At 12 h. 38 m. 28·96 s.

866. **Problem XIX.—To find the mean time of the moon's culmination at any given meridian.**

RULE I.—Find the sun's mean right ascension for the reduced time corresponding to the longitude, and find also the moon's right ascension for the same time; subtract the former from the latter, increased if necessary by 24 hours, and the remainder will be an approximate time.

Then as 1 hour diminished by the difference between the hourly variations in right ascension of the sun and moon is to the approximate time, so is this difference to a fourth term, which, added to the approximate time, will give the true time.

Let A' = the moon's R.A. at the reduced time,
A = " sun's " " " "
T' = " approximate time of culmination,
T = " true time of culmination,
v' = " hourly variation in R.A. of the moon,
v = " " " " " sun,
r = " fourth term.

Then $T' = A' - A$.

$$1-(v'-v) : T' = v'-v : r, \text{ and } T = T'+r.$$

867. The interval of time between two successive meridian passages of the moon is called the **moon's daily retardation.**

If the transit is required only to about a minute of accuracy, it can easily be found by the following rule :—

RULE II.—Find the difference between the times of the preceding and succeeding meridian passages—that is, the moon's daily retardation; then find the proportional part for the longitude in time; add this part to the first registered time, and the sum will be the required time.

Let P' = registered time of preceding passage,
P = required time of meridian passage,
v' = the moon's daily retardation,
v = " variation for the given longitude in time,
t = " longitude in time.

Then $24 : t = v' : v$, and $v = \frac{1}{24}v't$. And $P = P' + v$.

EXAMPLE.—Find the time of the moon's culmination in longitude = 40° 45′ W. on the 2nd of May 1854.

By the Second Method

Longitude 40° 45′ = 2 h. 43 m.

Time of reg. meridian passage on 2nd May	=	+ 4 h.	13 m.	0 s.
" " " 3rd "	=	5	3	12
Retardation in 24 h.	=	0	50	12
Hence retardation in 2 h. 43 m. . .	=	+ 0	5	41
Required time of transit	=	4	18	41

By the First Method

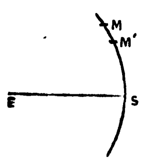
Sun's m. R.A. at m. noon on 2nd May	=	2 h.	40 m.	17·50 s.
Increase or acceleration in 2 h. 43 m. .	= +			26·77
Sun's R.A. at noon at given place, A	= −	2	40	44·27
Moon's reg. R.A. on 2nd May at 2 h. .	=	6	49	6·80
" " " 3 h. .	=	6	51	18·74
Increase in 1 h., or . . . v'	=	0	2	11·94
" 43 m.	= +	0	1	34·56
Moon's R.A. at noon at place, . A'	= +	6	50	41·36
Approximate time . $= A' - A = T'$	=	4	9	57·09
Also, by Table of accelerations, . v	=	0 m.	9·858 s.	
And by *Nautical Almanac*, . . v'	=	2	11·94	
Hence . . . . $v' - v$	=	2	2·08	

And $1 - (v' - v) : T' = v' - v : r.$

Or, 57 m. 57·92 s. : 4 h. 9 m. 57·09 s. = 2 m. 2·08 s. : r.

P.L., 57 m. 57·92 s.	=	1·39520
P.L., 4 h. 9 m. 57·09 s. . .	=	·76050
P.L., 2 m. 2·08 s. . . .	=	1·94701
		2·70751
P.L., 0 h. 8 m. 47·16 s. . .	=	1·31231

And	T' =	4	9	57·09	
Hence	T =	4	18	44·25	= mean time of transit.



Let the meridian be at S at mean noon at the given place, and the moon then at M′; then, if M be the position of the same meridian of the earth at the culmination of the moon, the meridian will have moved over the arc SM, while the moon has moved over M′M. Now, if arc SM′ = T′ in sidereal time = A′ − A,

SM = T″ in sidereal time,

T = the mean time corresponding to T″,

h, h' = a mean and sidereal hour respectively,

v, v' = the variations in R.A. of sun and moon in 24h.

Then $$24h' + v : v' = \text{SM} : \text{M'M} = \text{T}'' : \text{T}'' - \text{T}'.$$
Or, $$24h' - (v' - v) : 24h' + v = \text{T}' : \text{T}''.$$
But $$24h' + v : 24h' = \text{T}'' : \text{T}.$$
Therefore $$24h' - (v' - v) : 24h' = \text{T}' : \text{T}.$$

By this proportion the required time T can be found. Or if v, v' refer to 1 instead of 24 hours, then

$$1 - (v' - v) : 1 = \text{T}' : \text{T}.$$
Or, $$1 - (v' - v) : v' - v = \text{T}' : \text{T} - \text{T}'.$$
Or, $$1 - (v' - v) : \text{T}' = v' - v : r, \text{ if } r = \text{T} - \text{T}'.$$

This proportion is the rule, and the calculation can be made by proportional logarithms, taking those for 24 hours for the first two terms, and for three hours for the third and fourth.

Exercises

1. Find the time of the moon's meridian passage at a place in longitude = 168° 30′ W. on the 27th of November 1854, the sidereal time or sun's mean right ascension at noon on the 27th being = 16 h. 24 m. 17·70 s., and the moon's right ascension on the same day at 11 h. = 23 h. 17 m. 17·8 s., and at 12 h. = 23 h. 19 m. 23·67 s.
At 7 h. 5 m. 21·15 s.

2. Find the time of the moon's meridian passage at a place in longitude = 68° 30′ W. on the 5th of November 1841, the registered sidereal time on the 5th being = 14 h. 58 m. 10·35 s., and the moon's registered right ascension on the same day at 4 h. being 8 h. 35 m. 1·64 s., and at 5 h. = 8 h. 37 m. 22·13 s. . At 18 h. 17 m. 15·17 s.

868. **Problem XX.—To find the time of culmination of a planet at any given meridian.**

The rule is exactly the same as that in last problem; only, as the increase in right ascension of a planet for 24 hours is small, the right ascension is given, not for every hour, but only for every noon; and the meridian passages are given to the tenth of a minute. In the first formula of last problem, therefore, when adapted to this one, v and v' are the daily variations in right ascension of the sun and planet. Hence

$$24 - (v' - v) : \text{T}' = v' - v : r, \text{ and } \text{T} = \text{T}' + r.$$

When $v' < v$, r is negative, and $\text{T} = \text{T}' - r$. When v' is negative—that is, when the motion of the planet is retrograde—then r is also negative, and

$$24 - (v' + v) : \text{T}' = v' + v : r, \text{ and } \text{T} = \text{T}' - r.$$

Example.—Find the time of the meridian passage of Mars, at a place in longitude = 45° 30′ W., on the 6th of April 1854.

By the Second Method

Longitude in time = 3 h. 2 m.

Time of registered meridian passage on 6th .	= + 9 h.	6 m.	
" " " " 7th .	= + 9	1·9	
Retardation in 24 h.	= 0	4·1	
" " 3 h. 2 m.	= − 0	0·5	
Time of mer. passage at given place on 6th .	= 9	5·5	

By the First Method

Registered R.A. of Mars on 6th . .	= +	10 h.	5 m.	12·72 s.
" " " 7th . .	=	10	5	5·28
Decrease in 24 h.	=	0	0	7·44
" " 3 h. 2 m.	= −	0	0	0·94
R.A. at noon at given place, . A′	=	10	5	11·78
Registered R.A. of sun on 6th . .	=	0	57	47·09
Increase or acceleration in 3 h. 2 m. .	=	0	0	29·9
R.A. of sun at noon at given place, A		0	58	16·99
Approximate time . = T′ = A′ − A =		9	6	54·79

The motion of the planet being retrograde,

$$v' - v = -0 \text{ m. } 7{\cdot}4 \text{ s.} - 3 \text{ m. } 56{\cdot}6 \text{ s.} = -4 \text{ m. } 4 \text{ s.},$$

and
$$24 - (v' - v) = 24 \text{ h. } 4 \text{ m. } 4 \text{ s.}^*$$

P.L., 24 h.	4 m.	4 s.	= − ·00123 = − P.L., 23 h. 55 m. 56 s.	
P.L., 9	6	54·79	= + ·42045	
P′.L., 0	4	4	= + 1·64603	
P′.L., 0	1	32·41	= 2·06771	
T′ = 9	6	54·79		
T = 9	5	22·38 = T′ − r, the required time.		

Exercises

1. Find the time of Jupiter's transit over the meridian in longitude = 160° E. on the 10th of January 1854, the registered times of meridian passage on the 9th and 10th being 23 h. 20·8 m. and 23 h. 17·9 m. . . . . At 23 h. 19·5 m.

2. Find the time of the meridian passage of Jupiter on the 10th of September 1854 at a place in longitude = 160° E., its registered right ascension on the 9th and 10th being = 19 h. 17 m. 6·93 s. and 19 h. 17 m. 3·55 s., and that of the sun on the 10th being = 11 h. 16 m. 46·43 s. . . . . At 8 h. 0 m. 43·63 s.

* When $v' < v$, the first term, $24-(v'-v)$, exceeds 24 hours. But it will never exceed 24 hours by more than 10 minutes, and L1440−L1430=L1450−L1440, when carried only to 5 figures. Hence $-L[24-(v'+v)]$ may be taken instead of $+L[24+v'+v]$, and it must be added to the logarithms of the second and third terms.

869. **Problem XXI.—To find the meridian altitude of a celestial body at a given place, the declination of the body and the latitude of the place being given.**

RULE.—Find the declination of the body at its meridian passage at the given place; then take the sum or difference of the colatitude and declination, according as they are of the same or of different names, and the result will be the meridian altitude.

Let L, C = the latitude and colatitude,
D, P = " declination and polar distance,
A, A′ = " meridian altitudes at upper and lower culmination;

then $A = C \pm D$.

When the declination exceeds the latitude, the altitude then would exceed 90°, and the supplement is to be taken, which is the altitude from the opposite point of the horizon below the pole; or in this case $A = L + P$.

When the declination exceeds the colatitude, the lower meridian passage will be above the horizon, and the altitude then is

$$A' = D - C, \text{ or } A' = L - P.$$

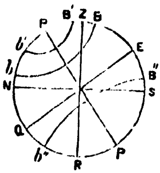
In all these formulæ the latitude and declination are of the same name, except in $A = C - D$.

EXAMPLE.—Required the meridian altitude of the moon at a place in latitude = 56° 20′ 10″ N., and longitude = 40° 45′ W., on the 12th of May 1854.

Longitude in time = 2 h. 43 m.						
Time of registered passage on 12th .	=	+12 h.	16·8 m.			
" " " 13th .	=	13	16·6			
Retardation in 24 h.	=	0	59·8			
" " 2 h. 43 m. . . .	=	+	6·7			
Time of meridian passage on 12th .	=	12	23·5			
Declination on 12th at 12 h. . .	=	19°	0′	39·5″	S.	
" " " 13 h. . .	=	19	12	24·4	S.	
Increase in 1 h.	=	0	11	45·9		
" " 23·5 m.	= +	0	4	36·5		
Hence declination at transit, .	D =	19	5	16·0	S.	
Colatitude, ...	C =	33	39	50·0	N.	
Meridian altitude, . .	A = C − D =	14	34	34·0		

If the apparent altitude were required, this altitude just found would require to be corrected for refraction and parallax.

Let NS be the horizon, EQ the equator, NZSR a meridian, and B, B′, B″ celestial bodies in the meridian, and Bb, B′b', and B″b'' parallels of declination; then ZE = L.; and hence ES = C, and A = BS = C + D. So when B″ is south of the equator, B″E = D, and B″S = A; hence A = C − D. For the body B′, D > L., and B′S = C + D is > 90, and A = B′N = 180° − (C + D) = L + P. When D > L., and of the same name, then A′ = b'N = D − C.



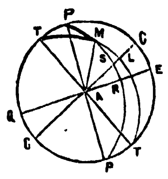
Exercises

1. Find the meridian altitude of Castor on the 11th of May 1854 at a place in latitude = 28° 30′ 25″, its declination being = 32° 12′ 10·7″ N. . . . . = 86° 16′ 30·1″.

2. Find the meridian altitude of Jupiter on the 10th of September 1854 at a place in latitude = 46° 35′ 28″ N., and longitude = 160° E., its registered declination on the 9th and 10th being = 22° 45′ 6·0″ and 22° 45′ 14·2″ S. (See 2nd exercise, Art. 868, for time of transit.) . . . . = 20° 39′ 19·7″.

870. **Problem XXII.—Of the obliquity of the ecliptic, the sun's longitude, declination, and right ascension, any two being given, to find the other two.**

Let PEQ be the solstitial colure, EQ the equator, CC′ the ecliptic, PRP′ a meridian through the sun's centre S. Then A is the first point of Aries, C of Cancer, C′ of Capricorn, and the point diametrically opposite to A is the first point of Libra; also,



AS = L is the sun's longitude,
AR = A " " right ascension,
SR = D " " declination;
angle SAR = O is the obliquity of the ecliptic.

Now, the triangle ARS is right-angled at R, and any two parts of it, except the right angle, being given, the other two can be found by the rules of right-angled spherical trigonometry.

Example.—The sun's registered longitude on the 11th of May 1854 was = 50° 25′ 39·4″, and the obliquity of the ecliptic = 23° 27′ 34·26″; find the sun's declination and right ascension at mean noon at Greenwich.

In the triangle ARS are given AS = L = 50° 25′ 39·4″, and angle SAR = O = 23° 27′ 34·26″.

To find AR = A		To find RS = D	
L, cot L . .	= 9·9172221	L, radius . .	= 10·
L, radius . .	= 10·	L, sin O . .	= 9·5999732
L, cos O . .	= 9·9625311	L, sin L . .	= 9·8869532
L, tan A . .	= 10·0453090	L, sin D . .	= 9·4869264

And A = 47° 59′ 0·12″ = 3 h. 11 m. 56·1 s., and D = 17° 52′ 10·2″.

When the longitude exceeds 90°, so will the right ascension. Since right ascension is reckoned from the vernal equinox, and since the equator and ecliptic intersect at two points diametrically opposite, it is evident that to any particular declination there belong four different right ascensions, and of these the one required must be determined by the time of the year.

Exercises

1. The sun's longitude on the 10th of June 1854 will be = 79° 13′ 1·8″; what will be its declination and right ascension, the obliquity of the ecliptic being = 23° 27′ 34·04″?

D = 23° 1′ 16·1″ N., and A = 5 h. 13 m. 5 s.

2. The obliquity of the ecliptic on the 20th of July 1854 being = 23° 27′ 34·47″, and the sun's declination = 20° 42′ 11·7″ N., required its right ascension and longitude.

A = 7 h. 57 m. 46·25 s., and L = 117° 22′ 23·3″.

3. The sun's right ascension and declination on the 8th of September 1854 will be = 11 h. 6 m. 30·79 s., and 5° 43′ 52·4″ N.; what will be its longitude and the obliquity of the ecliptic?

L = 165° 28′ 20·7″, and O = 23° 27′ 34·1″.

4. On the 27th of December 1854 the sun's longitude and right ascension will be = 275° 28′ 44·8″, and 18 h. 23 m. 52·67 s.; required its declination and the obliquity of the ecliptic.

D = 23° 20′ 49·3″, and O = 23° 27′ 37·7″.

871. **Problem XXIII.—Having given the longitude and latitude of a celestial body, to find its right ascension and declination; and conversely, the obliquity of the ecliptic being supposed known in both cases.**

Let M be the moon or any celestial body (last fig.), and TMT′ an ecliptic meridian; then

AL = L′ is its longitude, reckoning from the nearest preceding equinox,

L = the true longitude,
$ML = l$ is its latitude,
$AR = A'$ is its right ascension, reckoned from the nearest preceding equinox,
A = true right ascension,
$MR = D$ is its declination,
angle $SAR = O$ the obliquity of the ecliptic.

Let $AM = H$ its distance from preceding equinox,
angle $MAR = E$, and $MAL = C$;
then $E = O \pm C$, according as M is without or within the angle of the equator and ecliptic;
and $C = E \sim O$.

Example.—If the moon's longitude on the 2nd of August 1841, at noon at Greenwich, be $= 310° 50' 1''$, its latitude $= 0° 10' 1''$ N., and the obliquity of the ecliptic $= 23° 27' 42''$, required its right ascension and declination.

Here $L' = 130° 50' 1''$, $l = 0° 10' 1''$ N., and $O = 23° 27' 42''$, and M is between the equator and ecliptic.

In triangle MAL

To find H		To find C	
L, radius . .	= 10·	L, tan l . .	= 7·4644506
L, cos L′ . .	= 9·8154878	L, radius . .	= 10·
L, cos l . .	= 9·9999982	L, sin L′ . .	= 9·8788730
L, cos H . .	= 9·8154860	L, cot C . .	= 12·4144224
And $H = 130° 50' 0{\cdot}2''$.		And $C = 0° 13' 14{\cdot}4''$	

Hence, also, $E = O - C = 23° 14' 27{\cdot}6''$.

In triangle MAR

To find D		To find A	
L, radius . .	= 10·	L, cot H . .	= 9·9366113
L, sin H . .	= 9·8788749	L, radius . .	= 10·
L, sin E . .	= 9·5961565	L, cos E . .	= 9·9632462
L, sin D . .	= 9·4750314	L, tan A′ . .	= 10·0266359
And $D = 17° 22' 16''$.		And $A' = 133° 14' 38{\cdot}7''$.	

Hence $A = A' + 180 = 313° 14' 38{\cdot}7'' = 20$ h. 52 m. 58·58 s.

Exercises

1. On the 19th of May 1854, at noon at Greenwich, the moon's longitude was $= 331° 3' 6{\cdot}7''$, and its latitude $= 5° 17' 22{\cdot}3''$ S.; required its right ascension and declination, the obliquity of the ecliptic being $= 23° 27' 34''$.

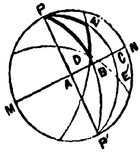
$A = 22$ h. 20 m. 11·52 s., and $D = 16° 2' 51{\cdot}3''$.

2. On the 17th of September 1854 the registered right ascension of the moon at noon will be 8 h. 10 m. 31·05 s., and its declination $=24° 47' 45·7''$ N.; what will be its longitude and latitude at the same time, the obliquity of the ecliptic being $=23° 27' 35·65''$?

$L=119° 24' 41·8''$, and $l=4° 36' 32·45''$.

872. **Problem XXIV.—The right ascensions and declinations, or the longitudes and latitudes, of two stars being given, to find their arcual distance apart.**

Let PNP′ be the solstitial colure, A the vernal equinox, MN the equator, P its pole; D, E two celestial bodies, of which AB, AC are the right ascensions, and DB, EC the declinations. Then in triangle DPE are given the codeclinations PD, PE, and angle $P=$the difference of their right ascensions; hence there are known two sides and the contained angle, and therefore the distance DE can be found.



When the latitudes and longitudes of two bodies are given, their distance is found exactly in the same way. When one of the bodies, as E′, is on the opposite side of MN, then $PE'=90+CE'$.

Let C, $C'=$the complements of their declinations,

$P=$ " difference of their right ascensions,

$D=$their distance;

then (Art. 779), $R : \cos P=\tan C : \tan\theta$ . . . . [1],

and $\cos\theta : \cos(C'-\theta)=\cos C : \cos D$ . . . [3].

Or D can be found, though not so concisely, by the method in Art. 774.

EXAMPLE.—Find the distance between Capella and Procyon on the 21st of January 1841, their right ascensions being $=5$ h. 4 m. 59·77 s. and 7 h. 31 m. 0·79 s., and their declinations $=45° 49' 58·2''$ N. and $5° 37' 37·9''$ N.

Here $P=2$ h. 26 m. 1·02 s. $=36° 30' 15·3''$.

$C=44° 10' 1·8''$, and $C'=84° 22' 22·1''$.

To find the arc θ		To find the distance D	
L, radius . .	$=10·$	L, cos θ . .	$=9·8966300$
L, cos P . .	$=9·9051549$	L, cos $(C'-\theta)$.	$=9·8386823$
L, tan C . .	$=9·9873727$	L, cos C . .	$=9·8557069$
L, tan θ . .	$=9·8925276$		19·6943892
And $\theta=37° 58' 55''$.		L, cos D . .	$=9·7977502$
		And $D=51° 7' 10·5''$.	

When the difference of the right ascensions exceeds 12 hours, add 24 hours to the less, and from the sum subtract the greater, and the difference will be the included angle at the pole.

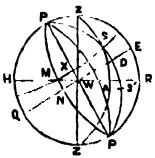
Exercises

1. When the latitudes of Sirius and Procyon were $=39° 34'$ S. and $15° 58'$ S., and their longitudes $=101° 14'$ and $=112° 56'$, what was their distance? . . . . $=25° 42' 52''$.

2. Find the distance of Rigel and Regulus on the 1st of August 1854, their right ascensions being 5 h. 7 m. 33 s. and 10 h. 0 m. 37 s., and their declinations $=8° 22' 30''$ S. and $12° 40' 34''$ N. $=75° 45' 44{\cdot}7''$.

873. **Problem XXV.—Given the latitude of the place, the declination and altitude of a celestial body, to find its azimuth.**

Let HZR be the meridian of the place, HR the horizon, and Z the zenith; EQ the equator, and P its pole; S the body; then



ZS = Z = zenith distance or coaltitude,

PS = P = body's polar distance or codeclination,

PZ = C = colatitude, and angle

PZS = A = supplement of azimuth AZR from south.

Here P, C, Z are given; and hence A can be found by Art. 769; thus, if $S=\frac{1}{2}(P+C+Z)$, then

$$2L \cos \tfrac{1}{2}A = L \sin S + L \sin (S-P) + L \operatorname{cosec} C + L \operatorname{cosec} Z - 20.$$

If the body's declination is south, as at S′, while the given latitude is north, the polar distance $PS' = 90° + DS'$.

Example.—When, in latitude $=44° 12'$ N., the sun's altitude was $=36° 30'$, its declination being $=15° 4'$ N., what was its azimuth?

Here	P = 74° 56′		L, sin S	= 9·9994498
	C = 45 48		L, sin (S − P) . .	= 9·3243657
and	Z = 53 30		L, cosec C	= ·1445350
	2)174 14		L, cosec Z	= ·0948213
Hence	S = 87 7			2)19·5631718
And	S − P = 12 11		L, cos ½A, ...	= 9·7815859

Hence $\frac{1}{2}A = 52° 47' 17{\cdot}3''$, and $A = 105° 34' 34{\cdot}6''$ = the azimuth from the north.

Exercises

1. When, in latitude $=48°\ 51'$ N., the sun's declination is $=18°\ 30'$ N., and its altitude $=52°\ 35'$, what is its azimuth from the north? . . . . $=134°\ 36'\ 7''$.

2. If, in latitude $=51°\ 32'$ N., the altitude of Arcturus was found to be $=44°\ 30'$, when its declination was $=20°\ 16'$ N., what was its azimuth from the north? . . . . $=117°\ 8'\ 28''$.

3. When, in latitude $=51°\ 32'$ N., the sun's altitude was $=25°$, and its declination $=4°\ 47'$ S., what was its azimuth from the north? . . . . $137°\ 17'\ 37''$.

METHODS OF DETERMINING TIME

874. **Problem XXVI.—Given the latitude of the place, the sun's declination and altitude, to find the hour of the day in apparent time.**

The angle P (last fig.) in triangle SPZ is evidently the horary angle. If this angle be denoted by H, then (Art. 769)

$$2\ \mathrm{L}\cos\tfrac{1}{2}\mathrm{H}=\mathrm{L}\sin\mathrm{S}+\mathrm{L}\sin(\mathrm{S}-\mathrm{Z})+\mathrm{L}\operatorname{cosec}\mathrm{P}+\mathrm{L}\operatorname{cosec}\mathrm{C}-20.$$

Example.—On the 8th of May 1854, at 5 h. 30 m. 32 s. P.M. per watch, in latitude $=39°\ 54'$ N., and longitude $=80°\ 39'\ 45''$ W., the altitude of the sun's lower limb was observed to be $15°\ 40'\ 57''$; required the error of the watch.

Given time	=	5 h.	30 m.	32 s.
Longitude in time	=	5	22	30
Greenwich time	=	10	53	11

1. To find the sun's declination

Sun's registered declination on 8th . .	=	17°	4′	36″
" " " 9th . .	=	17	20	45
Increase in 24 h.	=	0	16	9
Hence increase in 10 h. 53 m. 11 s. . .	=	0	7	19
Required declination	=	17	11	55

2. To find the sun's true altitude

Observed altitude	=		15°	40′	57″
Refraction	=	−	0	3	21
Semi-diameter	=	+	0	15	52
Contraction	=	−	0	0	3
Parallax	=	+	0	0	8
True altitude of centre	=		15	53	33

3. To find the horary angle

$Z = 74° \ 6' \ 27''$	L, sin S	=	9·9951982
$P = 72 \ 48 \ 5$	L, sin (S − Z) . .	=	9·6160088
$C = 50 \ 6 \ 0$	L, cosec P	=	·0198668
2)197 0 32	L, cosec C	=	·1151111
$S = 98 \ 30 \ 16$			2)19·7461849
$S - Z = 24 \ 23 \ 49$	L, cos $\frac{1}{2}$H	=	9·8730924

Hence $\frac{1}{2}H = 41° \ 42' \ 9{\cdot}5''$,

and $H = 5$ h. 33 m. 37·28 s. = apparent time.

Time by watch = 5 30 32

Watch slow by 0 3 5·28 for " "

Exercises

1. In latitude = 52° 12′ 42″ N., in the afternoon, the true altitude of the sun's centre was = 39° 5′ 28″, when its declination was = 15° 8′ 10″ N.; what was the apparent time of observation?

= 2 h. 56 m. 42·7 s.

2. In latitude = 24° 30′ N., in the forenoon, the true altitude of the sun's centre was found to be 33° 20′, when its declination was = 6° 47′ 50″ S.; required the apparent time of observation.

= 8 h. 45 m. 57·4 s. A.M.

875. **Problem XXVII.—Given the latitude and longitude of the place, the right ascension and declination of a fixed star and its altitude, to find the mean time.**

Rule.—Find the horary distance of the star from the meridian; then find the sun's mean right ascension at the preceding mean noon at the given place, and subtract it from the star's right ascension, increased if necessary by 24 hours; to this interval apply the horary angle by addition or subtraction, according as the star is west or east of the meridian, and the result is the sidereal interval from mean noon, and its corresponding interval of mean time will be the required time.

As in last problem,

$$2L \cos \tfrac{1}{2}H = L \sin S + L \sin (S - Z) + L \operatorname{cosec} P + L \operatorname{cosec} C - 20.$$

Let A = sun's mean R.A. at preceding noon at place,

A′ = star's right ascension,

d = the difference of A and A′,

s = " sidereal interval from mean noon, and m, r the corresponding mean time and retardation;

then $d = A' - A$, $s = d \pm H$, and $m = s - r$.

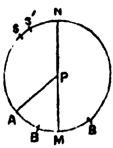
EXAMPLE.—At a place in latitude $=48° 56'$ N., and longitude $=66° 12'$ W., the true altitude of Aldebaran, which was west of the meridian, was $=22° 24'$ on the 10th of February 1854; what was the mean time of observation?

Star's dec. $=16° 12' 43''$ N, and R.A. $=4$ h. 27 m. 33·3 s.

$Z =$	$67° 36' 0''$	L, sin S	$=$	9·9999004
$P =$	73 47 17	L, sin (S − Z)	$=$	9·6029108
$C =$	41 4 0	L, cosec P	$=$	·0176222
	2)182 27 17	L, cosec C	$=$	·1824765
$S =$	91 13 38			2)19·8029099
$S - Z =$	23 37 38	L, cos $\frac{1}{2}$H,	$=$	9·9014549

Hence $\frac{1}{2}H = 37° 9' 21{\cdot}7''$, and $H = 4$ h. 57 m. 14·9 s.

			h.	m.	s.
Sun's reg. R.A. or sidereal time on 10th		$=$	21	20	56·6
Acceleration for long. 4 h. 24 m. 48 s. W.		$= +$	0	0	43·5
Sun's R.A. for noon at place,	A	$= +$	21	21	40·1
Star's " " "	A′	$=$	4	27	33·3
Hence	$d = A' - A$	$=$	7	5	53·2
and	$S = d + H$	$=$	12	3	8·1
and retardation		$= -$	0	1	58·5
Mean time required		$=$	12	1	9·6



Let A be the first point of Aries, MP the meridian of the place, B the place of the star, and S of the mean sun at time of observation. Then, if since noon the sun's increase of mean right ascension be SS′, the sidereal interval at noon between the sun and star is $=$ BS′, and as B has passed the meridian by the horary arc BM, the sidereal time from noon till the observation is expressed by $S'M = S'B + BM = d + H = A' - A + H$. And when the star is east of the meridian when observed, as at B′, the sidereal time from noon $= S'M = S'B' - B'M = A' - A - H$. And this interval, reduced to mean time, gives that required.

EXERCISES

1. If, at a place in latitude $=53° 24'$ N., and longitude $=25° 18'$ W., the altitude of Coronæ Borealis when east of the meridian was found to be $=42° 8' 50''$ on the 31st of January 1841, its right ascen-

sion being = 15 h. 27 m. 58 s., its declination = 27° 15′ 12″ N., and the registered mean right ascension of the sun = 20 h. 42 m. 8 s., find the mean time of observation.

H = 3 h. 40 m. 19·3 s., and the mean time = 15 h. 2 m. 45·8 s.

2. Find the hour of observation in mean time at which the altitude of Procyon was = 28° 10′ 13″, when east of the meridian in latitude = 7° 45′ N., its declination being = 5° 41′ 52″ S., its right ascension = 7 h. 29 m. 30 s., and that of the mean sun at mean noon = 11 h. 4 m. 40 s.

H = 4 h. 2 m. 0·5 s., and time = 16 h. 20 m. 8·5 s.

876. The **equation of equal altitudes** is a correction, generally of a few seconds (and seldom exceeding half-a-minute), that must be applied to the middle time between the instants of two observations at which the sun has equal altitudes in the forenoon and afternoon. It depends on the change of the sun's declination in the interval between the observations.

877. **Problem XXVIII.—To find the equation of equal altitudes.**

Let L = the latitude of the place,
I = " interval of time expressed in degrees, &c.,
P = " sun's polar distance,
v' = variation of declination in 24 hours in seconds,
v = " " interval I in seconds,
E = the equation of equal altitudes in seconds.

Then an arc θ, called arc first, is such that

$$\text{L, tan } \theta = \text{L, cot L} + \text{L, cos } \tfrac{1}{2}\text{I} - 10;$$

and if ϕ is another arc, called arc second, then $\phi = \text{P} - \theta$.

And $$\text{L, E} = \text{L, cot } \tfrac{1}{2}\text{I} + \text{L, cosec } \theta + \text{L, cosec P} + \text{L, sin } \phi + \text{L, I} + \text{L, } v' + \bar{5}\text{·}3645,$$

in which the quantity I in L, I is expressed in minutes. The logarithms require to be carried only to four places. This rule is approximate, but it will give the result correct to a small fraction of a second. The polar distance at the nearest noon may be used, as any small change in it or in the latitude produces a very small effect on the equation.

Example.—Find the equation of equal altitudes for an interval of 7 h. 45 m. 30 s., and latitude = 46° 30′ S., on the meridian of Greenwich, the sun's declination being = 7° 10′ N.

Here L = 46° 30′ S., I = 7 h. 45 m. 30 s., $\tfrac{1}{2}$I° = 58° 11′.
P = 97° 10′ N., I m. = 465·5 m., v' = 22′ 14″ = 1334″.

L, cot L . .	= 9·97725	Constant . .	= $\overline{45}$·3645
L, cos $\frac{1}{2}$I°. .	= 9·72198	L, cot $\frac{1}{2}$I° . .	= 9·7927
L, tan θ, 26° 35′	= 9·69923	L, cosec θ . .	= 10·3492
P	= 97 10	L, cosec P . .	= 10·0034
ϕ	= 70 35	L, sin ϕ . .	= 9·9746
		L, I m. . .	= 2·6679
		L, v' . . .	= 3·1251
		L, E 18·9 s. . .	= 1·2774

Hence the equation of equal altitudes is 18·9 s.

It is evident that when the declination of the sun has varied in one direction during the interval between two equal altitudes, the intervals between the meridian passage, or apparent noon, and the instants of the two observations are different. When it increases, the interval in the afternoon will exceed that of the forenoon, and conversely when it diminishes. For a demonstration of the rule, see Riddle's treatise on Navigation and Nautical Astronomy. A slight alteration has been made here which improves it a little. Instead of L, $\frac{1}{30}v$, where v is the variation due for the interval I, and which requires v to be separately calculated, there has been introduced above the constant $\bar{5}$·3645, L, I m. and L, v'.

For $\quad 24 : \mathrm{I} = v' : v$; and hence L, v = L, I + L, v' − L, 24.

And if v', v are in seconds of space, and 24 and I m. in minutes of time, then, since 24 h. = 1440 m., and L, $\frac{1}{30}v$ = L, v − L, 30; therefore, $\quad$ L, $\frac{1}{30}v$ = L, I m. + L, v' + $\bar{5}$·3645.

Exercise

If at a given place, when the sun's declination at noon was = 17° 54′ N., the sun had equal altitudes at an interval of 5 h. 40 m. 6 s., the latitude of the place being = 57° 10′, what was the equation of equal altitudes? . . . . = 14·36 s.

878. The **middle time** for the times of observation of two equal altitudes of the sun is half the sum of the times.

879. Problem XXIX.—To find the time by equal altitudes of the sun.

Rule.—Apply the equation of equal altitudes to the middle time by addition or subtraction, according as the polar distance is increasing or diminishing, and the result is the time shown by the

clock at apparent noon; find the mean time at apparent noon, and the difference between it and the preceding time will be the error of the clock.

When a chronometer is used for the times of observation, apply the longitude in time to the mean time at apparent noon, and the result is the mean time at Greenwich at that instant; and the difference between it and the time found by the chronometer for the same instant will be the error of the chronometer.

EXAMPLE.—At a given place the altitude of the sun was the same at 9 h. 34 m. 20 s. A.M., and 2 h. 32 m. 26 s. P.M.; required the error of the clock, the polar distance being decreasing, the equation of equal altitudes = 8·4 s., and the equation of time = 1 m. 57·6 s. to be added to apparent time.

Time of first observation . . .	=	21 h.	34 m.	20 s.
" second " . . .	=	2	32	26
		24	6	46
Middle time of observation . . .	=	12	3	43
Equation of equal altitudes . .	= −	0	0	8·4
Time by clock at apparent noon . .	=	12	3	34·6
Mean time " " " . .	=	12	1	57·6
Clock is fast " " " . .	=	0	1	37

Instead of only two observations being taken, several corresponding pairs may be taken, and the sum of the times of observation in the forenoon being divided by their number, and also the sum of the afternoon observations being similarly divided, the quotients are the mean times of observation, which are then to be treated as the two times of observation. The times of observation ought to be more than two hours distant from noon.

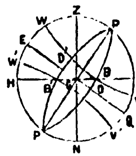
EXERCISES

1. At a given place the altitude of the sun was the same at 8 h. 4 m. 54 s. and 4 h. 2 m. 36 s.; required the error of the clock, the polar distance being increasing, the equation of equal altitudes = 12·4 s., and the equation of time = 4 m. 16·7 s. to be subtracted from apparent time. . . . . . . Clock fast 8 m. 14·1 s.

2. Suppose that at a given place the altitude of the sun was the same at 9 h. 40 m. 2 s. A.M. and 2 h. 10 m. 25 s. P.M.; required the error of the clock, the polar distance being decreasing, the equation of equal altitudes = 14·5 s., and the equation of time = 3 m. 50·2 s., to be added to apparent time. . . . Clock slow 8 m. 51·2 s.

880. **Problem XXX.—Given the latitude of the place and the declination of a celestial body, to find its amplitude and ascensional difference.**

Let HZR be the meridian of the place; EQ the equator, P its pole; HR the horizon, Z the zenith, and B the body in the horizon when rising; then in the triangle ODB, D is the right angle, O is the colatitude, BD the declination, OB the amplitude, OD the ascensional difference, and EO+OD the semi-diurnal arc. When the declination is south, OB′ is the amplitude, OD′ the ascensional difference, and EO−OD′ the semi-diurnal arc. When the body is setting, the figure is exactly similar.



It is evident that if any two parts of the right-angled triangle OBD are given, the other parts can be found.

Let angle BOD = C the colatitude,
BD = D „ declination,
OB = M „ amplitude,
OD = N „ ascensional difference,
BW = I „ semi-diurnal arc.

To find the amplitude OB

R . sin BD = sin O . sin OB, or sin C : R = sin D : sin M.

To find the ascensional difference OD

R . sin OD = cot O . tan DB, or R : tan D = cot C : sin N.

Then I = 6 h. ± N.

Example.—When the declination of a celestial body is = 14° 15′ N., what is its amplitude and ascensional difference in latitude = 36° 45′ N.?

Here C = 53° 15′, D = 14° 15′.

To find the amplitude M

L, sin C	=	9·9037701
L, radius	=	10·
L, sin D	=	9·3912057
L, sin M	=	9·4874356

And M = 17° 53′ 28″ N.

To find N

L, radius	=	10·
L, tan D	=	9·4047784
L, cot C	=	9·8731668
L, sin N	=	9·2779452

And N = 10° 55′ 56″.

The semi-diurnal arc I = 6 h. + N = 6 h. 43 m. 43·7 s.

Exercises

1. The declination of a celestial body is $=37° 34'$ N.; what is its amplitude and ascensional difference in latitude $=46° 8'$ N.?

$M=61° 37' 4{\cdot}5''$ N., and $N=3$ h. 32 m. 36·4 s.

2. The declination of a celestial body is $=26° 3' 53''$ S.; what is its amplitude and semi-diurnal arc in latitude $=55°$ N.?

$M=50°$ S., and $I=3$ h. 2 m. 45·4 s.

881. **Problem XXXI.—To find the apparent time at which the sun's centre rises or sets at a given place.**

Find the sun's zenith distance when its centre appears on the horizon; then, its polar distance and the colatitude being known, find the corresponding semi-diurnal arc, and this arc, converted into time, will be the time of rising or setting before or after apparent noon.

The sun's zenith distance, when the apparent altitude of its centre is zero, is found by subtracting its parallax from the sum of the dip and horizontal refraction, and adding the remainder to 90°.

The declination to be used is of course that at rising and setting, which can be found by first determining the semi-diurnal arc, as in last problem, supposing the declination to be that at noon at the given place; and then the approximate times of rising and setting are known, and the longitude being also known, the reduced time, and hence also the reduced declination, can be found.

Example.—Find the mean time of the apparent rising of the sun's centre on the 24th of May 1841 at a place in latitude $=55° 57'$ N., and longitude $=25° 30'$ W., the observer's eye being at the height of 24 feet.

Approximate apparent time of rising on 23rd	=	15 h. 38 m.
Longitude in time	=	1 42
Reduced time of rising on 23rd	=	17 20
Hence reduced declination	=	20° 44′ 29″ N.
Horizontal refraction	=	0 33 51
Depression	=	0 4 47
Horizontal parallax	= −	0 0 9
Depression of centre	=	0 38 29

By (Art. 769), $2 \text{ L} \cos \frac{1}{2}H = \text{L} \sin S + \text{L} \sin (S - Z) + \text{L} \operatorname{cosec} P + \text{L} \operatorname{cosec} C - 20.$

Here Z =	90° 38′ 29″	L, sin S . . .	= 9·9967739
P =	69 15 31	L, sin (S − Z) . .	= 9·0426439
C =	34 3 0	L, cosec P . . .	= 10·0291009
	2)193 57 0	L, cosec C . . .	= 10·2518770
S =	96 58 30		2)19·3203957
S − Z =	6° 20′ 1″	L, cos $\frac{1}{2}$H . . .	= 9·6601978
And $\frac{1}{2}$H =	62 47 8	And H = 8 h. 22 m. 17·8 s.	

Hence apparent time of rising on 24th is	3 h.	37 m.	42·2 s. A.M.
Equation of time . . . . =	− 0	3	30·9
Mean time of rising . . . =	3	34	11·3

Exercise

Find the mean time of the setting of the sun on the 20th of July 1841 in longitude = 35° 45′ E., and latitude = 55° 57′ N., the eye of the observer being 20 feet high, its registered declination on the 20th and 21st being = 20° 40′ 38″ and 20° 29′ 12″, and the equations of time = 5 m. 58·7 s. and 6 m. 2·1 s. . . At 8 h. 27 m. 9·3 s.

882. The time of a star's rising or setting may be found thus:— Compute the star's semi-diurnal arc, and it will be the sidereal interval from its rising to its culmination, which is to be reduced to the mean solar interval by Art. 858; then find the mean time of the star's culmination by Art. 865, and apply to it the preceding interval by subtraction for the mean time of rising, and by addition for the time of setting.

The time of the moon's rising or setting may be found thus:— Find approximately its semi-diurnal arc, considering its declination and horizontal parallax to be that at the nearest noon; and find the time of its meridian passage; then the approximate time of its rising or setting is known. Compute its declination and parallax for the reduced approximate time of rising, and find again its semi-diurnal arc; then 24 hours is to the semi-diurnal arc found as the daily retardation to a fourth term, which, added to the preceding arc, will give the interval between the rising or setting and the meridian passage in mean time. For a sidereal day is to any sidereal arc as a lunar day (expressed in mean time) is to the corresponding lunar arc (expressed also in mean time). Then the sum or difference of this interval and the time of transit will be the time of rising or setting.

Let H = semi-diurnal arc in sidereal time,
H′ = corresponding lunar arc in mean time,
R = moon's daily retardation " "
r = " retardation for arc H′ in mean time,
t' = the mean time of transit,
t = " " rising or setting;

then $24 : R = H : r$, or P.L, r = P.L, R + P.L, H,

and $H' = H + r$; then $t = t' \pm H'$,

where the upper sign refers to the time of setting, and the lower to the time of rising.

The same method applies in finding the rising or setting of the planets; but when $v' < v$, or when v' is negative (Art. 868), r is negative, and $H' = H - r$.

Exercise

Find the mean time of the rising of the moon for the data of the example in Art. 867; having also given the moon's declination on the 2nd of May at noon = 26° 21′ 52·9″ N., its horizontal parallax on the 2nd at noon and midnight = 54′ 9·5″ and 54′ 10·8″, and its declination on the 1st at 21 h. = 26° 21′ 49″, and at 22 h. = 26° 21′ 58″, the horizontal refraction being = 33′ 50″, and the latitude of the place = 54° 30′ S. . . = 7 h. 1 m. 46·7 s. A.M.

METHODS OF FINDING THE LATITUDE

883. **Problem XXXII.—Given the declination of a celestial body, and its meridian altitude, to find the latitude of the place of observation.**

Call the true zenith distance of the object *north* or *south*, according as the zenith is north or south of the body; then, when the zenith distance and declination are of the same name, their sum is the latitude also of the same name; but when of different names, their difference is the latitude of the same name as the greater.

When the body is at the lower culmination, the latitude is equal to the sum of the altitude and polar distance, and is of the same name as the latter.

That is, $L = Z \pm D$, or $L = A' + P$, where A′ is the altitude for the lower culmination.

Example.—On the 2nd of May 1854, in longitude = 50° 15′ W., the observed meridian altitude of the sun's lower limb was 70° 31′ 18″ S., the height of the eye being = 15 feet; what is the latitude?

Observed altitude,	A'' =	70° 31′ 18″ S.
Depression,	d = −	0 3 45
		70 27 33
Refraction,	r = −	0 0 20
True altitude of lower limb.	=	70 27 13
Sun's semi-diameter	= +	0 15 53
Parallax	= +	0 0 3
True altitude of centre,	A =	70 43 9
Longitude 50° 15′ W. = 3 h. 21 m.		
Registered declination on 2nd May	=	15° 21′ 52·2″ N.
Increase in 3 h. 21 m.	=	0 2 28·9
Declination at given time,	D =	15 24 21·1 N.
Zenith distance,	P =	19 16 51
Latitude,	L =	34 41 12·1 N.

The principle of the rule is easily explained by a reference to the figure in Art. 869. Let B be the body, then the zenith distance ZB and the declination BE are of the same name, whether P be the south or the north pole; and the latitude EZ is their sum, and of the same name. If B″ be the body, then the zenith distance B″Z and declination B″E are of different names, and the latitude EZ = B″Z − B″E. So when B′ is the body, Z and D are of different names, and EZ = EB′ − ZB′. Let b' be the body at the lower culmination, A′ = b'N, and P = b'P; then L = A′ + P.

Exercises

1. If the true altitude of Aldebaran, at a place in longitude = 48° 30′ W., on the 20th of May 1854, was = 54° 20′ 35″ S., required the latitude, the star's declination being = 16° 12′ 45″ N.

= 51° 52′ 10″ N.

2. If the meridian altitude of the moon's centre on the 2nd of May 1841, in longitude = 40° 45′ W., was = 25° 13′ 45·2″ S., when its declination was = 8° 26′ 3·8″ S., what was the latitude of the place?

= 56° 20′ 10″ N.

884. Problem XXXIII.—Given the sun's declination and altitude, and the hour of the day, to find the latitude of the place.

Let the parts of the triangle PZS in Art. 873 represent the same quantities as in that problem; then the polar distance P = PS, the zenith distance Z = ZS, and the horary angle H = SPZ are given, to find the colatitude C = PZ.

The side PZ can be obtained by the method in Art. 776, or, more concisely, by that in Art. 781. In it the quantities a, b, A, and c are respectively the same as Z, P, H, and C in this problem. Hence $\quad R \cdot \tan\theta = \tan P \cdot \cos H$,

and $\quad \cos P : \cos Z = \cos\theta : \cos(c \sim \theta)$.

The algebraic signs of the terms indicate whether these arcs are greater or less than quadrants.

EXAMPLE.—On the 8th of May 1854, at 5 h. 33 m. 33·4 s. apparent time, the altitude of the sun's lower limb was observed to be $=15°\ 40'\ 57''$, the longitude of the place being $=80°\ 39'\ 45''$ W.; what was the latitude?

The declination of the sun at the time of observation is found to have been $=17°\ 12'$, and the true altitude of its centre $=15°\ 53'\ 37''$; hence $\quad P=72°\ 48'$, $Z=74°\ 6'\ 23''$, and $H=83°\ 23'\ 21''$.

To find the segment θ		To find the segment $(c-\theta)$	
L, radius . . =	10·	L, cos P . . =	9·4708631
L, cos H . . =	9·0611695	L, cos Z . . =	9·4375159
L, tan P . . =	10·5092668	L, cos θ . . =	9·9718687
L, tan θ . . =	9·5704363		19·4093846
Hence $\theta = 20°\ 24'\ 2''$.		L, cos $(c-\theta)$. =	9·9385215
		Hence $(c-\theta)$. =	29° 46′ 21″
		θ =	20 24 2

Therefore the colatitude, or . . . . . C = 50 10 23

And therefore the latitude, ... L = 39 49 37

EXERCISES

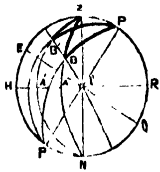
1. At a given place in south latitude, when the sun's declination was $=15°\ 8'\ 10''$ S., its true altitude was $=39°\ 5'\ 28''$ at 2 h. 56 m. 42·7 s. P.M.; required the latitude of the place. $=52°\ 12'\ 42''$.

2. At a place in north latitude, when the sun's declination was $=6°\ 47'\ 50''$ S., its true altitude was $=33°\ 20'$ at 8 h. 46 m. A.M.; required the latitude. . . . . $=24°\ 31'\ 10{\cdot}3''$

885. Problem XXXIV.—Given two altitudes of the sun or of a star, and the interval of time between the observations; or the altitudes of two known stars, taken at the same instant, to find the latitude of the place.

Let P be the pole, Z the zenith, and B, B′ the body in two different positions, or two different bodies. Then PB, PB′ are the polar distances, and ZB′, ZB the zenith distances; these four quantities being given. Also, when B and B′ are the sun in two

different positions on the same day, or of a star on the same night, or of two different stars at the same instant, the angle BPB′, which measures the elapsed time, or the difference of right ascensions of two different stars, is known. But in the second case, when B, B′ are two different stars, and the elapsed time between the observations is measured in mean time, it must be reduced to sidereal time. Hence the latitude may be found thus :—



Let P, P′ = the polar distances PB, PB′,
Z, Z′ = " zenith " BZ, B′Z,
H = " angle BPB′,
E = " side BB′,
and L, C = " latitude and colatitude ZE and ZP.

1. To find angle B′ in triangle PBB′

By Art. 774, $\sin \frac{1}{2}(P+P') : \sin \frac{1}{2}(P \sim P') = \cot \frac{1}{2}H : \tan \frac{1}{2}(B \sim B')$,
and $\cos \frac{1}{2}(P+P') : \cos \frac{1}{2}(P \sim P') = \cot \frac{1}{2}H : \tan \frac{1}{2}(B+B')$.
From which B and B′ can be found.

2. To find E in triangle PBB′

$$\operatorname{Sin} \tfrac{1}{2}(B \sim B') : \sin \tfrac{1}{2}(B+B') = \tan \tfrac{1}{2}(P \sim P') : \tan \tfrac{1}{2}E.$$

3. To find angle B′ in triangle BZB′

By Art. 769, $2\ L \cos \frac{1}{2}B' = L \sin S + L \sin (S - Z) + L \operatorname{cosec} E + L \operatorname{cosec} Z' - 20$.

4. To find angle B in triangle PB′Z

$$B' = BB'Z \sim PB'B.$$

5. To find C in triangle PB′Z

By Art. 779, $R : \cos B' = \tan Z' : \tan \theta$,
and $\cos \theta : \cos (P' - \theta) = \cos Z' : \cos C = \sin L$.

When B and B′ are the same star at the times of the two observations, P is $=$ P′; and when they represent the sun, if its declination for the middle time between the observations be taken, PB and PB′ may be considered equal to this declination P. The solution may then be simplified, for PBB′ will be an isosceles triangle, and a perpendicular from P on BB′ will bisect it, and will form two equal right-angled triangles. Hence, instead of the preceding formulæ at No. 1 and 2, take these two :—

1. To find B′ in one of the right-angled triangles

$$\operatorname{Cot} \tfrac{1}{2}H : R = \cos P : \cot B'.$$

2. To find ½E in the same triangle

R : sin P = sin ½H : sin ½E.

EXAMPLE.—If in the forenoon, when the sun's declination was = 19° 30′ N., at the middle time between two observations of its altitude, these altitudes corrected were = 38° 19′ and 50° 25′, what was the latitude, the place of observation being north, and the interval between the observations one hour and a half?

Here P = 70° 21′, Z = 51° 41′, and Z′ = 39° 35′.

H = 1 h. 30 m., and ½H = 11° 15′, and PZ = C.

1. To find angle B′ in PBB′

L, cot ½H . .	=	10·7013382
L, radius . .	=	10·
L, cos P . .	=	9·5266927
L, cot B′ . .	=	8·8253545

And B′ = 86° 10′ 24″.

2. To find E in PBB′

L, radius . .	=	10·
L, sin P . .	=	9·9739422
L, sin ½H . .	=	9·2902357
L, sin ½E . .	=	9·2641779

And E = 21° 10′ 26″.

3. To find angle B′ in BZB and PB′Z

Z . .	=	51° 41′ 0″		
L, cosec Z′ . .	=	39 35 0 . . .	=	10·1957243
L, cosec E . .	=	21 10 26 . . .	=	10·4422527
		2)112 26 26		
L, sin S . . .	=	56 13 13 . . .	=	9·9196958
L, sin (S − Z) . .	=	4 32 13 . . .	=	8·8981869
				2)19·4558597
L, cos ½B′, . .	=	57° 41′ 29·4″ . . .	=	9·7279298
		2		
B′ . .	=	115 22 59		
PB′B . .	=	86 10 24		
4. PB′Z . .	=	29 12 35		

5. To find C in PB′Z

L, radius . .	=	10·
L, cos B′ . .	=	9·9409342
L, tan Z′ . .	=	9·9173911
L, tan θ . .	=	9·8583253

And θ = 35° 48′ 58″.

P − θ = 34 32 2.

L, cos θ ($a.c$)* .	=	0·0910331
L, cos (P − θ) .	=	9·9158171
L, cos Z′ . .	=	9·8868846
L, sin L . .	=	9·8937348

And L = 51° 31′ 54″.

* Here ($a.c$) means the arithmetical complement of L. cos θ (see Table of 'Numbers of Frequent Use in Calculation,' p. 620).

The following method is somewhat simpler when only one star is observed, or when the mean declination of the sun is used, as in the last example.*

Let P, H, and L denote the same quantities as in the preceding rule; let $S = A + A'$, the sum of the true altitudes,

$D = A \sim A'$ " difference of the true altitudes;

and let M, N, O, Q, R denote what are called the first, second, third, fourth, and fifth arcs; then,

1. $\text{L sin M} = \text{L sin P} + \text{L sin } \tfrac{1}{2}\text{H} - 10.$
2. $\text{L cos N} = \text{L cos P} + \text{L sec M} - 10.$

And N is of the same species as P.

3. $\text{L sin O} = \text{L sin } \tfrac{1}{2}\text{D} + \text{L cos } \tfrac{1}{2}\text{S} + \text{L cosec M} - 10.$
4. $\text{L cos Q} = \text{L cos } \tfrac{1}{2}\text{D} + \text{L sin } \tfrac{1}{2}\text{S} + \text{L sec M} + \text{L sec O} - 10.$
5. $\text{R} = \text{N} \pm \text{Q}.$

When the zenith and elevated pole are on the same side of the great circle that passes through the two positions of the sun or star, $R = N \sim Q$; otherwise $R = N + Q$.

6. $\text{L sin L} = \text{L cos O} + \text{L cos R} - 10.$

The preceding example, computed by this rule, is added here:—

$P = 70° 21'$	$A = 38° 19'$	$\tfrac{1}{2}S = 44° 22'$
$\tfrac{1}{2}H = 11\ \ 15$	$A' = 50\ \ 25$	$\tfrac{1}{2}D = 6\ \ 3$

1. To find M

L, sin P . . =	9·9739422
L, sin ½H . . =	9·2902357
L, sin M . . =	9·2641779

M = 10° 35′ 13″.

2. To find N

L, cos P . . =	9·5266927
L, sec M . . =	10·0074565
L, cos N . . =	9·5341492

N = 69° 59′ 44″.

3. To find O

L, sin ½D . . =	9·0228254
L, cos ½S . . =	9·8542329
L, cosec M . =	10·7358221
L, sin O . . =	9·6128804

O = 24° 12′ 38″.

4. To find Q

L, cos ½D . . =	9·9975743
L, sin ½S . . =	9·8446310
L, sec M . . =	10·0074565
L, sec O . . =	10·0399840
L, cos Q . . =	9·8896458

Q = 39° 8′ 22″.

5. $R = 30° 51' 22'' = N \sim Q.$

6. To find L

L, cos O . . . . . . =	9·9600160
L, cos R . . . . . . =	9·9337191
L, sin L . . . . . . =	9·8937351

And L = 51° 31′ 54″, as before.

* For the demonstration of this rule, consult any good treatise on Navigation and Nautical Astronomy.

In this problem it is not necessary to know the times of observation, but merely the interval between them; but the latitude being found, the time of either observation could be calculated, and if the time was taken with a chronometer, the longitude could also be found.

Exercises

1. At a place in north latitude, when the sun's declination was $=23° 29'$ N., its true altitude at 8 h. 54 m. A.M. was $48° 42'$, and at 9 h. 46 m. A.M. it was $=55° 48'$; required the latitude.

$=49° 49' 23''$ N.

2. At a place in north latitude, when the sun's declination was $=2° 46'$ S., its true altitude was $=33° 11'$ at 9 h. 20 m. A.M. and $42° 44'$ at 1 h. 20 m. P.M.; required the latitude. $=40° 50' 7''$ N.

3. Find the time at which the first altitude was taken in the example to this problem, and the azimuth.

The time $=8$ h. 30 m. 2 s., and the azimuth from N. $=107° 47' 42''$.

4. On the 6th of October 1830, at a place in north latitude, the true altitude of the sun at 7 h. 5 m. 49 s. A.M. mean time was $=8° 37' 42{\cdot}6''$, and at 1 h. 2 m. 47·8 s. it was $=33° 43' 46{\cdot}1''$, and the sun's declination for the middle time was $=5° 9' 48{\cdot}1''$ S.; required the correct mean time of the first observation, and the latitude, the equation of time to be subtracted from apparent time being $=11$ m. 42·1 s.

The latitude $=48° 42' 42{\cdot}9''$ N.; the time $=7$ h. 5 m. 39·8 s.

In the following example the first method must be employed, as the polar distances are different. Also, as Atair is to the east of Arcturus, and the altitude of the former was taken some minutes later than that of the latter, this elapsed time, converted to sidereal time, must be subtracted from the difference of their right ascensions in order to obtain the angle H.

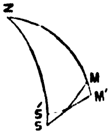
On the 19th of September 1830 the zenith distance of Arcturus was found to be $=73° 19' 26{\cdot}5''$ at 8 h. 2 m. 47·8 s. mean time, and that of Atair was $=40° 53' 56{\cdot}3''$ at 8 h. 22 m. 3 s.; the polar distance of the former was $=69° 55' 36{\cdot}4''$, and that of the latter $=81° 34'$; required the latitude, and the correct time of the first observation, the sun's mean right ascension at mean noon being $=11$ h. 51 m. 29·76 s.

The latitude $=48° 42' 12''$, and the time $=8$ h. 4 m. 19·6 s.

LUNAR DISTANCES

886. **Problem XXXV.—To find the true angular distance between the moon and the sun or a star, having given their altitudes and apparent distances.**

Let M′, S′ be the apparent places of the centres of the moon and the sun, or a star, or a planet, and M, S their true places, and Z the zenith; then M will be above M′, because the moon's parallax exceeds the refraction due to its height; but S will be below S′, in consequence of the refraction exceeding the parallax of the body. M′S′ is the apparent distance, and MS the true distance.



Let h = the apparent height of the moon's centre = the complement of M′Z,

h' = the apparent height of the centre of sun or star = the complement of S′Z,

H = the true height of the moon's centre = the complement of MZ,

H′ = the true height of the centre of sun or star = the complement of SZ,

d = the apparent distance of the centres = S′M′,

D = the true distance of the centres = SM,

s = $h+h'$, and S = H + H′.

Then, by Spherical Trigonometry (Art. 748, d), we have

$$\cos Z = \frac{\cos d - \sin h \,.\, \sin h'}{\cos h \,.\, \cos h'} = \frac{\cos D - \sin H \,.\, \sin H'}{\cos H \,.\, \cos H'};$$

$$\therefore 1 + \frac{\cos d - \sin h \,.\, \sin h'}{\cos h \,.\, \cos h'} = 1 + \frac{\cos D - \sin H \,.\, \sin H'}{\cos H \,.\, \cos H'},$$

or
$$\frac{\cos d + \cos s}{\cos h \,.\, \cos h'} = \frac{\cos D + \cos S}{\cos H \,.\, \cos H'};$$

hence
$$\frac{2 \cos \tfrac{1}{2}(s+d) \cos \tfrac{1}{2}(s \sim d)}{\cos h \,.\, \cos h'} = \frac{\cos D + \cos S'}{\cos H \,.\, \cos H'} \quad . \quad . \quad (a).$$

But
$$1 + \cos S = 2 \cos^2 \tfrac{1}{2}S,$$
and
$$1 - \cos D = 2 \sin^2 \tfrac{1}{2}D;$$
$$\therefore \cos D + \cos S = 2 \cos^2 \tfrac{1}{2}S - 2 \sin^2 \tfrac{1}{2}D.$$

Substituting in (a) the above value of cos D + cos S, and dividing both sides by 2, we have

$$\frac{\cos \tfrac{1}{2}(s+d) \cos \tfrac{1}{2}(s \sim d)}{\cos h \,.\, \cos h'} = \frac{\cos^2 \tfrac{1}{2}S - \sin^2 \tfrac{1}{2}D}{\cos H \,.\, \cos H'};$$

hence $\sin^2 \frac{1}{2}D = \cos^2 \frac{1}{2}S - \dfrac{\cos H . \cos H' . \cos \frac{1}{2}(s+d) . \cos \frac{1}{2}(s \sim d)}{\cos h . \cos h'}$

$$= \cos^2 \tfrac{1}{2}S\left(1 - \frac{\cos H . \cos H' \cos \frac{1}{2}(s+d) . \cos \frac{1}{2}(s \sim d)}{\cos h . \cos h' \cos^2 \frac{1}{2}S}\right);$$

assume $\quad \sin^2 \theta = \dfrac{\cos H . \cos H' . \cos \frac{1}{2}(s+d) . \cos \frac{1}{2}(s \sim d)}{\cos h . \cos h' . \cos^2 \frac{1}{2}S};$

then $\quad \sin \frac{1}{2}D = \cos \theta . \cos \frac{1}{2}S.$

Or logarithmically—

$$L \sin \theta = \tfrac{1}{2}\left\{\begin{matrix} L \cos H + L \cos H' + L \cos \frac{1}{2}(s+d) \\ + L \sec h + L \sec h' + L \cos \frac{1}{2}(s \sim d) \end{matrix}\right\}$$
$$- (L \cos \tfrac{1}{2}S + 10) \quad . \quad . \quad . \quad . \quad [1],$$

and then $\quad L \sin \frac{1}{2}D = L \cos \theta + L \cos \frac{1}{2}S - 10 \quad . \quad . \quad . \quad [2].$

EXAMPLE.—On the 14th of December 1818, at 12 h. 10 m. nearly, latitude = 36° 7′ N., longitude by account = 11 h. 52 m. W., the following observations were made, the height of the observer's eye being = 19·5 feet, in order to find the distance between the moon and Regulus.

Ob. dis. of moon's nearest l. and Regulus, d' = 33° 15′ 25″.

Ob. alt. moon's l. l. .	=	61° 26′ 12″		
Depression	=	− 4 18		
		61 21 54		
Moon's semi-diameter .	=	0 14 56		
	h =	61 36 50		
Refraction	=	0 0 31·1		
		61 36 18·9	L. sec.	= ·32274
Hor. par. 53′ 59″,			P.L.	= ·52301
Par. in alt.	=	0 25 40·5	P.L.	= ·84575
	H =	62 2 0 nearly.		

Observed altitude of Regulus.	=	28° 29′ 17″
Depression	=	−0 4 18
	h' =	28 24 59
Refraction	=	0 1 45
	H′ =	28 23 14
	d' =	33 15 25
Moon's semi-diameter	=	0 14 56
Hence	d =	33 30 21

Then L, sec h	61° 36′ 50″	= 10·3229308	
L, sec h'	28 24 59	= 10·0557580	
L, cos $\frac{1}{2}(s+d)$	61 46 5	= 9·6748997	
L, cos $\frac{1}{2}(s-d)$	28 15 44	= 9·9448723	
L, cos H	62 2 0	= 9·6711338	
L, cos H′	28 23 14	= 9·9443616	
		2)59·6139562	
		29·8069781	
L, cos $\frac{1}{2}$S + 10		= − 19·8478853	
L, sin θ = 65° 31′ 13″		= 9·9590928	
L, cos θ		= 9·6173895	$\left.\right\}$ − 10
L, cos $\frac{1}{2}$S		= 9·8478853	
L, sin $\frac{1}{2}$D = 16° 58′ 24·2″		= 9·4652748	

Hence D = 33 56 48·4.

The calculation of the time of observation, and of the longitude of the place, is performed in the example to the next problem.

Exercises

1. Given the apparent altitudes of the centres of the sun and moon = 32° 0′ 1″ and 24° 0′ 8″, their true altitudes = 31° 58′ 38″ and 24° 51′ 48″, and their apparent distance = 68° 42′ 15″; required their true distance. . . . . = 68° 19′ 34″.

2. Suppose that on the 6th of April 1821, in latitude = 47° 39′ N., and longitude = 57° 16′ W., by account, at 3 h. 56 m. P.M. per watch, it was found that the apparent altitudes of the centres of the sun and moon were = 26° 9′ 7″ and 46° 34′ 44″, their true altitudes = 26° 7′ 19″ and 47° 14′ 19″, and the apparent distance of their centres = 76° 0′ 7″; required the true distance and the true apparent time of observation, the sun's declination at the time being = 6° 32′ 12″ N.
The distance = 75° 45′ 43″, and time = 3 h. 51 m. 24 s.

3. If in longitude = 11° 15′ W. by account, at 3 h. 45 m. A.M. per watch, the apparent altitude of the centre of the moon was = 24° 29′ 33″, and that of Regulus = 45° 9′ 12″, and their true altitudes = 25° 16′ 50″ and 45° 8′ 15″, and the apparent distance of Regulus from the moon's centre = 63° 35′ 4″; required the true distance. = 63° 4′ 54″.

4. On the 9th of April 1837, at 5 h. 29 m. 36·8 s. mean time, suppose that the nearest limbs of the sun and moon were = 54° 30′ 12″ distant; that the apparent height of the sun's centre was = 21° 50′ 14″, and that of the moon's = 61° 10′ 10″; the moon being east of the sun, and both west of the meridian; the latitude by account was = 41° 47′ N., and the longitude = 2 h. 10 m. W.; the horizontal

equatorial parallax was $=55' \ 31 \cdot 1''$, and the semi-diameters of the sun and moon $=15' \ 59''$ and $15' \ 7 \cdot 8''$; required the true distance. $=55° \ 10' \ 43''$.

5. On the 6th of May 1840, at a place in latitude $=36° \ 40'$ N., and longitude by account $=39°$ W., at 7 h. 40 m. A.M., the apparent altitudes of the centres of the sun and moon were $=30° \ 33' \ 0 \cdot 2''$ and $53° \ 15' \ 40 \cdot 9''$, their true altitudes $=30° \ 31' \ 27 \cdot 5''$ and $53° \ 50' \ 31 \cdot 3''$, and the apparent distance of their centres $=62° \ 0' \ 9 \cdot 1''$; required their true distance. . . . . $=61° \ 52' \ 40 \cdot 8''$.

6. At a place in latitude $=10° \ 1' \ 50''$ N., and longitude by account $=30° \ 5'$ W. of Paris, on the 17th of December 1823 at 14 h. 59 m. 48·8 s. P.M., the apparent altitudes of the moon's centre and of Regulus were $=48° \ 0' \ 49''$ and $70° \ 34' \ 9''$, their true altitudes $=48° \ 40' \ 38''$ and $70° \ 33' \ 49''$, and their apparent distance was $=58° \ 25' \ 36''$; what was their true distance? . . . . $=57° \ 47' \ 12 \cdot 6''$.

7. Required the distance between the centres of the sun and moon from these data:—

Distance of nearest limbs of the two bodies .	$=83° \ 26' \ 46''$
Altitude of lower limb of sun	$=48 \ \ 16 \ \ 10$
" " upper " moon	$=27 \ \ 53 \ \ 30$
Semi-diameter of sun	$= \ 0 \ \ 15 \ \ 46$
" " moon, including augmen. .	$= \ 0 \ \ 15 \ \ \ 1$
Correction of sun's altitude, including dip . .	$= \ 0 \ \ \ 5 \ \ 27$
" " moon's " " . .	$= \ 0 \ \ 46 \ \ 43$

The dip being $=4' \ 24''$, the latitude $=10° \ 16' \ 40''$, and longitude by account $=149°$ E., and the observations taken on the 5th of June 1793, about 1 h. 30 m. P.M. . . . . . $=83° \ 20' \ 55''$.

THE LONGITUDE BY LUNAR DISTANCES

887. **Problem XXXVI.—Given the true angular distance between the moon and the sun or a star, and the time of observation, to find the longitude.**

The time, when not previously known, can be calculated by means of the altitude of one of the bodies, as in Art. 874.

The time at Greenwich can be found thus:—Take from the *Nautical Almanac* the two distances to which the given distance is intermediate; then the difference between the registered distances is to that between the first registered distance and the given distance as three hours is to a fourth term, which is to be added to the time of the first registered distance, to give the required time.

Then the difference between the time at the place and that found at Greenwich will be the longitude.

Let t' = time of the first registered distance,
d' = the difference between the two registered distances—that is, for intervals of three hours,
d = the difference between the first registered distance and the given distance,
t = the interval of time corresponding to d,
T = " time required at Greenwich;

then $d' : d = 3 \text{ h.} : t$; hence $t = \frac{3d}{d'}$, and $T = t' + t$.

Or, $L\,t = L\,3 + L\,d - L\,d'$, or $P.L\,t = P.L\,d - P.L\,d'$.

Since the moon moves over 360° in about 30 days, therefore an error of 10″ in measuring the lunar distance will cause an error of about 5′ on the longitude. For

$$360° : 10'' = 30 \text{ d.} : x, \text{ and } x = \frac{1 \text{ m.}}{3} = \frac{15'}{3} = 5'.$$

When only the first differences of the lunar distances are taken, the result will be a few seconds of time wrong. Thus, in the fourth of the following exercises, the correction found, when the second differences are used, is 5·8 s., or about 1·4′; the correct longitude being = 2 h. 0 m. 16·9 s.

EXAMPLE.—Find the time of observation and the longitude of the place of observation from the data of the example to the preceding problem.

1. To find the time at Greenwich

Distance at 0 h. . .	= 33° 58′ 7″	33° 58′ 7″
" " 3 h. . .	= 32 30 3	D = 33 56 48·4
d' . .	= 1 28 4	d = 0 1 18·6

and $d' : d = 3 \text{ h.} : t$, or 1° 28′ 4″ : 1′ 18·6″ = 3 h. : 2 m. 40·6 s.,
and $T = t' + t$ = 0 h. + 2 m. 40·6 s. = 0 h. 2 m. 40·6 s.

2. To find the time at the place

By Art. 875 (the declination of star being = 12° 50′ 53·4″),

Z = 61° 36′ 46″	L, sin S . . .	= 9·9973493
P = 77 9 6·6	L, sin (S − Z) . .	= 9·7554481
C = 53 53 0	L, cosec P . .	= ·0110120
2)192 38 52·6	L, cosec C . .	= ·0926862
S = 96 19 26·3		2)19·8564956
S − Z = 34 42 40·3	L, cos ½H . . .	= 9·9282478

		h.	m.	s.
And $\frac{1}{2}H = 32° 2' 11·1''$, and	H =	4	16	17·5
And star's right ascension	=	9	58	43·6
Right ascension of meridian	=	5	42	26·1
Sun's R.A. at noon at place	=	17	27	14·7
		12	15	11·4
Acceleration	=	0	2	0·4
Time at place on 14th	=	12	13	11·0
" Greenwich on 15th	=	0	2	40·0
Longitude of place	=	11	49	29·0 W.

The time at the place could also be found from the observed altitude of the moon.

The principle on which the rule is founded is so simple as to require no explanation. It proceeds, however, on the hypothesis that the moon's motion is uniform, which it is so nearly for three hours that the error arising from this assumption amounts at most only to a few seconds. When extreme accuracy is required, what is called the equation of second differences, which depends on the differences of the first differences, is used as a correction.*

Exercises

1. Find the true longitude for the data in the third exercise of last problem, supposing the true time of observation to be the 24th of January 1813, at 3 h. 45 m. A.M., the true distance = 63° 4′ 54″, and that the distance of the centre of the moon from Regulus on the 23rd at 15 h. = 62° 20′ 8″, and at 18 h. = 63° 48′ 54″.

= 11° 26′ 45″ W.

2. Find the longitude from Paris for the data in the fourth exercise of the last problem, the exact mean time of observation at the place being = 5 h. 29 m. 36·8 s., the true lunar distance = 55° 17′ 20″, and the registered distance at 6 h. in the *Connaissance des Temps* being = 54° 11′ 36″, and the difference for 3 h. = 1° 25′ 55″. . . . . = 2 h. 48 m. 6·2 s. W.

3. Required the longitude west of Paris for the fifth exercise in last problem, the exact mean time of observation at the place being = 7 h. 40 m. A.M., the true lunar distance = 61° 52′ 35·4″, and the distances registered in the *Connaissance des Temps* being on the 5th at 21 h. = 61° 6′ 22″, and on the 6th at 0 h. = 62° 45′ 51″.

= 2 h. 43 m. 38 s. W.

4. Find the longitude for the data in the sixth example of the preceding problem, the true distance of the moon's centre from

* See *Nautical Almanac*.

Regulus being $=57^\circ\ 47'\ 12{\cdot}6''$, and their registered distances in the *Connaissance des Temps* being on the 17th at 15 h. $=59^\circ\ 2'\ 7''$, and at 18 h. $=57^\circ\ 9'\ 45''$. . . . . $=2$ h. 0 m. $11{\cdot}2$ s.

5. Find the true apparent time at the place, and the longitude for the true lunar distance $=83^\circ\ 20'\ 55''$ in the seventh exercise of the preceding problem, the latitude being $=10^\circ\ 16'\ 40''$ S., and the sun's declination $=23^\circ\ 22'\ 48''$ N.; also the next less and greater registered lunar distances being at 15 h. apparent time $=83^\circ\ 6'\ 1''$, and at 18 h. $=84^\circ\ 28'\ 26''$.

Time $=1$ h. 39 m. $38{\cdot}5$ s.; longitude $=10$ h. 7 m. 6 s.

NAVIGATION

888. The department of navigation that belongs to Practical Mathematics consists in the solution of the problems of determining the direction and distance of the intended port from the port left, or from the place of the ship at any time, and also the determining of the ship's place at any instant during the voyage. The principles of plane trigonometry, modified in their application according to circumstances, are sufficient for the solution of these problems.

The ship is navigated, as nearly as possible, by the path which is the shortest distance between the two places, but, from contrary winds and intervening land, it is generally necessary to sail in a track of a zigzag form; the distance sailed in each direction being known, as also the direction, the ship's place can always be found, as will be afterwards explained.

DEFINITIONS OF TERMS

889. When a vessel is obliged to sail to the right or left of the direction of the intended port, she is said to **tack.** When the ship is tacking towards the left, and the wind consequently on the right, she is said to be on the **starboard** tack; and when she is tacking towards the right, she is said to be on the **larboard** tack.

890. A ship does not sail exactly in the direction of her keel or longitudinal axis, but deviates towards the side that is opposite to the wind; and the angle contained between the apparent and

real direction is called **leeway.** The real direction is observable by the track of the vessel in the water, called the ship's **wake,** or by the direction of the log-line; and the leeway can therefore be estimated.

891. The angle formed by the meridian and the direction of the ship's track is called the **course.**

892. A line cutting all the meridians at the same angle is called a **rhumb-line,** which when continued approaches nearer and nearer to the pole, in a spiral form, but without ever reaching it; it is also called a **loxodrome**; whereas the arc of a great circle, which is the shortest distance between two places, is called the **orthodrome.**

893. The portion of a rhumb-line intercepted between two places is called their **nautical** distance.

894. The distance of a ship from the meridian left, reckoned on the parallel of latitude of the ship's place, is called her **meridional distance.**

895. If the nautical distance is supposed to be divided into an indefinite number of minute equal parts, the sum of all the meridional distances belonging to these parts is called the **departure.**

896. The **difference of latitude** of two places is an arc of a meridian, intercepted between the parallels of latitude passing through these places.

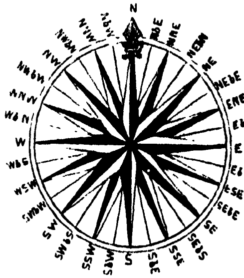
897. The **difference of longitude** of two places is an arc of the equator intercepted between their meridians.

INSTRUMENTS USED IN NAVIGATION

898. The **mariner's compass** is the instrument by which the course is measured. This compass consists of a circular card suspended horizontally on a point, and having for one of its diameters a small magnetised bar of steel, called **the needle.** The circumference of the card is divided into 32 equal parts, called **points of the compass**; and each point is divided into 4 equal parts, called **quarter points.** The point of the card which coincides with the north end of the needle is called the **magnetic north**; the opposite point, the **magnetic south**; and the middle points between these, on the extremities of the diameter perpendicular to the needle, are called the **magnetic east** and **west.** These are called the **cardinal magnetic points,** and the other

points are named from their situation in reference to these points. The true cardinal points are consequently the **north, south, east,** and **west.** Since there are 8 points in each quadrant, therefore a point is = an angle of 11° 15′.

At the same place the needle points nearly in the same direction for many years, but in different places its direction is not towards the same part of the horizon. The angular difference between the magnetic and true north is called the **variation** of the compass, being **west** or **east** according as the magnetic north is towards the left or right of the true north.



The compass needle may be affected sensibly by the attraction of iron placed near it, and even by a great mass of iron at a considerable distance, as in a ship-of-war by the guns. When the metal is symmetrically distributed in reference to the longitudinal axis, the needle is not affected when the direction of this axis coincides with the magnetic meridian or vertical plane passing through the needle; and its local attraction produces the greatest error in the true variation when the direction of the axis of the ship is perpendicular to the former direction. The variation of the needle at London is at present about 24½°.

The points of the compass are seen in the foregoing figure. The middle point between N. and E. is called NE.; that between N. and NE. is called NNE.; and so on.

899. The **log** is a piece of wood, of the form of a circular sector, which is nearly quadrantal; and the arc of it is loaded with lead, so that it floats vertically with the central point uppermost. The line called the **log-line** is so attached to the log that when the line is drawn gently the log turns its flat side towards the ship, so that it remains nearly immovable while the line is unwound from the reel.

The **log-line** is about 100 fathoms long, and is divided into equal parts called **knots**, each of which is generally subdivided into fathoms. A knot is the 120th part of a nautical mile, or of 6079 feet, and ought therefore to be 50 feet 8 inches. In practice, however, 50 feet is usually made the length of a knot, for the log being drawn a small way towards the vessel during the operation of

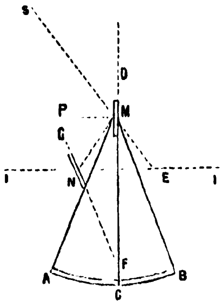
estimating the ship's rate, or, as it is called, of heaving the log, the distance given by this line is nearer the truth; and, besides, it is safer that the reckoning should be in advance of the ship, or **ahead** of it, as it is termed.

The time, when observing the ship's rate by the log-line, is estimated by a **sand-glass,** which measures half-minutes—that is, it runs out in 30 seconds.

Since 30 seconds is the same part of an hour that a knot is of a mile, the number of knots run out in 30 seconds shows that the rate of the vessel is just the same number of miles per hour.

Sometimes the sand-glass and log-line, from various causes, become incorrect, and therefore the rate measured by them, or the distance sailed, must be corrected.

900. The angular instruments used in navigation are Hadley's **quadrant** and **sextant.** The principles on which these instruments are constructed will be understood from the adjoining figure.



The graduated arc AB is the **limb** of the instrument, CM an **index,** movable about an axis at M, with a **vernier** at its extremity C. M is a small mirror attached to the index CM, and placed perpendicularly to the plane ABM of the instrument; N is a similar small plate of glass, called the **fore horizon glass,** one half of which is a mirror; and it is placed parallel to the mirror M when the index coincides with MB, or rather with the zero point at B, and is fixed in this position. When the angular distance between two objects, as two stars, at S and I is to be measured, the plane of the instrument is first placed in the same plane with the objects, and in such a position that one of them, I, is visible through the glass N to the eye situated at E, and then the index CD is moved till the image of S, after two reflections from M and N, appears to coincide with I, seen directly through the plate; and the angle subtended by their distance—namely, angle E—is then measured by double the arc BC.

The ray SM proceeding from S is reflected in the direction MN by the mirror M, and then at N by the mirror N, in the direction NE, so that, PM being perpendicular to MD, the angle of **reflection** PMN is equal to that of **incidence** PMS, or the inclination

of the incident ray SMD is equal to that of the reflected ray NMC, and also angle GNM is equal to FNE. From these relations of the angles, it is easily proved that the angle E of the triangle MNE is equal to twice the angle F of the triangle NMF. But angle F is equal to FMB, as GN is parallel to MB; hence the double of angle CMB, which is measured by twice the arc BC, is the measure of angle E.

In Hadley's quadrant the arc AB is an octant—that is, the eighth part of a circle—and therefore it contains only 45°; the **sextant** differs from the quadrant merely in having its limb AB a sextant, or the sixth part of a circle. The sextant is furnished with a small telescope, to show with more precision when the image of one of the objects coincides with the other. The arcs of the sextant and quadrant are both graduated, so as to give the reading of the true angle, though they are only the measure of half that angle.

PRELIMINARY PROBLEMS

901. **Problem I.—Given the distance sailed as determined by the reckoning, and the error of the log-line and sand-glass, to find the true distance.**

I. When only the log-line is incorrect—as the correct length of the knot or 50 feet is to the incorrect length, so is the incorrect distance to the true distance.

II. When only the sand-glass is incorrect—the number of seconds run by the glass is to 30 seconds as the incorrect distance to the true distance.

III. When both the log-line and sand-glass are incorrect, multiply six times the measured length of the log-line by the observed distance, and divide the product by ten times the seconds the glass takes to run out.

Let k, k' = the true and incorrect lengths of a knot,
s, s' = " " number of seconds,
d, d' = " " distances;

then, for I. $k : k' = d' : d$, and $d = \frac{k'}{k}d' = \frac{1}{50}k'd'$,

II. $s' : s = d' : d$, and $d = s\frac{d'}{s'} = 30\frac{d'}{s'}$,

$$\text{III. } d = \frac{k'}{k} \cdot \frac{s}{s'} \cdot d' = \frac{3}{5} \cdot \frac{k'}{s'} \cdot d' = \frac{6}{10} \cdot \frac{k'}{s'} \cdot d'$$

For if $k : k' = d' : d''$, then $d'' = \frac{k'}{k} d'$, and $s' : s = d'' : d$;

hence $$d = \frac{s}{s'} d'' = \frac{k'}{k} \cdot \frac{s}{s'} d' = \frac{6}{10} \cdot \frac{k'}{s'} \cdot d'.$$

EXAMPLE.—The distance by reckoning is $= 92$ miles, the length of the knot $= 51$ feet, the seconds by the sand-glass $= 28$; what is the true distance?

$$d = \frac{6}{10} \cdot \frac{k'}{s'} \cdot d' = \frac{6}{10} \cdot \frac{51}{28} \times 92 = 100{\cdot}5 \text{ miles.}$$

EXERCISES

The true distance is required from the data in the first three columns :—

	Distance by Log	Length of a Knot	Seconds by Glass	Answer : True Distances
1.	245 miles	48 feet	30	235·2
2.	156 "	50 "	32	146·2
3.	126 "	46 "	27	128·8
4.	164 "	49 "	33	146·1

902. Problem II.—Given the magnetic course—that is, the course per compass—and the variation, to find the true course.

RULE.—Apply the variation to the magnetic course towards the left when the variation is W., and towards the right when E.

EXAMPLE.—What is the true course when the compass course is NW., and the variation 2 points W.?

The variation, being W., must be applied to the left of the course, which will therefore increase it by 2 points, and the true course is therefore WNW.

EXERCISES

Find the true courses from the magnetic courses and variations given in these exercises :—

	Magnetic Course	Variation	Answer : True Course
1.	NE.	1 W.	NE. *b* N.
2.	SW. *b* W.	2 W.	SW. *b* S.
3.	N. *b* E.	3 E.	NE.
4.	SSW. ½ W.	2 ¼ E.	SW. ¾ W.
5.	WNW. ½ W.	1 ½ W.	W.
6.	SE. ¼ S.	1 ¼ E.	SSE. ½ E.

903. **Problem III.—Given the true course and the variation, to find the magnetic course.**

This problem is solved exactly as the last, only the variation is applied in the opposite direction to the true course. The true courses and variations in the exercises to the preceding problem may be taken as data for exercises to this problem, and the corresponding magnetic courses will be the answers.

904. **Problem IV.—Given the compass course, the variation, and leeway, to find the true course.**

Apply the variation, then apply the leeway in a direction from the wind—that is, to the left when the vessel is on the starboard tack, and to the right when on the larboard tack.

EXAMPLE.—The magnetic course is NE. *b* E. on the larboard tack; required the true course, the variation being 2 points W., and the leeway 5 points.

EXERCISES

Find the true course in the following exercises, the compass course, leeway, and variation being given:—

	Compass Course	Tack	Variation Points	Leeway Points	Answer: True Course
1.	NE. *b* N.	Larboard	2 W.	2	NE. *b* N.
2.	SE. *b* E.	"	2 W.	$1\frac{1}{2}$	ESE. $\frac{1}{2}$ S.
3.	WNW.	Starboard	3 W.	$2\frac{1}{2}$	SW. $\frac{1}{2}$ W.
4.	N. $\frac{3}{4}$ E.	"	5 E.	$3\frac{1}{4}$	NNE. $\frac{1}{2}$ E.

905. **Problem V.—Given the latitudes and longitudes of two places, to find their difference of latitude and longitude.**

RULE.—When the latitudes are of the same denomination find their difference, but when they are of different names take their sum; and the remainder in the former case, or the sum in the latter, will be the difference of latitude.

Find the difference of longitude in the same manner as that of latitude, observing that when the longitudes are of different names, and their sum exceeds 180°, it must be subtracted from 360°, and the remainder will be the difference of longitude.

EXAMPLE.—What is the difference of latitude and longitude of Quito and Canton?

Canton, . . Lat.	=	23° 8′ 9″ N.	Long. =	113° 16′ 54″ E.
Quito, . . "	=	0 14 0 S.	" =	78 45 6 W.
Difference of lat. .	=	23 22 9		192 2 0
		60		360 0 0
Dif. of long. . .	=	1402·15 miles		167 58 0
				60
				= 10078 miles.

The difference of longitude in miles is estimated on the equator.

Exercises

Find the difference of latitude and longitude of the places stated in each of the following exercises :—

1. Liverpool, lat. = 53° 24′ 40″ N., long. = 2° 58′ 55″ W.; and New York, lat. = 40° 42′ 6″ N., long. = 73° 59′ W.

Dif. of lat. = 762·57 miles; dif. of long. = 4260·08 miles.

2. Valparaiso, lat. = 33° 1′ 55″ S., long. = 71° 41′ 15″ W.; and Manila Cathedral, lat. = 14° 35′ 26″ N., long. = 120° 59′ 3″ E.

Dif. of lat. = 2857·35 miles; dif. of long. = 10039·7 miles.

906. **Problem VI.—Given the latitude and longitude of the place left, and the difference of latitude and longitude made by the ship, to find the latitude and longitude of the place reached.**

Rule.—Apply the difference of latitude and longitude respectively to the latitude and longitude left by addition or subtraction, according as they are of the same or different denominations.

When the longitude and difference of longitude are of the same name, and their sum exceeds 180°, subtract it from 360°, and the remainder is the longitude of a contrary denomination from that left.

Example.—The latitude and longitude of the place left are = 24° 36′ N. and 174° 40′ W. respectively; and after sailing SW. for some time, the differences of latitude and longitude made were found to be = 245 miles and 384 miles; what are the latitude and longitude in?

Lat. left .	=	24° 36′ N.	Long. left .	=	174° 40′ W.
Dif. lat. 245 .	=	4 5 S.	Dif. long. 384 .	=	6 24 W.
Lat. in . .	=	20 31 N.			181 4 W.
					360
			Long. in .	=	178 56 E.

Exercises

Find the latitude and longitude arrived at in the following exercises :—

1. Lat. left = 34° 4′ S., long. left = 12° 5′ E.; dif. of lat. = 145 miles S., dif. of long. = 365 miles W.

Lat. in = 36° 29′ S.; long. in = 6° 0′ E.

2. Lat. left = 20° 40′ N., long left = 178° 14′ W.; dif. of lat. = 216 miles S., dif. of long. = 420 miles W.

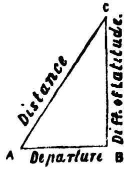
Lat. in = 17° 4′ N.; long. in = 174° 46′ E.

Navigation is divided into different branches, according to the methods of calculation employed.

PLANE SAILING

907. In plane sailing the surface of the earth is considered to be a plane, the meridians being equidistant lines, and the parallels of latitude also equidistant, cutting the meridians perpendicularly. This supposition, though incorrect, will lead to no error, so far as the nautical distance, difference of latitude, and departure are concerned; for, as appears from the explanation following the example given below, these elements will be the same whether they are lines drawn on a plane or equal lines similarly related drawn on a sphere. As the north is on the upper side of the figure of the mariner's compass, and the upper side of maps, the top of a page is considered to be directed towards the north; therefore the upper parts of diagrams in navigation are considered to be the northern parts of the figure.

Hence a vertical line, BC, will denote the difference of latitude; a horizontal line, AB, the departure; the oblique line or hypotenuse, AC, the nautical distance; angle C the course, and A the complement of the course. Hence—



908. If any two of the four parts - namely, the nautical distance, departure, difference of latitude, and course—are given, the other two can be found by the rules of right-angled trigonometry.

There will therefore be six cases, of which the first, however, is the most important. These cases may also be calculated very easily, and with sufficient accuracy, by means of the Table of **the difference of latitude and departure**, or, as it is sometimes called, a **Traverse Table**; this method of solution is called **inspection.** They can also be solved by

construction, as in the problems from Art. 136 to 139, or by means of **logarithmic lines,** as explained from Art. 154 to 157.

909. **Problem VII.—Of the course, distance, difference of latitude, and departure, any two being given, to find the other two.**

EXAMPLE.—A ship from a place in latitude = 56° 14′ N. sails SW. ½ W. 425 miles; required the latitude in and the departure.

The proportions are the same as in the second case of right-angled trigonometry, only the nautical terms are used for the angles and sides of the triangle.

Construction

Let BC be the meridian, and make angle C = 4½ points = 50° 37′, and CA = 425, and draw AB perpendicular to BC. Then measure AB and BC.

By Calculation

1. To find the departure AB

Rad. : sin C = AC : AB, or

Radius	=	10·
Is to sin course 4½ points	=	9·888185
As distance 425	=	2·628389
To departure 328·53	=	2·516574

2. To find the difference of latitude BC

Rad. : cos C = AC : BC, or

Radius	=	10·
Is to cos course	=	9·802359
As distance 425	=	2·628389
To difference of latitude 269·6 . . .	=	2·430748

Latitude left	=	56° 14′ N.
Dif. of lat. 269·6	=	4 30 S.
Latitude in	=	51 44 N.

By Gunter's Logarithmic Lines

When the course is given in points, use sine rhumbs or tangent rhumbs instead of the lines of sines and tangents.

1. To find the departure

The distance from radius, or 90° on the line **S. Rhumb,** to 4½ points will extend on the line of **numbers** from 425 to 328, the departure.

2. To find the difference of latitude

The distance from 90° to the complement of the course $3\frac{1}{2}$ points (as sine $3\frac{1}{2}$ points is = cosine $4\frac{1}{2}$ points) on the line **S. Rhumb** will extend on the line of **numbers** from 425 to 270, the difference of latitude.

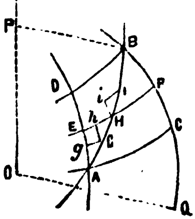
By Inspection

In the Traverse Table in the page containing the course $4\frac{1}{2}$ points, and opposite to the distance 425, is the departure 328·5 and the difference of latitude 269·6.

As the distance in the Table is not greater than 300, take out first the difference of latitude and departure for 300, and then for 125, and their sum will give the above; or take the difference of latitude and departure corresponding to one-fifth of the distance, and multiply them by 5.

When the course is not given, the problem cannot be conveniently solved by inspection.

Let AC, BD be the parallels of the latitude left and reached, BC, DA their meridians, and AGB their nautical distance, which therefore is at every point equally inclined to the meridian. Let the distance AB be divided into a great number of minute equal parts AG, GH,...and let Gg, Hh,...be portions of parallels of latitude, and Ag, Gh,...portions of meridians passing through the points A, G, H. Then, since these parts differ insensibly from straight lines, and the angles GAg, HGh,...are equal, therefore the parts AG, GH,...are proportional to Ag, Gh;...and hence $AG : Ag = AG + GH + \ldots : Ag + Gh + \ldots$ or as $AB : AD$. But $AG : Ag \ldots = \text{rad.} : \cos \text{course}$; hence



$$AB : AD = \text{rad.} : \text{cosine course}.$$

It is similarly shown that $AG : Gg = AG + GH + \ldots : Gg + Hh + \ldots$ = distance : departure; and hence

$$\text{rad.} : \sin \text{course} = \text{distance} : \text{departure}.$$

910. The distance, difference of latitude, departure, and course are therefore related as the sides and angles of a plane right-angled triangle, and their various relations are therefore determinable in the same manner as those of the sides and angles of the triangle.

The following exercises, which illustrate the six cases, are to be

performed by construction, calculation, and logarithmic lines, and by inspection :—

Exercises

1. A ship from a place in latitude $=49° 57'$ N. sails SW. *b* W. 244 miles ; required the departure and latitude.

Departure $=203$; latitude $=47° 41{\cdot}4'$ N.

2. A ship sails SE. *b* E. from a place in $1° 45'$ north latitude, and is then found by observation to be in $2° 46'$ south latitude ; required the departure and distance. Departure $=405{\cdot}6$; distance $=487{\cdot}8$.

3. A ship sails NE. *b* E. $\frac{3}{4}$ E. from a port in latitude $=3° 15'$ S., till her departure is 406 miles ; what is the distance sailed and the latitude in? . . . . Distance $=449$; latitude $=0° 3'$ S.

4. A ship sails between the south and east 488 miles from a port in latitude $=2° 52'$ S., and then by observation she is found to be in latitude $=7° 23'$ S. ; what course has she steered, and what departure has she made? The course $=56° 16'$ or SE. *b* E. ; departure $=405{\cdot}8$.

5. A ship has sailed between the north and west from the island of Bermuda, in latitude $=32° 25'$ N., till her distance is 488 miles and departure 405 miles ; what has been her course, and what is the latitude? . The course N. $=56° 6'$ W. ; latitude $=36° 57'$ N.

6. A ship sails between the north and west till her difference of latitude is 271 miles, and departure 406 miles ; what is the course and distance sailed?

Course N. $=56° 17'$ W. or NW. *b* W., and distance $=488{\cdot}2$.

TRAVERSE SAILING

911. **Problem VIII.—Given several successive courses and distances sailed by a ship between two places, to find the single course and distance by which she would have arrived at the same place.**

Find the difference of latitude and departure for each course and distance, and then the whole difference of latitude and departure, and the course and distance corresponding to these two elements.

The difference of latitude and departure for each course and distance are to be found by the last problem, the method by the Traverse Tables being the most expeditious ; then these are arranged in a table called a **Traverse Table**, the courses being in the first column, the distances in the second, the north and south differences of latitude, marked N. and S., in the third and fourth, and the east and west departure, marked E. and W., in the fifth and sixth columns.

The difference between the sums of the columns N. and S., or of the **northings** and **southings**, will be the whole difference of latitude of the same name as the greater; and the difference between the sums of the columns E. and W., or of the **eastings** and **westings**, will be the whole departure of the same name as the greater.

EXAMPLE.—A ship from Cape Clear, latitude = 51° 25′ N., sails S. *b* W. 20 miles, SE. 12 miles, SW. *b* S. 18 miles, WNW. ½ N. 14 miles, and SSW. 24 miles; required the equivalent course and distance and the latitude in.

TRAVERSE TABLE

Courses	Distances	Difference of Lat.		Departure	
		N.	S.	E.	W.
S. *b* W	20	...	19·6	...	3·9
SE.	12	..	8·5	8·5	...
SW. *b* S.	18	...	15·	...	10·
WNW. ½ N.	14	6·6	...	...	12·3
SSW.	24	...	22·2	...	9·2
		6·6	65·3	8·5	35·4
			6·6		8·5
			58·7		26·9

Latitude left	=	51° 25′ N.
Difference of latitude	=	0 58·7 S.
Latitude in	=	50 26

The whole difference of latitude and departure being now known —namely, 58·7 S. and 26·9 W.—the corresponding course and distance can be found, as in the sixth example of the last problem.

1. To find the course		2. To find the distance	
Dif. of lat. 58·7 .	= 1·768638	Sin course . .	= 9·619662
Is to dep. 26·9 .	= 1·429752	Is to radius . .	= 10·
As radius . .	= 10·	As dep. 26·9 . .	= 1·429752
To tan. course 24° 37′	= 9·661114	To dist. 64·6 . .	= 1·810090

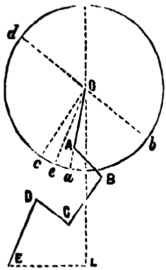
The course is therefore S. = 24° 37′ W., and the distance = 64·6 miles.

These two proportions can also be performed by Gunter's scale as formerly.

Construction

The different courses and distances may also be drawn as in the annexed diagram, and the equivalent course and distance measured.

Describe a circle *bcd*, and let OL represent the meridian; draw the radii O*a*, O*b*, O*c*, O*d*, O*e*, making angles with OL equal to the given courses; then on O*a* lay off the corresponding distance OA = 20; draw AB parallel to O*b*, and = 12; BC parallel to O*c*, and = 18; CD parallel to O*d*, and = 14; and DE parallel to O*e*, and = 24. Draw EL perpendicular to OL, then OL is the whole difference of latitude, EL the whole departure, and (supposing O and E joined) angle EOL is the equivalent course, and OE the equivalent distance.



Exercises

1. A ship takes her departure from the Lizard W. light in latitude = 49° 58′ N., which then bears NNW., its distance being = 15 miles, and sails SE. 34 miles, W. *b* S. 16, WNW. 39, and S. *b* E. 40; what is the latitude in, and the bearing and distance of the Lizard?

Latitude in = 48° 53′ N.; bearing of Lizard N. = 12° 16′ E.; and its distance = 66·8.

2. A ship's place is in north latitude = 50° 36′, and she sails during 24 hours in the following manner:—SSW. 54 miles, W. *b* S. 39, NW. *b* N. 40, NE. *b* E. 69, and NNW. 60; what is the latitude in, and the equivalent course and distance from the former place?

Lat. in = 51° 45′; course N. = 33° 57′ W. or NW. *b* W.; and distance = 83·8.

912. If the ship has sailed in a current during any time, its effect for that time is allowed for as a separate course and distance. For instance, if the ship has been sailing for 10 hours under the influence of a current setting NE. at the rate of 2½ miles per hour, the effect is the same as if the ship had sailed NE. 25 miles, and should be entered as an additional course.

GLOBULAR SAILING

913. In globular sailing the methods of calculation are derived on the supposition that the earth is of a spherical form, and they

apply with sufficient accuracy for the determination of the ship's place at any time, and the bearing and distance of the port bound for or of that left.

CASE 1.—When the ship sails between two places on the same meridian.

The difference of latitude is just the distance sailed, and the course is due north or south, and there is no difference of longitude.

CASE 2.—When the ship sails on the equator.

The distance sailed is the difference of longitude, the course is due east or west, and there is no difference of latitude.

CASE 3.—When the ship sails on the same parallel of latitude.

914. To find the distance when the latitude is given, and the longitudes of the two places.

Radius is to the cosine of the latitude as the difference of longitude to the distance.

Rad. : cos lat. = dif. long. : distance.

915. To find the difference of longitude when the latitude and distance on the same parallel are given.

Radius is to the secant of the latitude as the distance to the difference of longitude.

Rad. : sec lat. = distance : dif. long.

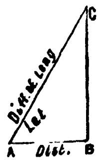
916. To find the latitude when the distance and difference of longitude are given.

The difference of longitude is to the distance as radius to the cosine of the latitude.

Dif. long. : distance = radius : cosine lat.

This case is sometimes called **parallel sailing.**

The proportions in this case can be represented by this construction :—



ABC is a right-angled triangle, of which B is the right angle, AB the distance, AC the difference of longitude, and angle A the latitude. Then, when AC is radius,...

Dif. long. : distance = radius : cos lat.

And when AB is radius,

Rad. : sec lat. = distance : dif. long.

EXAMPLE.—The longitudes of two places in the latitude of 56° S. are = 140° 20′ and 148° 45′; find the distance.

Dif. of long. = 8° 25′ = 505 miles.

To find the distance

L, radius	=	10·
L, cos lat. 56°	=	9·747562
L, dif. long. 505	=	2·703291
L, distance 282·4	=	2·450853

The proportion can be derived from the figure in Art. 909. Let O be the centre of the earth's equator, and OQ its radius; P the centre of the parallel of latitude at B, and PB its radius; then the distance between two meridians, measured on the equator, is to their distance on the parallel at B as OQ : PB—that is, dif. long. : dist. = radius : cos lat.

EXERCISES

1. A ship in latitude = 49° 30′ sails due E. till her difference of longitude is = 3° 30′; what is the distance sailed? . = 136·4 miles.

2. A ship sails 136·4 miles due W. on the parallel of latitude = 49° 30′; required the difference of longitude made. = 210 miles.

3. A ship sails 136·4 miles due E., and her difference of longitude is then = 3° 30′; on what parallel of latitude did she sail?

Latitude = 49° 30′.

CASE 4.—When the course is compound, to find the difference of latitude and longitude.

917. METHOD I.—The first method of solution is by **middle latitude sailing.**

This method combines plane and parallel sailing; and in it, it is supposed that the **departure** made by a ship is equal to the **meridional distance** on the middle parallel—that is, the meridional distance EF (fig. to Art. 909) on the parallel of latitude in the middle between the latitudes of A and B, the latitude left and that arrived at, is equal to the sum of the elementary meridian distances G*g*, H*h*,...which it nearly is. There are two proportions used—namely,

918. The difference of latitude is to the difference of longitude as the cosine of the middle latitude to the tangent of the course.

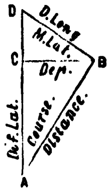
Diff. lat. : dif. long. = cos mid. lat. : tan course.

919. The cosine of the middle latitude is to the sine of the course as the distance to the difference of longitude.

Cos mid. lat. : sin course = distance : dif. long.

920. These two proportions can be obtained by means of two right-angled triangles, ABC, DBC, having their right angles at C.

In the triangle ABC angle A is the course, AB the distance, BC the departure, and AC the difference of latitude; and in the triangle BCD, BC is the departure, angle B the middle latitude, and BD the difference of longitude. ABC is a triangle in plane sailing, and BCD in parallel sailing; and from the triangle ABD the proportion in Art. 919 is easily derived, while the two proportions



Dif. lat. : dep. = radius : tan course,

Dep. : dif. long. = cos mid. lat. : rad.,

being compounded, give Art. 918,

Dif. lat. : dif. long. = cos mid. lat. : tan course.

921. METHOD II.—The second method of solution is by **Mercator's sailing.**

In this method the surface of the earth is considered to be plane, the meridians being parallel lines, and also the parallels of latitude, as in plane sailing; and since by this hypothesis the distance between the meridians is increased, except at the equator, the lengths of the arcs of the meridians are increased in the same proportion, so that the distances between the parallels of latitude for every successive minute are continually increasing with the latitude; and the **relative** bearings of places are thus preserved. This method is so accurate that it may be used without sensible error for any distance on the earth's surface.

The lengths of the meridians from the equator to any latitude are thus increased, and the increase is greater the higher the latitude. For instance, the increased distance of the parallel of 10°, instead of being 600 miles, is found to be 603 miles; and that of the latitude of 50°, instead of $50 \times 60 = 3000$ miles, is 4527 miles. The increased lengths of the meridians, from the equator to any latitude, are called the **meridional parts**, from the manner in which they are computed; and their numerical values are contained in tables.

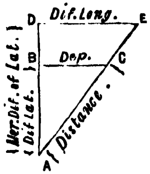
922. The difference of the meridional parts for any two latitudes is called the **meridional** difference of latitude; and the true difference of latitude is sometimes, for distinction, called the **proper** difference of latitude.

The analogies peculiar to this method are :—

923. The difference of latitude is to the departure as the meridional difference of latitude to the difference of longitude.

Dif. lat. : dep. = mer. dif. lat. : dif. long.

924. The meridional difference of latitude is to the difference of longitude as radius to the tangent of the course.



Mer. dif. lat. : dif. long. = radius : tan course.

These proportions can be obtained from two right-angled triangles, ABC, ADE, in which AB and AD are the proper and the meridional differences of latitude, BC the departure, DE the difference of longitude, AC the distance, and A the course.

925. **Problem IX.—Given the place left and that bound for, to find the course and distance.**

1. *By Middle Latitude Sailing*

To find the course

Dif. lat. : dif. long. = cos mid. lat. : tan course.

To find the distance

Radius : sec course = dif. lat. : distance.

2. *By Mercator's Sailing*

To find the course

Mer. dif. lat. : dif. long. = radius : tan course.

To find the distance

Radius : sec course = dif. lat. : distance.

The second proportion in the two methods is the same.

Example.—Required the bearing and distance of New York from Liverpool.

By Middle Latitude Sailing		*By Mercator's Sailing*	
Liverpool, lat. .	= 53° 25′ N.	Mer. parts . . .	= 3806
New York, lat. .	= 40 42 N.	Mer. parts . . .	= 2678
Dif. lat. . .	= 12 43	Mer. dif. lat. . .	= 1128
Mid. lat. . .	= 47 4	P. dif. lat. = 12° 43′ =	763 miles.
Liverpool, long. .	= 2° 59′ W.	Dif. long. = 71° 0′ =	4260 „
New York, long.	= 73 59 W.		

1. To find the course

Dif. lat. 763 (*a . c*)	= 7·117475	Mer. dif. lat. 1128 .	= 3·052309
Dif. long. 4260	= 3·629410	Dif. long. 4260 .	= 3·629410
Cos mid. lat. 47° 4′	= 9·833241	Radius . . .	= 10·
Tan course 75° 16′	**= 10·580126**	**Tan course 75° 10′ .**	**= 10·577101**

2. To find the distance

Radius . . .	$=10\cdot$	Radius . . .	$=10\cdot$
Sec course 75° 16′ .	$=10{\cdot}594618$	Sec course 75° 10′ .	$=10{\cdot}591746$
Dif. lat. 763 . .	$=2{\cdot}882525$	Dif. lat. 763 . .	$=2{\cdot}882525$
Distance 3000 . .	$=3{\cdot}477143$	Distance 2980 .	$=3{\cdot}474271$

EXERCISES

1. What is the bearing and distance of a place in latitude =71° 10′ N., longitude=26° 3′ E., from another place in latitude =60° 9′ N., and longitude=0° 58′ W.?

Course N. = 45° 18′ E., distance = 940 miles, by mid. lat. sailing.

Course N=44° 49′ E., distance=931·8 miles, by Mercator's sailing.

2. A ship having arrived at a place in latitude=49° 57′ N., longitude=5° 14′ W., is bound for another place in latitude =37° N., longitude=25° 6′ W.; required the bearing and distance of the latter place from the former.

Course S.=48° 4′ W., distance=1162·7 miles, by mid. lat. sailing.

Course S.=47° 54′ W., distance=1159 miles, by Mercator's sailing.

926. **Problem X.—Given the place sailed from, the course and distance, to find the place arrived at.**

1. *By Middle Latitude Sailing*

To find the difference of latitude

Radius : cos course = distance : dif. lat.

To find the difference of longitude

Cos mid. lat. : sin course = distance : dif. long.

2. *By Mercator's Sailing*

To find the difference of latitude

The analogy is the same as in the preceding method.

Or, Radius : cos course = distance : dif. lat.

To find the difference of longitude

Radius : tan course = mer. dif. lat. : dif. long.

By the first proportion in these two methods the difference of

latitude is found, and by the second the difference of longitude; and hence the latitude and longitude of the place in are known.

EXAMPLE.—A ship from a place in latitude = 25° 40′ S. and longitude = 35° 12′ W. sails SW. *b* S. 246 miles; required the latitude and longitude in.

To find the difference of latitude by both methods.

Radius	=	10·
Cos course 3 points	=	9·919846
Distance 246	=	2·390935
Dif. lat. 204·6	=	2·310781

Lat. left	= 25° 40′ S.		Mer. parts	= 1594·3
Dif. lat.	= 3 25 S.		Mer. parts	= 1825·2
Lat. in	= 29 5 S.		Mer. dif. lat.	= 230·9
Mid. lat.	= 27 22			

By Middle Latitude Sailing

To find the dif. long.

Cos mid. lat. (*a.c*)	= 0·051547
Sin course 3 points	= 9·744739
Distance 246	= 2·390935
Dif. long. 153·9	= 2·187221

By Mercator's Sailing

To find the dif. long.

Radius	= 10·
Tan course 3 points	= 9·824893
Mer. dif. lat. 231	= 2·363612
Dif. long. 154·4	= 2·188505

Longitude left	=	35° 12′ W.
Different longitude	=	2 34 W.
Longitude in	=	37 46 W.

The two methods give the differences of longitude to within less than a mile of each other. The place arrived at is in latitude = 29° 5′ S., and longitude = 37° 46′ W.

EXERCISES

1. A ship from a place in latitude = 50° 30′ N. and longitude = 145° 20′ W. sails 450 miles SSW.; find the latitude and longitude in.

   Latitude = 43° 34′ N., longitude = 149° 33′ W., both by mid. lat. and Mercator's sailing.

2. A ship from latitude = 51° 15′ N. and longitude = 9° 50′ W. sails SW. *b* S. till the distance run is 1022 miles; what are the latitude and longitude in?

   Latitude = 37° 5′ N., longitude = 23° 2′ W., by mid. lat. sailing, and = 23° 8′ W. by Mercator's sailing.

927. **Problem XI.—Given the latitude left, the difference of latitude and departure, to find the difference of longitude.**

By Middle Latitude Sailing

Cos mid. lat. : radius = dep. : dif. long.

By Mercator's Sailing

Dif. lat. : dep. = mer. dif. lat. : dif. long.

EXAMPLE.—A ship on a course between the south and west from latitude = 54° 24′ N. and longitude = 36° 45′ W. has made 346 miles of difference of latitude and 243 miles of departure; what is the latitude and longitude in?

Lat. left . .	= 54° 24′ N.	Mer. parts . .	= 3905·7
Dif. lat. 346 .	= 5 46 S.	Mer. parts . .	= 3348·7
Lat. in . .	= 48 38 N.	Mer. dif. lat. .	= 557
Mid. lat. . .	= 51 31		

To find the difference of longitude

By Middle Latitude Sailing		*By Mercator's Sailing*	
Cos mid. lat. 51° 31′	= 9·793991	Dif. lat. 346 (*a.c*).	= 7·460924
Radius . . .	= 10·	Dep. 243 . .	= 2·385606
Dep. 243 . .	= 2·385606	Mer. dif. lat. 557 .	= 2·745855
Dif. long. 390·5 .	= 2·591615	Dif. long. 391·2 .	= 2·592385
Long. left . .	= 36° 45′ W.	Long. left . .	= 36° 45′ W.
Dif. long. 390·5 .	= 6 31 W.	Dif. long. 391·2 .	= 6 31 W.
Long. in . .	= 43 16 W.	Long. in . .	= 43 16 W.

The place arrived at is therefore in latitude = 48° 38′ N. and longitude = 43° 16′ W.

EXERCISE

A ship from latitude = 37° N., longitude = 48° 20′ W., sails between the north and east till her difference of latitude and departure are 855 and 564 miles; required the latitude and longitude in.

Latitude = 51° 15′ N., longitude in 35° 14′ W., by mid. lat. sailing, and 35° 8′ W. by Mercator's.

928. **Problem XII.—To perform a traverse or compound course by middle latitude and Mercator's sailing.**

RULE.—Form a Traverse Table, and find by it the whole difference of latitude and the departure; then find the latitude in and

the course made good, as in Art. 911. Find then the middle latitude between that left and that arrived at, or find the meridional difference of latitude for these two latitudes; then, to find the difference of longitude,

Cos mid. lat.: radius = dep.: dif. long. by mid. lat. sailing.

Or, rad.: tan course = Mer. dif. lat.: dif. long. by Mer. sailing.

EXAMPLE.—Find the longitude and latitude of the place of the ship arrived at, after sailing the various courses and distances given in the example of a traverse in plane sailing in Art. 911, supposing the longitude left to be = 23° 40′ W.

Construct the traverse as in that example, and it will be found that the difference of latitude is = 58·7 S., and the departure = 26·9 W. Hence—

Lat. left . .	= 51° 25′ N.	Mer. parts . .	= 3608·7
Dif. lat. . .	= 0 59 S.	Mer. parts . .	= 3515·1
Lat. in . .	= 50 26	Mer. dif. lat. .	= 93·6
Mid. lat. . .	= 50 55		

Then dif. lat.: dep. = radius: tan course, and, as found in that example, the course is S. 24° 37′ W.

To find the difference of longitude

By Middle Latitude Sailing		*By Mercator's Sailing*	
Cos mid. lat. 50° 55′	= 9·799651	Radius . . .	= 10·
Radius . . .	= 10·	Tan course 24° 37′ .	= 9·661043
Dep. 26·9 . .	= 1·429752	Mer. dif. lat. 93·6 .	= 1·971276
Dif. long. 42·7 .	= 1·630101	Dif. long. 42·9 .	= 1·632319
Long. left . .	= 23° 40′ W.	Long. left . .	= 23° 40′ W.
Dif. long. . .	= 0 43 W.	Dif. long. . .	= 0 43 W.
Long. in . .	= 24 23 W.	Long. in . .	= 24 23 W.

929. Since the difference of latitude, distance, and course are the same parts of a right-angled triangle in plane sailing that the departure, difference of longitude, and middle latitude are in middle latitude sailing; and the difference of latitude, departure, and course are the same parts of the triangle in plane sailing that meridional difference of latitude, difference of longitude, and course are in Mercator's sailing; therefore the difference of longitude for the last two proportions can be found, by *inspection*, in the Table of the difference of latitude and departure.

In Plane Sailing		*In Mid. Lat. Sailing*
Course	corresponds to	Mid. lat.
Dif. lat.	" "	Departure.
Distance	" "	Dif. long.

In Plane Sailing		*In Mercator's Sailing*
Course	corresponds to	Course.
Dif. lat.	" "	Mer. dif. lat.
Departure	" "	Dif. long.

Thus, for the proportion above by middle latitude sailing, in the Table of difference of latitude and departure in the page for course $=51°$, and departure 26·9 in the difference of latitude column, there is 42 in the distance column for the difference of longitude, as above; and for the proportion by Mercator's sailing, in the same Table for course $=25°$ (for $24° 37'$), and meridional difference of latitude 93·6 in the difference of latitude column, there is 44 for the difference of longitude in the departure column.

930. When great accuracy is required, or when sailing in high latitudes, it is necessary to calculate the difference of longitude for each course and distance, supposing the distances not to exceed a few miles, instead of merely finding the difference of longitude on a whole day's sailing. This method is called a globular traverse.

EXERCISES

1. A ship from a place in latitude $=50° 6'$ N. and longitude $=5° 55'$ W. is bound to a port in the island of St Mary's in latitude $=36° 58'$ N., and longitude $=25° 12'$ W., and steers the following courses: S. *b* W. 24 miles, WSW. 32, NW. ½ W. 41, SSE. ¼ E. 49, ENE. ¾ E. 19, W. 21, NE. ½ E. 36, S. 41, SSW. 92, and N. 36; what is the latitude and longitude in, and also the direct course and distance to the intended port?

Latitude in $=48° 9'$, and longitude in $=7° 19'$, by mid. lat. and by Mercator's sailing.

Course $=42° 26'$, and dist. $=990·4$, by mid. lat. sailing.
Or, " $=42° 19'$, " $=988·6$, " Mer. sailing.

In the following exercise the difference of longitude is found on each course, as explained in Art. 930—that is, by the globular traverse :—

2. A ship from latitude $=68° 38'$ N. and longitude $=8° 40'$ E. is bound for the North Cape in latitude $=71° 10'$ N., and longitude $=26° 3'$ E., and sails the courses and distances in the subjoined

Table; what is the latitude and longitude in, and the direct course and distance of the Cape?

Courses	Distance	Latitude N.	Latitude S.	Departure E.	Departure W.	Lat. in	Dif. Long. E.	Dif. Long. W.
						68° 38′		
NE. *b* N.	63	52·4	...	35·0	...	69 30	97·2	...
NE.	38	26·9	...	26·9	...	69 57	77·6	...
NNE.	56	51·7	...	21·4	...	70 49	64·2	...
N.	30	30·0	...	...	...	71 19	...	...
NW. *b* N.	25	20·8	...	...	13·9	71 40	...	43·8
NNW. ½ W.	36	31·8	...	...	17·0	72 12	...	55·2
N. *b* E.	40	39·2	...	7·8	...	72 51	25·8	...
NE. *b* E. ½ E.	72	33·9	...	63·5	...	73 25	219·1	...
SE.	50	...	35·4	35·4	...	72 50	120·5	...
ENE.	65	24·9	...	60·1	...	73 15	206·9	...
		311·6	35·4	250·1	30·9		811·3	99·0
		35·4		30·9			99·0	
		276·2		219·2			712·2	

Latitude in = 73° 14′, longitude in = 20° 32′ E.

The course required is S. = 38° 1′ E., distance = 157·4, by mid. lat. sailing.

Or, The course is S. = 37° 59′ E., distance = 157·3, by Mercator's sailing.

3. A ship in latitude = 67° 30′ N., longitude = 8° 46′ W., sails NE. 64 miles, NNE. 50, NW. *b* N. 58, WNW. 72, W. 48, SSW. 38, S. *b* E. 45, and ESE. 40; what is the latitude and longitude in?

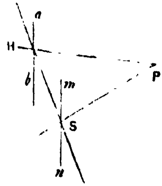
By the plane traverse, the lat. in is = 68° 43′ N., and longitude = 11° 3′ W.; and by the globular traverse, the long. in is = 11° 37′ W. by mid. lat., and = 11° 42′ by Mercator's sailing.

Departure of a Ship

The place of **departure** of a ship—that is, the place from which the beginning of a voyage is reckoned—is generally some promontory or other convenient object whose latitude and longitude are known; and as the vessel is usually some miles distant from it, observations must be taken to determine this distance.

931. **Problem XIII. —Given the bearing of a headland from a ship at two places, and the distance and direction sailed between them, to find the distance of the promontory from the ship.**

Let P be the promontory, S and H the two places of the ship, *mn* and *ab* parts of the meridians through H and S; then angle *m*SP, the bearing of P from S, is given; and angle *m*SH or *b*HS, the ship's course; and also angle PH*b*, the bearing of P from H.



Therefore, in the triangle PHS, the angle at $S = mSP + mSH$ is known, and that at $H = PHb - bHS$ is also known, and the side HS; therefore the distances PH and PS can be found (Art. 186).

Exercises

1. A headland was observed from a ship to bear NE. *b* N., and after sailing 7·5 miles on a NNW. course, the headland then bore ESE.; required the distance of the headland from both places of the ship. . . . . $=5·4$ and 6·36 miles.

2. A lighthouse was observed to bear from a ship NNE., and after sailing 15 miles on a WNW. course its bearing was found to be NE. *b* E.; required its distance from the last place of the ship. $=27$ miles.

3. A cape was observed to bear E. *b* S. from a ship, and after sailing NE. 18 miles, its bearing was SE. *b* E.; what was the distance of the cape from the second place of the ship? $=39·1$ miles.

932. After taking the departure, the next important problem is to find the bearing and distance of the port bound for, which is solved by Art. 925. After performing a day's sailing, as nearly as possible in the proper direction, the place of the ship is then to be determined by Art. 928; and then again, if necessary, the bearing and distance of the intended port; and these problems are to be successively repeated during the voyage. The latitude and longitude of the ship, determined in this manner, are said to be the latitude and longitude by **account.** As the place of a ship determined in this manner cannot be depended upon on a long voyage, on account of the errors occasioned by unknown currents, storms, and the unavoidably imperfect means of measuring the courses and distances, it becomes necessary to employ the prin-

ciples of practical astronomy to determine the latitude and longitude with greater accuracy. This method of determining the various elements in navigation is called **nautical astronomy.**

MISCELLANEOUS EXERCISES ON NAVIGATION

1. A gunboat of a blockading squadron lies 4 miles to the south of a harbour, and observes that a ship leaves the harbour in a direction E. 30° S. If the blockading ship sails 12 miles an hour, find in what direction she must go so as to cross the course of the other ship in three-quarters of an hour. . . =E. 7° 21′ 45″.

2. From a given point the position of two gunboats is found to be 29° east and 56° west. Supposing them to occupy a position in line distant from the given point 80 yards, and that their line is at right angles to one produced from the given point, find the distance between them. . . . . . . =163 yards nearly.

3. A yacht which is known to be sailing due east at the rate of 12 miles an hour was observed at noon to be 150° to the east of south at 1 h. 30 m. after noon. She was seen in the south-east. Determine the distance of the ship at noon. . . =25·45 miles.

4. From a ship a rock and a headland are observed to bear 18° E. of N. The ship sails 6 miles NW., and then the rock is due east, and the headland NE. What is the distance between the rock and the headland? . . . . =8·755 miles.

5. A ship sailing out of harbour is watched by an observer from the shore, and at the instant she disappears below the horizon he ascends to a height of 20 feet, and thus retains her in sight 40 minutes longer. Find the rate at which the ship is sailing, assuming the earth to be a sphere of 4000 miles radius and neglecting the height of the observer. . =8·257 miles per hour.

6. An observer from the deck of a ship 20 feet above the level of the sea can just see the top of a distant lighthouse, and on ascending to the masthead, which is 60 feet above the deck, he sees the door, which he knows to be one-fourth of the height of the lighthouse above the level of the sea. Find his distance from the lighthouse, and its height, assuming the earth to be a sphere of 4000 miles radius. . . . =Height, 80 ft.; distance, 87196·32 ft.

7. Two ships sail at the same time from the same port, and sail for 5 hours at the respective rates of 8 and 10 knots an hour in straight lines inclined to each other at an angle of 60°. They then sail directly towards each other. Find the inclination of their new course to their original courses. . . . 70° 55′ 36″; 49° 6′ 24″.

8. A buoy is moored 9 miles north of a port from which a yacht

sails in a direction ENE. She tacks and sails towards the buoy until the port is SW. of her, when she tacks and sails into port. Prove that the length of the course is about 16 miles.

NAUTICAL ASTRONOMY

933. By the principles of nautical astronomy the time at the ship's place, the variation of the compass, the latitude and longitude, and various other elements used in navigation can be determined. As the complete solutions of these problems have already been given in the problems in practical astronomy, excepting the circumstances peculiar to navigation, by which the solutions are in some cases modified, it will be necessary here merely to add the methods of calculating the effect of these circumstances.

934. **Problem XIV.—To find the variation of the compass.**

RULE.—Find the azimuth or amplitude of some celestial object by the methods formerly given in Art. 873 and 880; and find also its bearing per compass, and the difference between the azimuth and bearing will give the variation of the compass.

EXERCISES

1. The azimuth of the sun was found to be S. = 48° 54′ E. when its true bearing was S. = 77° 1′ E.; what was the variation of the compass? . . . . =28° 7′, or 2½ points E.

2. The amplitude of a star was found to be E. = 10° 15′ N. when its true bearing was S. = 84° 12′ E.; what was the declination of the needle? . . . . = 16° 3′ W.

3. The azimuth of a star was found to be N. = 68° 10′ E. when its true bearing was NE. *b* E.; what was the variation of the compass? . . . . = 11° 55′ E.

935. **Problem XV.—Having given two altitudes of a celestial body, the ship having sailed for several hours during the interval, to reduce the first altitude to the place at which the second was taken.**

RULE.—Find the angle of inclination between the ship's course and the bearing of the body at the first place of observation, or its supplement when greater than a right angle; then

The radius is to the cosine of this angle as the distance run to the correction in minutes; which is to be applied by addition or subtraction to the first altitude, according as the inclination is less or greater than a right angle.

If d = the distance, i = the inclination, c = the correction, a' = the first altitude, and a = the first altitude reduced to the second place, then Rad. : cos $i = d : c$, and $a = a' \pm c$.

It is evident that c can be found by inspection of the Table of difference of latitude and departure, by considering i as the course, and d the distance; then c will be found in the latitude column.

Exercises

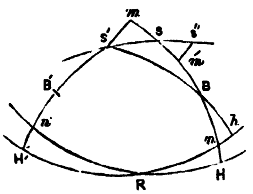
1. The altitude of a star when east of the meridian was observed to be $= 20° \ 40'$, and its bearing at the time was SE. *b* S.; and after sailing 40 miles W. *b* S. its altitude was again observed when it was west of the meridian; what would have been its altitude at the time of the first observation if it had been taken at the place of the second observation?

Here $i = 3 + 7 = 10$ pts. $= 112° \ 30'$, $d = 40$, $c = 15{\cdot}3'$, and $a = 20° \ 24{\cdot}7'$.

2. The sun's altitude was observed to be $= 30° \ 41{\cdot}5'$, and its bearing was SE. *b* E.; and after sailing 48 miles E. *b* S. its altitude was again taken; required the sun's altitude at the latter place of the ship at the time of the first observation.

Here $i = 2$ pts. $= 22° \ 30'$, $d = 48$, $c = 44{\cdot}3'$, and $a = 31° \ 25{\cdot}8'$.

The principle of the rule may be proved thus:—Let S be the zenith of the place of the first observation, and S′ that of the second, B and B′ the positions of the body at these two instants of time, SS′ the intermediate distance sailed by the ship in minutes of space, HRn' and H′Rn the horizons of S and S′; then BH and B′H′ are the two altitudes. Now, to find the altitude Bh of the body when at B, supposing the altitude to be taken then at S′, produce BS to m, and from S′ draw the perpendicular mS′



from S′ on Sm; then, since SS′, and consequently mS, is a small distance, mn may be considered as differing insensibly from S′h, at least for ordinary nautical purposes; but S′h is a quadrant, as also SH; hence mn = SH nearly, therefore mS = nH nearly. Consequently Bh, the altitude of B, when taken at S′, which is nearly = Bn, is *less* than BH by mS. Now, angle BSS′ is evidently $= i$, SS′ $= d$, mS $= c$, and radius : cos $i = d : c$, which is the rule. If the ship had sailed from S to S″ instead of S′, it could in the same manner be

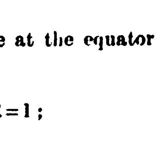
shown, by drawing $S''m'$ perpendicularly to SH, that Sm' would require to be *added* to the altitude of B, taken at S, in order to obtain its altitude at the same time if it were taken at S″.

CONSTRUCTION OF MAPS AND CHARTS

936. Maps and charts are representations of portions or of the whole of the surface of the earth, with meridians and parallels of latitude at some convenient distance from each other, as at 5 or 10 degrees. The principal kinds of construction are the **plane construction**, the method of **conical projection**, the **stereographic projection**, and **Mercator's projection.**

PLANE CONSTRUCTION

937. In the **first method** of plane construction the meridians are parallel straight lines, as are also the parallels of latitude. It is used for a very small portion of the earth's surface, extending only a few degrees in length and breadth, as for a portion of a kingdom.



The breadth from north to south, AC, is the number of degrees of latitude, each of which is equal to 60 geographical or 69·02 imperial miles; and the length from east to west, AB, is just the length of the number of degrees of longitude contained in it, estimated on the parallel EF of middle latitude by the proportion in Art. 914.

Let L', L = the lengths of a degree of longitude at the equator and at the middle latitude,

l = the middle latitude;

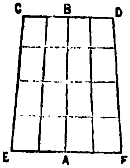
then rad. : $\cos l = L' : L$, and $L = L' \cos l$, if $R = 1$;

and since $L' = 60$ geographic miles, $L = 60 \cos l$.

If $l = 56°$, then $L = 60 \times {\cdot}5592 = 33{\cdot}552$ miles.

938. In the **second method** of plane construction the parallels of latitude are parallel straight lines, and the meridians are converging straight lines. This method is used for projecting larger portions of the earth's surface, as for a kingdom.

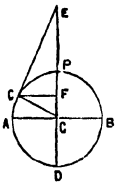
The breadth of the map in this case is AB, which is equal to



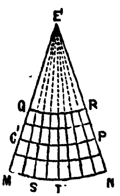
the length of the degrees in the latitude contained in the map, as in the preceding method. The parallels of latitude are perpendicular to AB, which is the middle meridian. The lengths of the degrees of longitude in the parallel CD are the lengths for the latitude of B found as those in EF in the preceding method; and the lengths for the parallel EF are also those corresponding to the latitude of A, and the corresponding divisions of EF and CD are joined by straight lines, which are the meridians.

CONICAL PROJECTION

939. The method of conical projection is used for still larger portions of the earth's surface, as for a continent. This method is thus derived:—A conic surface is supposed to touch the earth's surface along the parallel of the middle latitude, and the former surface is supposed to coincide with that of the earth for a few degrees of latitude on both sides of the middle parallel.



Let APB be the earth, C the middle latitude, and CE a tangent, meeting the axis DP produced; then E is the vertex of the cone, CE its slant side, and ED its axis. When this conic surface is developed on a plane it will be represented by E′MN, in which the points E′, C′ correspond to E and C in the above figure. The breadth of the map MQ, therefore, is just the length of the number of degrees of latitude contained in the map, as in those of the last two articles; and the



length of the middle parallel C′P is just the same length as on the earth, which is computed exactly in the same manner as the length of the middle parallel EF in Art. 938; and the meridians E′S, E′T, &c. are drawn through points of division on C′P from the vertex E′, which is the centre of the parallels of latitude.

The length of the radius E′C′ or EC is just the cotangent of the latitude, if the radius CG=1; for in triangle ECG, angle G is the complement of AGC, which is the latitude of the middle parallel C′P, and

$$GC : CE = \text{rad.} : \tan G, \text{ or } a : r = 1 : \cot l;\ \therefore\ r = a \cdot \cot l,$$

where a=CG, r=CE, and l=middle latitude.

Since the length of C′P is just the length of the middle parallel on the sphere APBD, whose radius CF is $=\cos l$, or $r'=a\cos l$, if $CF=r'$; and the radius EC is $r=a\cot l$; therefore the number of degrees in the angle ME′N will be to the number of degrees of longitude in the map inversely as the radii r and r'. Or, if v, v' denote the number of degrees in angle ME′N, and in the longitude, then $$r:r'=v':v,\text{ or } a\cot l : a\cos l = v' : v;$$

and hence $$v=\frac{\cos l}{\cot l}v'=v'\sin l.$$

Thus, if $l=56°$, and $v'=15°\ 12'$, and $a=246$,

$$r=a\cot l=246\times{\cdot}6745=165{\cdot}927,$$

and $$v=v'\sin l=15{\cdot}2°\times{\cdot}829=12{\cdot}6°=12°\ 36'.$$

STEREOGRAPHIC PROJECTION

940. This projection is used for large portions of the earth's surface, as for a hemisphere. The meridians and parallels of latitude are projected according to the methods explained in the problems of this projection, beginning with Art. 716.

I. When the projection is made on the plane of a meridian.

The meridians are described in the same manner as AGC in Case 1, Art. 722. If, for instance, 12 meridians are to be described in the hemisphere ABCD—that is, at the distance of every 15°—for the meridian next to ADC, the angle FAE will be 15°; for the next it will be 30°; for the next, 45°; for the next, 60°; and so on. Or the meridians may be described thus:—Let the projection of a meridian, inclined to the primitive ACBD (fig. to Art. 718), by an angle measured by the arc AF, be required. Join DF, and through C, G, D describe the circle CGD, for the meridian required. When the meridian is to be inclined 15° to CAD, make AF = 15°; when the inclination is to be 30°, make AF = 30°; and so on.

The parallels of latitude are described thus:—Let CE or CM (first fig. to Art. 719) be the distance of one of the parallels of latitude from the pole; then draw the tangent EL to meet the polar diameter CD produced, and EL is the radius, and L the centre of the projected parallel MNE. Or, join AE by a straight line, and it will cut CP in N; then if a circle is described through MNE, it is the projection required. If the circle is 30° from the pole C—that is, if it is the parallel of latitude of 60°—make CE and CM each = 30°; for the latitude of 50°, make CE and CM each = 40°; and so on.

II. When the projection is to be made on some circle, as the horizon of a place, which cuts the meridian in the east and west points.

Let ACBD (first fig. to Art. 719) be the horizon of the place, AB the meridian of the place, C, D the east and west points, and AF = the latitude. Join FD, and G will be the projected pole, for AG is the projection of the latitude (Art. 720). The projection of any meridian, inclined to the meridian AGB of the place, is described in the same manner as the circle IFK (second fig. to Art. 722), F being the pole, and BFD the meridian of the place. When the meridian is inclined 50° to BFD, make angle LFH = 50°; and similarly for any other meridian.

The parallels of latitude are in this case described as in Case 3, Art. 719.. If P is the pole, AB the meridian of the place, B the north point, and D, C the west and east points; then, to describe the parallel of latitude 30°, draw CP, and produce it to E; make EG and EF each = 60°, the complement of the latitude; draw CG and CF, cutting AB in I and H, and on IH describe the circle IKH for the required projection; and in a similar manner describe the projections of the other parallels of latitude.

941. There is a method of construction called the **globular**, which is not properly a projection, but it is useful, as it represents the magnitudes of different portions of the earth very nearly in their proper proportions. In this method the radii of the polar and equatorial diameters AB, CD (first fig. to Art. 717) are divided into the same number of equal parts—into nine, for instance—when the meridians and parallels are respectively 10° distant, and the meridians pass through the poles A, B, and the divisions of CD; also the four quadrants AC, CB, BD, DA being divided into the same number of equal parts as the radii, the parallels for the northern hemisphere, for instance, pass through the corresponding divisions of the quadrants AC, AD, and radius AE; and similarly for the southern hemisphere.

If a point is taken for the projecting point in that equatorial diameter produced, which is perpendicular to CD, at three-fourths of the radius above the sphere, the projections of the meridians and parallels of latitude on the meridian ACBD would very nearly coincide with those of the globular construction.

942. In the **orthographic** projection the half-meridians, as APB, are semi-ellipses, and the parallels of latitude are straight lines parallel to the equator CD, and passing through the equal divisions of the quadrants AC, AD.

MERCATOR'S CONSTRUCTION

943. In Mercator's construction the meridians and parallels of latitude are parallel straight lines, the former being equidistant for equal differences of longitude, but the distance between the latter, for equal differences of latitude, increases with the latitude. When these distances are very small, as 1′, they are increased in the ratio of the cosine of latitude to radius, or of radius to the secant of latitude. (See Art. 921.)

Let e = an elementary part of a terrestrial meridian--that is, a minute portion of latitude, as 1′,

e' = the enlargement of e on the projection,

l = „ latitude, and R = the radius,

then $\quad R : \sec l = e : e'$, or $e' = e \sec l$, when $R = 1$;

and if $\quad e = 1'$, then e' in minutes $= \sec l$.

Hence, if (1′), (2′), (3′),...denote the enlargement of 1′ in the latitudes of 1′, 2′, 3′,...respectively, and if $m_1, m_2, m_3, .$ denote the meridional parts for these latitudes respectively, then is

$$m_1 = (1'),\ m_2 = (1') + (2'),\ m_3 = (1') + (2') + (3'),\ \&c.$$

Or, $$m_1 = (1'),\ m_2 = m_1 + (2'),\ m_3 = m_2 + (3'),\ \&c.$$

At the latitude of 40° 1′, for example, if m' = the meridional parts for 40°, and m = the same for 40° 1′ or 241′, then is

$$m = m' + (241').$$

This method is not rigorously correct, for the enlargement of 1′ is made uniform at any particular latitude, whereas the minute portions of a minute that are farthest from the equator ought to be increased in a higher ratio than the other portions of it; but the difference is trifling, for in the latitude of 45° the error is only about 0·2′; the meridional parts by the above, or Wright's method, being 3030·127, and its true value being 3029·939. If 1″ were taken for the elementary part of the latitude the error would be less, but the labour of calculation of the meridional parts would be much increased.

To construct a map by this projection.

1. **When the map contains the equator.**

Draw a straight line to represent the equator, and lay off on it from a convenient scale the number of degrees of longitude in the map. Through every 10th degree on this line draw perpendiculars to it for the meridians. Find the meridional parts corresponding to the extreme latitude on the north, for instance; and as it is given in geographical miles, divide it by 60, and lay off the quotient from the same scale as the degrees of longitude on one

of the meridians, and through its extremity draw a parallel to the equator for the extreme parallel of latitude. Find the meridional parts for the latitudes 10°, 20°, 30°,...lay them off in the same manner, and draw the corresponding parallels.

2. When the map is limited by two parallels of latitude of the same name.

Draw a line to represent the lower parallel of latitude in the same manner as the line representing the equator in the preceding case, and draw also the meridians in the same way; then find the meridional difference of latitude for the two extreme latitudes; divide it by 60, and lay off the quotient on a meridian as in the preceding case, and draw the extreme parallel as before. Find then the meridional difference of latitude corresponding to the lower latitude, and the latitude 10° higher; and, dividing it by 60, lay off the quotient from the lower parallel on a meridian, and it will reach the point through which the corresponding parallel passes; proceed in the same way for the parallels of 20, 30, 40,... degrees, and the map will be constructed. (See Art. 922).

GEODETIC SURVEYING

944. The method of **geodetical surveying** is employed when a large portion of the earth's surface, extending several degrees, is to be accurately measured. The method consists in forming a series of large primary triangles, connecting the summits of high edifices and mountains, so that the sides of any one triangle serve as bases for three contiguous triangles. All the angles of these triangles being measured, and one side, the other sides are calculated by the principles of spherical trigonometry, or more simply by means of methods deduced from these principles. The object of the survey may be either the construction of an accurate map or the determination of the elements of the figure of the earth (see Art. 950 and 994).

945. When the angles at the stations, subtended by any other two stations, are taken by means of a sextant, a repeating circle, or any other instrument by which the inclined

angles in the planes of the objects are measured, the angles at the different stations must, by computation, be reduced to the corresponding horizontal angles, which are just the angles of the spherical triangles; but when a theodolite is used in the survey this reduction is unnecessary, as by means of it the horizontal angle is directly measured.

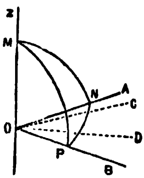
946. The method of a system of triangulation composed of plane triangles may be adopted in a common survey of a large estate, or any small district, over the extent of which the surface of the earth may be conceived to be plane, without any material error; but the triangles of a geodetic survey ought to be spheroidal triangles formed on the spheroidal surface of the earth. On account of the complexity of the direct computation of the parts of such triangles, those of the survey are conceived to be spherical triangles formed on the surface of an imaginary sphere nearly concentric with the earth; and the rather tedious methods of spherical trigonometry are avoided by using more simple and expeditious methods of approximation, founded on the fact that the sides of the triangles are exceedingly small compared with the radius of the earth; and the errors of these methods, being much less than those of observation, are practically insensible. These methods would be inapplicable to comparatively large triangles; but the lengths of the sides of the triangles in a geodetic survey are generally considerably less than 100 miles, or even less than $\frac{1}{100}$ part of the earth's diameter. The lengths of the sides of the spherical triangles being found, those of the corresponding spheroidal triangles can be more readily computed.

The imaginary sphere on which the triangles are conceived to be formed is one whose centre is the centre of curvature of the elliptical meridian at the place, its surface being on a level with the sea; so that the sides of the triangles represent the distances of the stations referred to this sphere. It is evident, however, that when two consecutive stations are

considerably elevated above the surface of this sphere, their distance will be greater than the corresponding side of the spherical triangle; and when the former is known, the latter is to be computed from it as in Art. 955.

947. **Problem I.—Given the zenith distances of two stations observed at a third station, and the inclined angle at the latter, subtended by the distance between the former, to find the corresponding horizontal angle.**

RULE.—Consider the two zenith distances and the inclined angle to be the three sides of a spherical triangle; and find the spherical angle contained by the two former, and it will be the required horizontal angle.



Let OAB be the three stations, OC, OD horizontal lines at the station O, and OZ a vertical line; also, let MNP be a spherical triangle, whose centre is O. Let MN $=z=$ the zenith distance of A, MP $=z'=$ the zenith distance of B, PON $=o$, the inclined angle at O, COD $=$ O, the horizontal angle at O; then, if $s=$ half the sum of the sides by Art. 769,

$$\text{Sin}^2\tfrac{1}{2}\text{O}=\frac{\sin(s-z)\,.\,\sin(s-z')}{\sin z\,.\,\sin z'}\quad.\quad.\quad.\quad[1].$$

Or,
$$\text{Cos}^2\tfrac{1}{2}\text{O}=\frac{\sin s\,.\,\sin(s-o)}{\sin z\,.\,\sin z'}\quad.\quad.\quad.\quad.\quad[2].$$

948. When the zenith distances differ by only 2 or 3 degrees in excess or defect from 90 degrees—that is, when two objects are elevated or depressed by only 2 or 3 degrees—the following formulæ may be employed:—If e, $e'=$ the elevations or depressions of A and B, and $d=\text{O}-o$; then $e=90-z$, $e'=90-z'$, and $\text{O}=o+d$; also, if d, e, e' are expressed in parts of the radius,

$$d=\tfrac{1}{4}(e+e')^2\tan\tfrac{1}{2}o-\tfrac{1}{4}(e-e')^2\cot\tfrac{1}{2}o\,.\quad.\quad.\quad[3].$$

949. But if d, e, e' denote the number of minutes in these arcs, then $d=\frac{1}{13751}\{(e+e')^2\tan\frac{1}{2}o-(e-e')^2\cot\frac{1}{2}o\}$ . . [4].
Or, denoting the terms within the parentheses respectively by m and n, $d=\frac{1}{13751}(m-n)$, or $\text{L}d=\text{L}(m-n)-4\cdot13833$.

In formulæ [3] and [4], when logarithms are used, they require to be carried only to 5 decimal places.

When e is expressed in minutes, then, since $1'=\cdot000290888$ when radius $=1$, therefore the reciprocal of $1'=3437\cdot75$; and hence, if in the former expression [3] d, e, e' are divided by this number, the

formula will be changed into the latter expression, for $4 \times 3437 \cdot 75 = 13751$, as in [4].

In the formula, e and e' are considered to be elevations; when either of them is an angle of depression its sign must be changed.

The solutions for the formulæ [1] and [2] are exactly the same as in Art. 769; it will therefore be necessary only to give an example of the application of the last formula.

EXAMPLE.—Given e an angle of elevation $= 1° \; 0' \; 10 \cdot 92''$, e' an angle of depression $= -7' \; 49 \cdot 54''$, and $o = 61° \; 33' \; 20 \cdot 59''$, to find O.

Here $e = 60 \cdot 182'$, $e' = -7 \cdot 826'$, $e + e' = 52 \cdot 356$, $e - e' = 68 \cdot 008$, and by [4],

$2L(e+e')$ . . .	$= 3 \cdot 43793$	$2L(e-e')$ . . .	$= 3 \cdot 66512$
L, tan $\frac{1}{2}o$ . . .	$= \underline{9 \cdot 77495}$	L, cot $\frac{1}{2}o$ . . .	$10 \cdot 22505$
L, m 1632·6 . . .	$= 3 \cdot 21288$	L, n 7765·5 . . .	$= 3 \cdot 89017$

L$(m-n)$6132·9 . . . . . $= 3 \cdot 78766$
Constant, L . . . . . . $= -\overline{4} \cdot 13833$

L, 0·4460 . . . . $= \overline{1} \cdot 64933$

Hence $d = -0 \cdot 446' = -26 \cdot 76''$, which is negative, because $m < n$; and $O = o - d = 61° \; 33' \; 20 \cdot 59'' - 26 \cdot 76'' = 61° \; 32' \; 53 \cdot 83''$.

The same example may be solved in the following manner as an illustration of formula [3]:—

The length of $1° = \cdot 01745$ when radius $= 1$; and hence the lengths of e and e' can be found by means of the preceding or by Art. 279; or, more readily, by means of a Table of the lengths of circular arcs. It is thus found that

$$e = \cdot 01750, \; e' = -\cdot 00228, \; e + e' = \cdot 01522, \; e - e' = \cdot 01978.$$

$2L(e+e')$ . . .	$= \overline{4} \cdot 36482$	$2L(e-e')$ . . .	$= \overline{4} \cdot 59246$
L, tan $\frac{1}{2}o$ . . .	$= \underline{9 \cdot 77495}$	L, cot $\frac{1}{2}o$ . . .	$= \underline{10 \cdot 22505}$
L, m ·00013797 .	$= \overline{4} \cdot 13977$	L, n ·00065691 .	$= \overline{4} \cdot 81751$

And $\frac{1}{4}(m-n) = d = -\cdot 00012973 = -26 \cdot 75''$.

950. The horizontal angle found by this problem is just the spherical angle at the station at which the angle is formed, contained by arcs of two great circles of the earth, considered as a sphere, passing through that point.

EXERCISES

1. Given $z = 88° \; 12'$, $z' = 88° \; 39'$, and $o = 63°$, to find O by [1].
$O = 63° \; 1' \; 30 \cdot 4''$.

2. Given $e = 25' \; 47 \cdot 2''$, $e' = -1''$, and $o = 66° \; 30' \; 38 \cdot 9''$, to find O by [4]. . . . . $O = 66° \; 30' \; 36 \cdot 37''$.

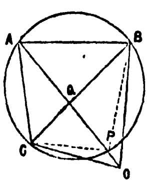
3. Given $e=1° 30'$, $e'=-1° 6'$, and $o=97° 36'$, to find O by [2] or [4]. $O=97° 34' 29{\cdot}94''$.

4. Given $e=-41' 50{\cdot}68''$, $e'=-57' 31{\cdot}91''$, and $o=56° 38' 33{\cdot}34''$, to find O. . . . . $O=56° 38' 54{\cdot}68''$.

5. Given $e=25' 11{\cdot}9''$, $e'=1° 15' 42{\cdot}97''$, and $o=61° 48' 10{\cdot}61''$, to find O. . . . . $O=61° 48' 18{\cdot}6''$.

951. **Problem II.—To reduce angles taken out of the centre of a station to their corresponding angles at the centre.**

Let ABC be three stations, the centre of an object at C being chosen for the centre of that station, at which it is impossible to place an angular instrument for determining the angle ACB; and let an angle AOB be taken at a point near to this station.



The angle ACB is called the **central** angle, AOB the **observed** angle, AOC the **angle of direction,** OC the **central distance,** and BC, AC the **right** and **left** distances, and AB the **base.**

Let ACB=C, AOB=O, AOC=d, OC=c, BC=r, and AC=l;

then $$C-O=\pm\frac{c}{\sin 1'}\left(\frac{\sin(O+d)}{r}-\frac{\sin d}{l}\right) \quad . \quad . \quad [1],$$

in which the correction C − O is expressed in minutes.

When the sum of the angles AOB, AOC is less than 180° the sign + must be used, and when they are together greater than 180° the sign − must be taken.

952. If the angle of the triangle ABC at the right station B be denoted by R, a more concise expression can be found—namely,

$$C-O=\frac{c}{\sin 1'}\cdot\frac{\sin(R-d)\sin O}{r\sin R} \quad . \quad . \quad . \quad [2],$$

where C − O is again expressed in minutes. But this formula, which is only approximative, requires that the correction of the angle C should be small, or that PB should be nearly=BC, supposing a circle ABPC to be described through A, B, and C.

When O falls within the circle, but not within the triangle ABC, then $d>R$, and $\sin(R-d)$ becomes negative. When O lies to the left of C, take BOC=d, CAB=R, and CA=r, supposing the letters A, B to remain unaltered.

The value to be substituted for $\frac{1}{\sin 1'}$ is 3437·75, the logarithm of which is 3·5362743. The values of r and l must be known, but

approximate values are sufficient; as, for example, such as would be obtained by solving a spherical triangle ABC of the magnitude common in geodetical operations, as a plane triangle.

953. When O lies to the left of C the letters A and B may be interchanged, and then the formulæ [1] and [2] apply without alteration.

EXAMPLE.—Given $c=12$, $l=4581{\cdot}8$, $r=5000$, $O=74°\ 32'$, and $d=139°\ 39'$, to find C, O being to the left of C.

Here, though the sum of the observed angles is greater than 180°, since the second includes the first, the second side is plus;

hence
$$C-O=+\frac{c}{\sin 1'}\left(\frac{\sin(O-d)}{r}+\frac{\sin d}{l}\right)$$
$$=\frac{12}{\sin 1'}\left(-\frac{\sin 65°\ 7'}{5000}+\frac{\sin 139°\ 39'}{4581{\cdot}8}\right).$$

L, sin 65° 7'	=	9·95769	L, cos 49° 39'	=	9·81121
L, 5000	=	3·69897	L, 4581·8	=	3·66104
L, m ·00018143,	=	$\bar{4}$·25872	L, n ·00014131,	=	$\bar{4}$·15017

L, $(m-n)$ ·00004012	=	$\bar{5}$·60336
L, 12	=	1·07918
Constant L	=	3·53627
L, 1·655'	=	0·21881

Or, $C-O=-1'\ 39{\cdot}3''$, and $C=74°\ 30'\ 29{\cdot}7''$.

To calculate $C-O$ by the second method, there must be given angle R—that is, ABC (supposing A and B interchanged, as O is to the left of C), and also angle d and r. By observation, let $R=56°\ 7'\ 45''$, then here $r=4581{\cdot}8$ (the l above), and $O=74\ 32'$ as above; but the value of d is the excess of the d given above over O, or $d=65°\ 7'$.

$$C-O=-\frac{c}{r}\,\frac{\sin(R-d)\sin O}{\sin 1'.\sin R}$$
$$=\frac{12}{4581{\cdot}8}\cdot\frac{\sin 8°\ 59'\ 15''.\sin 74°\ 32'}{\sin 1'.\sin 56°\ 7'\ 45''}.$$

L, 12	=	1·07918	L, r	=	3·66104
L, sin $(R-d)$	=	9·19373 − 10	L, R	=	9·91923 − 10
L, sin O	=	9·98398 − 10			3·58027
Constant log.	=	3·53627			3·79316
		3·79316	L, 1·6327	=	·21289

Or, $C-O=1'\ 37{\cdot}96''$; that is, 1·33'' less than the value found by the former method, which is the more correct, and requires only three more logarithms than the latter.

The first formula is easily proved by means of two analogies obtained from the triangles BCO, ACO—namely,

$$r : c = \sin (O+d) : \sin CBO, \text{ and } \sin CBO = \frac{c}{r} \sin (O+d)$$

$$l : c = \sin d : \sin CAO, \text{ and } \sin CAO = \frac{c}{l} \sin d.$$

But CBO, CAO are small angles, and are therefore nearly proportional to their sines; and thus the number of minutes in CBO and CAO respectively will be very nearly $\frac{\sin CBO}{\sin 1'}$ and $\frac{\sin CAO}{\sin 1'}$. Also, angle CAO + C = AQB = CBO + O, and C − O = CBO − CAO

$$= \frac{c}{\sin 1'}\left(\frac{\sin (O+d)}{r} - \frac{\sin d}{l}\right).$$

Were the central distance c considerable, it would be necessary to find the angles CBO, CAO by means of the analogies in the preceding paragraph, and then their difference would give C − O.

By describing a circle about A, B, C, cutting AO in P, the second formula can be proved by means of the triangles CPO, PBO, provided PB be very nearly equal to BC or r; which it must be when c is very small in relation to r. In the triangle CPO, the angle PCO = (R − d), and POC = d; also, in the triangle BPO, the exterior angle APB = C; therefore the angle PBO = C − O. Now, from the triangle CPO we have

$$\sin R : \sin (R-d) = c : PO \; ; \; \therefore \; PO = \frac{c \sin (R-d)}{\sin R};$$

and from the triangle BPO we have

$$r : PO = \sin O : \sin PBO = \sin (C-O);$$

$$\therefore \sin (C-O) = \frac{PO \, . \sin O}{r} = \frac{c \sin (R-d) \sin O}{r \sin R};$$

hence
$$C - O = \frac{c \, . \sin (R-d) \sin O}{r \, . \sin 1' . \sin R}.$$

Exercises

1. The place of observation being to the right of the centre of the station, and O = 60°, d = 58° 41′ 0·6″, l = 19000, r = 20000, and c = 10, to find C. . . . . C = 59° 59′ 57·7″.

2. The data being as in the preceding exercise, except that l and r are ten times less, what is C? . . . . C = 59° 59′ 37·33″.

3. In this example the angles are expressed according to the centesimal or the new division of the circle—namely, O = 44′

25′ 92·6″, $d = 38°\ 14'\ 81·5''$, and $r = 4596·27$, $l = 4041·89$, and $c = 41·69$ metres; find C, supposing O to be situated on the right of the station C. . . . . $C = 44°\ 44'\ 44·4''$.

4. Given O, d, c, and r, as in the preceding example, and $R = 66°\ 66'\ 66·7''$; find C by the formula [2]. . . . $C = 44°\ 44'\ 42·6''$.

The answer to the third example is the correct one, and it therefore appears that with a distance c equal to about $\frac{1}{10}$ of r or l, the error of the second method is 1·8″, or 0·648″ of the sexagesimal division.

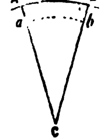
5. Given $O = 45°\ 42'$, $d = 50°\ 18'$, $r = 32656$, $l = 31052$, and $c = 46$ feet; find C, the position of O being still to the right of C.

$C = 45°\ 42'\ 53·85''$.

954. Before proceeding with the computations of geodetical measurements, it is necessary that the mean diameter of the earth should be known with considerable accuracy. A mean value, however, is sufficient for most of these purposes. If a base—for instance, AB (next fig.)—is measured at a level above that of the sea, and is to be reduced to its length ab, if measured at the mean level of the sea; then, supposing this base to be measured at a height of 3000 feet, and to be 40 miles long, the error on the correction of the base, arising from taking the mean diameter, cannot exceed at its greatest $\frac{1}{20}$ of a foot, and will be much less if the place be situated in a medium latitude (see Art. 996).

955. Problem III.—Given the length of a base, and its height above the sea, to reduce it to the level of the sea.

Rule.—Multiply together the base and height, and divide the product by the radius of the earth, and the quotient will be the correction to be deducted from the base.



Let B, b = the measured and reduced bases in feet;
r, h = the earth's radius and the given height in feet,

and $c = B - b$;

then $c = \dfrac{Bh}{r+h} = \dfrac{Bh}{r}$ very nearly,

and $b = B - c$.

Or, taking $r = 20899318$ feet, its logarithm is $= 7·3201319$, and its arithmetical complement $= \bar{8}·6798681$, which may be added for the term $-Lr$, then $L(B - b) = LB + Lh + \bar{8}·6798455$.

EXAMPLE.—Let $B=62546$ feet, and $h=624$, to find b.

$$L(B-b)=4\cdot7961995+2\cdot7951846+\bar{8}\cdot6798681+0\cdot2712522.$$

Hence $B-b=1\cdot8675$, and $b=62544\cdot1225$.

If a, b denote the polar and equatorial diameters of the earth, r the radius of a sphere of equal volume, and r' the equidifferent mean between $\frac{1}{2}a$ and $\frac{1}{2}b$, then

$$\begin{aligned} a&=41706000 \text{ feet}=7898\cdot87 \text{ miles};\\ b&=41845000 \quad " \quad =7925\cdot19 \quad "\\ 2r&=41798636 \quad " \quad =7916\cdot405 \quad "\\ \text{and} \quad 2r'&=41775500 \quad " \quad =7912\cdot03 \quad " \end{aligned}$$

The value of r is obtained thus:—Let v, v' be the volumes of the earth and of the equivalent sphere,

then $\cdot5236\, ab^2=v=v'=\cdot5236\, (2r^3)$;

hence $(2r)^3=ab^2$, and $3L2r=La+2Lb$;

and as a and b are supposed to be known, r can be found.

EXERCISES

1. A base measured at the height of 1245 feet was $=25086$ feet long; required the reduced base. . . . . . $=25084\cdot5056$.
2. Find the reduced base corresponding to a base of 12543 feet measured on a plane elevated $=996$ feet. . . . . $=12542\cdot402$.

956. When the three angles of any of the spherical triangles of a system of triangulation have been observed, they can be verified by means of the **spherical excess**—that is, the excess of the three angles of a spherical triangle above two right angles. This excess can be calculated when the area of the triangle is known, and it can be found when any three parts of the triangle are given. Conversely, the area of a spherical triangle is easily and accurately found when the spherical excess is known.

957. Problem IV.—Given the spherical excess, to find the area of the spherical triangle.

RULE.—As 180 is to the spherical excess in degrees, so is $\frac{1}{4}$ of the surface of the sphere to the area of the triangle.

Let $E=$ the spherical excess,

S, $T=$ " surfaces of the sphere and of the triangle;

then $180:E=\frac{1}{4}S:T$, or $T=\frac{1}{4}\cdot\frac{1}{180}E.S$.

EXAMPLE.—The three angles of a spherical triangle are $=60°$, $65°$, and $85°$ on a sphere whose diameter is $=20$; required the area of the triangle.

$$E=60+65+85-180=30,\ \tfrac{1}{4}S=\pi r^2=314\cdot16;$$

and $T=\frac{1}{180}E\,.\,\frac{1}{4}S=\frac{30}{180}\times314\cdot16=52\cdot36$ feet.

EXERCISES

1. Find the area of a spherical triangle described on a sphere of 15 feet radius, its angles being $= 75°$, $50°$, and $85°$. $= 117\cdot81$ feet.

2. The diameter of a sphere is $= 50$, and the angles of a spherical triangle described on it are $= 75° \ 15'$, $82° \ 12'$, and $35° \ 3'$; find the area of the triangle. . . . . $= 136\cdot354$.

958. When the accurate values of the angles are unknown, the area of a spherical triangle, by means of which the spherical excess is found, cannot be accurately calculated unless by methods that are tedious; but as the spherical excess of triangles whose sides are very small compared with the radius of the sphere, such as those formed in a geodetical survey, amounts only to a few seconds—from 2 to 10 generally, and at most to about 30″—this excess will be determined with sufficient accuracy by considering the spherical triangle as a plane triangle, whose given parts—one of which at least must be a side—are equal to those of the spherical triangle. An error of 1″ on an angle would produce an error of little more than 1 foot on the opposite side of a triangle whose sides are 20 miles long.

When the two sides a, b and the contained angle C are given, the area will be obtained nearly by the formula,

$$T = \tfrac{1}{2}ab \sin C.$$

When the side c and the angles A and B are given,

$$\sin C = \sin (A + B), \ a = c \frac{\sin A}{\sin (A + B)}, \ b = c \frac{\sin B}{\sin (A + B)};$$

and hence

$$T = \tfrac{1}{2}c^2 \frac{\sin A \ . \sin B}{\sin (A + B)}.$$

959. **Problem V.—Given three parts of a spherical triangle, one of which is a side, to find the spherical excess.**

RULE.—From the logarithm of the area of the triangle in feet subtract the number 9·3267737, and the remainder will be the logarithm of the spherical excess in seconds.

$$L.E = L.T - 9\cdot3267737.$$

EXAMPLE.—In a spherical triangle LHS, the stations of which are Leith Hill, Hanger Hill, and Severndroog Castle, the observed angle at L is $= 35° \ 23' \ 14''$, and the containing sides LS and LH are 144760 and 127660.

Let the sides opposite to the angles L, H, S be denoted by l, h, s; then

$$T = \tfrac{1}{2}hs \sin L.$$

L, $\frac{1}{2}h$ 72380 . .	= 4·85962	L, T . .	= 9·72842
L, s 127660 . .	= 5·10605		= 9·32677
L, sin L 35° 23′ 14″	= 9·76275 − 10	L, E . .	= ·40165

Hence E = 2·522 = 2·52″.

960. In calculating the spherical excess, it is generally sufficient to carry the logarithms to the fifth decimal place inclusive, and therefore an error of a few seconds in the given angles, or of several feet in the sides, if they do not much exceed the usual limits, will not produce an error on the spherical excess amounting to $\frac{1}{100}$ part of 1″.

EXERCISES

1. In a spherical triangle WSL, the sides LW, SW are = 158840 and 75014, and angle W = 65° 26′ 48″; what is the spherical excess? = 2·55″.

2. In a triangle CSW, CS = 63489, CW = 133640, and angle C = 16° 35′ 2″; what is the spherical excess? . . . = 0·57″.

3. In a triangle HLW, the side HL = 127660, angle H = 84° 59′ 57″, and angle L = 17° 42′ 37″; what is the spherical excess? . = 1·19″.

Use the formula (Art. 958), $T = \frac{1}{2}c^2 \frac{\sin A \cdot \sin B}{\sin (A+B)}$.

961. The excess of the observed angles of a triangle above 180 may be called the **observed** spherical excess.

962. Problem VI. — Having given the three observed angles and the spherical excess, to find the true values of the angles.

RULE.—Subtract the observed from the true spherical excess, and add one-third of the remainder to each of the observed angles, and the sums will be the true spherical angles.

When the observed exceeds the true spherical excess the remainder is negative—that is, the third of the difference is to be subtracted from each observed angle. When the observed spherical excess is negative—that is, when the sum of the observed angle is less than 180—the remainder will be the sum of the two spherical excesses.

The rule applies only to the case in which the three angles have been observed under equally favourable circumstances; when one of the angles is considered to be perfectly correct, the correction must be distributed equally between the other two; and if two angles are quite correct, the whole correction must be applied to the third. When the angles, or any two of them, have not been

observed under equally favourable circumstances the correction must be distributed among them, according to the degrees of probability of accuracy, or what is called the **weight** of each observation, as determined by the principles of probability, which will be afterwards explained.

Let E and E′ denote the true and observed spherical excess, and E_1 the error or whole correction;

then $$E_1 = E - E'.$$

Let A_1, B_1, C_1 be the observed angles, and α, β, γ their corrections respectively, and A, B, C the true angles; then $\alpha + \beta + \gamma = E_1$; and when the corrections are to be equally distributed $\alpha = \frac{1}{3}E_1$, $\beta = \frac{1}{3}E_1$, $\gamma = \frac{1}{3}E_1$.

And $$A = A_1 + \tfrac{1}{3}E_1,\ B = B_1 + \tfrac{1}{3}E_1,\ C = C_1 + \tfrac{1}{3}E_1.$$

Or generally, $$A = A_1 + \alpha,\ B = B_1 + \beta,\ C = C_1 + \gamma.$$

In the exercises the correction is understood to be equally distributed, unless otherwise expressed.

EXAMPLE.—The observed angles A_1, B_1, C_1 are $= 65°\ 40'\ 10{\cdot}82''$, $40°\ 30'\ 45{\cdot}94''$, and $73°\ 49'\ 5{\cdot}69''$; find the true angles, the true spherical excess being $= 6{\cdot}42''$.

Here
$$\begin{aligned} A_1 &= 65°\ 40'\ 10{\cdot}82'' \\ B_1 &= 40\ \ 30\ \ 45{\cdot}94 \\ C_1 &= 73\ \ 49\ \ \ 5{\cdot}69 \\ &\ 180\ \ \ 0\ \ \ 2{\cdot}45 \end{aligned}$$

Hence $$E_1 = E - E' = 6{\cdot}42'' - 2{\cdot}45'' = 3{\cdot}97''.$$

And $$\tfrac{1}{3}E_1 = 1{\cdot}32''.$$

Therefore,
$$\begin{aligned} A = A_1 + \alpha &= 65\ \ 40'\ 12{\cdot}14'' \\ B = B_1 + \beta &= 40\ \ 30\ \ 47{\cdot}26 \\ C = C_1 + \gamma &= 73\ \ 49\ \ \ 7{\cdot}01 \\ &\ 180\ \ \ 0\ \ \ 6{\cdot}41 \end{aligned}$$

EXERCISES

1. The three observed angles of a triangle are $= 28°\ 40'\ 32{\cdot}26''$, $54°\ 28'\ 17{\cdot}35''$, and $96°\ 51'\ 12{\cdot}95''$; required the true angles, the spherical excess being $= 4{\cdot}3''$.

$= 28°\ 40'\ 32{\cdot}84''$, $54°\ 28'\ 17{\cdot}93''$, and $96°\ 51'\ 13{\cdot}53''$.

In the two following examples A_1, B_1, C_1, and E are given to find A, B, and C; angle A_1 in the last being correct:—

2. $A_1 = 50°\ 42'\ 36{\cdot}8''$, $E = 2{\cdot}13''$, $A = 50°\ 42'\ 38{\cdot}43''$.
$B_1 = 79\ \ 49\ \ 32{\cdot}25$, $B = 79\ \ 49\ \ 33{\cdot}88$.
$C_1 = 49\ \ 27\ \ 48{\cdot}19$, $C = 49\ \ 27\ \ 49{\cdot}82$.

3. $A_1 = 39°\ 40'\ 17{\cdot}24''$, $E = 1{\cdot}04''$, $A = 39°\ 40'\ 17{\cdot}24''$.
$B_1 = 68\ \ 14\ \ 32{\cdot}12$, $B = 68\ \ 14\ \ 31{\cdot}10$.
$C_1 = 72\ \ \ 5\ \ 13{\cdot}72$, $C = 72\ \ \ 5\ \ 12{\cdot}70$.

963. When three parts of a spherical triangle in a geodetic survey are given, the other parts can be calculated by means of the rules of spherical trigonometry; but the method of Legendre, by an **equal-sided plane triangle**, and that of Delambre, by means of the triangle of the chords or the **chordal triangle**, are more expeditious. Of these the following by Legendre's method is the most simple.

If from each of the angles of a spherical triangle one-third of the spherical excess is deducted, the remainders are the angles of a plane triangle, the lengths of whose sides are equal to those of the spherical triangle. This is Legendre's theorem, on which he founds his method, which is adopted in the following problems.

964. **Problem VII.—Given the three angles of a spherical triangle, to find the angles of the equal-sided plane triangle.**

RULE.—From each of the angles of the spherical triangle deduct one-third of the spherical excess, and the remainders are the required angles.

Let A, B, C be the angles of the spherical triangle; A′, B′, C′ those of the plane triangle; and E the spherical excess; then $A' = A - \frac{1}{3}E$, $B' = B - \frac{1}{3}E$, and $C' = C - \frac{1}{3}E$.

EXAMPLE.—The three angles of a spherical triangle are $= 48°\ 12'\ 30{\cdot}02''$, $55°\ 17'\ 36{\cdot}31''$, and $76°\ 29'\ 55{\cdot}8''$; find the angles of the equal-sided plane triangle.

$$\begin{aligned} A &= 48°\ 12'\ 30{\cdot}02'' \\ B &= 55\ \ 17\ \ 36{\cdot}31 \\ C &= \underline{76\ \ 29\ \ 55{\cdot}8\ \ } \\ & \ \ \ 180\ \ \ 0\ \ \ 2{\cdot}13 \end{aligned}$$

Hence $E = 2{\cdot}13''$, and $\frac{1}{3}E = 0{\cdot}71''$.

Therefore,
$$\begin{aligned} A' &= A - \tfrac{1}{3}E = 48°\ 12'\ 29{\cdot}31'' \\ B' &= B - \tfrac{1}{3}E = 55\ \ 17\ \ 35{\cdot}6 \\ C' &= C - \tfrac{1}{3}E = \underline{76\ \ 29\ \ 25{\cdot}09} \\ & \qquad\qquad\ \ 180\ \ \ 0\ \ \ 0 \end{aligned}$$

EXERCISES

In the following exercises the angles A, B, C of the spherical triangle are given to find the angles A′, B′, C′ of the equal-sided plane triangle:—

Given				Answers			
1.	A = 76°	13′	4·32″	A′ = 76°	13′	3·27″
	B = 48	15	13·8	B′ = 48	15	12·75
	C = 55	31	45·04	C′ = 55	31	43·99
2.	A = 49°	12′	13·02″	A′ = 49°	12′	12·21″
	B = 72	0	14·83	B′ = 72	0	14·02
	C = 58	47	34·58	C′ = 58	47	33·77
3.	A = 65°	14′	10·3″	A′ = 65°	14′	8·62″
	B = 70	10	19·08	B′ = 70	10	17·4
	C = 44	35	35·66	C′ = 44	35	33·98

965. When all the angles of a spherical triangle have been observed with equal accuracy, the angles of the equal-sided plane triangle can be obtained by deducting one-third of the observed spherical excess from the observed angles. The exercises given above will exemplify this rule, if the angles A, B, C be considered to be observed angles.

Since the angles of the equal-sided plane triangle can be thus obtained from the observed angles, if one side of the spherical triangle is accurately known, the other two sides, being equal to those of the plane triangle (Art. 963), can be easily computed. The computation of the spherical excess is therefore of little practical utility, except as a means of testing the accuracy of the observed angles; and when they are thus verified, the surveyor is confident of the accuracy of his operations.

966. **Problem VIII.—Given three parts of a spherical triangle, one of them being the correct value of a side, but the given angles being only observed angles, to find accurately the parts of the triangle.**

RULE.—By means of the parts given, compute the spherical excess; find the true values of the angles, and then the angles of the equal-sided plane triangle; and in the latter triangle, by means of its angles and the given side, find its other sides, and they will also be the required sides of the spherical triangle.

EXAMPLE.—Given the three observed angles of a spherical triangle—namely, $A_1 = 35° \ 23' \ 13{\cdot}87''$, $B_1 = 83° \ 26' \ 23{\cdot}6''$, and $C_1 = 61° \ 10' \ 24{\cdot}18''$, and the side $a = 84383{\cdot}12$ feet—to find the true angles and the other two sides of the triangle.

There is only one side given; hence, supposing the triangle

plane (Art. 965), use the formula (Art. 958) $T=\frac{1}{2}c^2 \cdot \frac{\sin A \cdot \sin B}{\sin (A+B)}$,

which here becomes $T=\frac{1}{2}a^2\frac{\sin B_1 \cdot \sin C_1}{\sin (B_1+C_1)}$.

To find the area of the triangle		To find E	
2L, a . .	= 9·85251	L, 2T	= 10·02947
L, sin B_1 . .	= 9·99715 − 10	L, 2	= 0·30103
L, sin C_1 . .	= 9·94255 − 10	L, T	= 9·72844
	19·79221		9·32677
L, sin (B_1+C_1)	= 9·76274 − 10	L, E	= ·40167
L, 2T . .	= 10·02947		

Therefore, $E=2·522$ or $2·52''$, and $\frac{1}{3}E=0·84''$.

And $\frac{1}{3}E_1=\frac{1}{3}(E-E')=\frac{1}{3}(2·52-1·65)=0·29''$.

			And (Art. 962)			
A_1 = 35°	23′	13·87″	A = 35°	23′	14·16″
B_1 = 83	26	23·6	B = 83	26	23·89
C_1 = 61	10	24·18	C = 61	10	24·47
180	0	1·65	180	0	2·52

Also (Art. 964)

A′ = 35°	23′	13·32″
B′ = 83	26	23·05
C′ = 61	10	23·63
180	0	0

To calculate the sides b, c of the plane triangle A′, B′, C′

L, sin A′ . .	= 9·7627510	L, sin A′ . .	= 9·7627510
L, sin B′ . .	= 9·9971470	L, sin C′ . .	= 9·9425445
L, a . . .	= 4·9262556	L, a . . .	= 4·9262556
	14·9234026		14·8688001
L, b . . .	= 5·1606516	L, c . . .	= 5·1060491
Hence $b=144761$.		And $c=127658·32$.	

The sides b, c are also sides of the spherical triangle, so that its angles and sides are now all accurately known.

967. Since the observed angles are supposed to be equally accurate, the angles A′, B′, C′ of the plane triangle could have been obtained from them by deducting from each of these angles $\frac{1}{3}E'=\frac{1}{3}\times 1·65''=0·55''$.

EXERCISES

In the following exercises one side of the triangle is given; and all the angles by observation, except in the third and fourth, in which only two angles are given. In the third, two angles of the plane triangle are found by deducting one-third of the spherical

excess from the two given angles, and its third angle is then easily found; and in the same manner the fourth exercise is calculated. The elements required are: the spherical excess, the true angles of the spherical triangle, the angles of the plane triangle—that is, the angles for calculation—and the other two sides of the spherical triangle, which are just those of the plane triangle. The spherical excess for the third exercise has been already calculated in the exercises in Art. 959. In the first line of each exercise the accurate value of one side is given, and in the second column are the observed angles; these are the data in each exercise; the required parts are: the spherical excess given in the third column, the angles of the equal-sided plane triangle given in the fourth column, and the other two sides given in the last column and placed opposite to the angles to which they are opposite in the triangles :—

1. Severndroog Castle, from Leith Hill Station, 144760·96 feet

Names of Stations	Observed Angles	Spherical Excess	Angles for Calculation	Distances in Feet
Wrotham .	65° 26′ 47·68″	...	65° 26′ 46·85″	...
Severndroog Castle .	86 25 58·40	...	86 25 57·57	158844·33
Leith Hill .	28 7 16·42	...	28 7 15·58	75014·27
	180 0 2·50	2·56″	180 0 0	

2. Wrotham Station, from Severndroog Castle, 75014·27 feet

Names of Stations	Observed Angles	Spherical Excess	Angles for Calculation	Distances in Feet
Chingford .	16° 35′ 1·77″	...	16° 35′ 2·00″	...
Severndroog Castle .	149 26 13·36	...	149 26 13·58	133640·58
Wrotham .	13 58 44·20	...	13 58 44·42	63488·87
	179 59 59·33	0·57″	180 0 0	

3. Hanger Hill, from Leith Hill Station, 127658·32 feet

Names of Stations	Observed Angles	Spherical Excess	Angles for Calculation	Distances in Feet
Westminster Abbey .	...	...	77° 17′ 27·37″	...
Hanger Hill .	84° 59′ 56·81″	...	84 59 56·41	130366·33
Leith Hill .	17 42 36·62	...	17 42 36·22	39809·03
		1·2″	180 0 0	

4. Westminster Abbey, from Leith Hill, 130366·33 feet

Names of Stations	Observed Angles	Spherical Excess	Angles for Calculation	Distances in Feet
Severndroog Castle .	62° 33′ 57·67″	...	62° 33′ 57·22″	...
Westminster Abbey .	...	...	99 45 26·38	144760·03
Leith Hill .	17 40 36·85	...	17 40 36·40	44601·10
		1·35″	180 0 0	

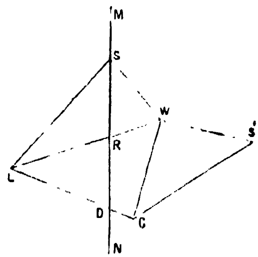
5. Leith Hill, from Wrotham, 158844·33 feet

Names of Stations	Observed Angles	Spherical Excess	Angles for Calculation	Distances in Feet
Crowborough	87° 5′ 15·01″	...	87° 5′ 14·32″	...
Leith Hill .	38 56 55·95	...	38 56 55·25	99982·56
Wrotham .	53 57 51·13	...	53 57 50·43	128615·27
	180 0 2·09	3·03″	180 0 0	

6. Wrotham, from Crowborough, 99982·55 feet

Names of Stations	Observed Angles	Spherical Excess	Angles for Calculation	Distances in Feet
Stede Hill .	44° 44′ 52·83″	...	44° 44′ 51·73″	...
Crowborough	41 58 20·94	...	41 58 19·84	94980·97
Wrotham .	93 16 49·54	...	93 16 48·43	141790·65
	180 0 3·31	2·23″	180 0 0	

The triangles for the first, fifth, and sixth exercises are given in the adjoining figure; the letters at the angular points are the initials of the names of the stations, except S′, which is Stede Hill. The bearing of Leith Hill from Severndroog Castle—that is, angle LSN—being 43° 5′ 51″ SW., MSR will be the direction of the meridian. The computation of the system of triangles is continued in the same manner to any extent. Sometimes the distance of some two of the stations is calculated by means of several triangles of which it is a common side, and the results serve as a means of verifying each other. Thus, for instance, the distance between Chingford Station and Severndroog Castle was calculated by means of five triangles of which it was a side, and the extremes of the computed distances differ by only 1·04 feet; and the greatest difference between any of them and the mean, which is 63489·32 feet, or about 12 miles, is only 0·49 of a foot. It may be observed here, as a proof of the extraordinary precision with which the original base of a geodetic survey is measured, that the base of the survey in Ireland, which is on a level plane near Londonderry, is more than 7 miles long, and that the greatest possible error on its length is considered to be within 2 inches.

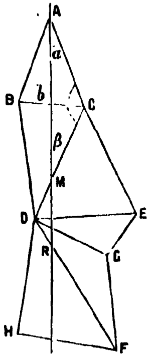


968. When the object of the survey is merely to determine the length of a degree of the meridian, the system of triangulation is extended as nearly as possible in the direction of the meridian.

The method of computing this length is explained in the next problem. The triangles composing a system of triangulation are called **primary** triangles; and other triangles formed within the system for determining the positions of objects for other purposes are called **secondary** triangles.

969. Problem IX. — Having determined the sides and angles of the primary triangles of a system of triangulation lying nearly in the direction of the meridian, to determine the length of that portion of the meridian over which it extends.

Let ABCDEGFDH be a system of primary triangles, whose angles and sides are known as in the preceding problem; find the azimuth a of the station C when observed from A, then angle $\beta = 180 - (a + \gamma)$ nearly, γ denoting angle ACD. Therefore the spherical excess of the triangle ACM can be found (Art. 959); and hence its true angles can be found, and also its sides AM, MC (Art. 966).



But CD is known; hence in triangle DMR the side DM = CD − CM is known, and angle D = CDE + EDG + GDF, and also angle M = AMC. Therefore, angle R is approximately known; consequently the spherical excess for this triangle can be found, and its true angles and sides computed; and hence MR can be found.

Proceeding in this manner, the lengths of the successive portions AM, MR,...of the meridian can be computed; and hence their sum, or the length of the portion of the meridian comprehended between the extreme stations, can be determined.

The latitudes of the extreme stations can be determined by the method in Art. 883, and then the difference of latitude is known, which is just the number of degrees in the arc of the meridian; and the length of one degree can then be found by a simple proportion.

It may be observed here that though the angles at M are equal in the two triangles ACM, DMR, yet the corresponding angles of their equal-sided plane triangles are not equal, for the spherical excesses of these triangles are not equal unless their areas are so.

EXAMPLE. — Let the azimuth $\alpha = 16° 46' 27·59''$, ACM $= \gamma = 143° 13' 41·52''$, and the side AC $= 27458·6$ metres; also angle MDR $= 134° 11' 14·78''$, and CD $= 35164·08$ metres, to find the portions AM, MR of the meridian.

The spherical excess for triangle AMC is found to be $= 0·96''$; hence $M = \beta = 180° 0' 0·96'' - (\alpha + \gamma) = 19° 59' 51·85''$.

Hence the angles of the equal-sided plane triangle, found by taking $0·32''$ from the angles α, β, γ, are $\alpha' = 16° 46' 27·27''$, $\beta' = 19° 59' 51·53''$, $\gamma' = 143° 13' 41·2''$.

The other two sides found by means of this plane triangle (Art. 966) are AM $= 48065·64$, CM $= 23172·57$ metres.

Hence, in triangle MDR, DM $=$ CD $-$ CM $= 35164·08 - 23172·57 = 11991·51$ metres, $M = \beta = 19° 59' 51·85''$, $D = 134° 11' 14·78''$; and computing the spherical excess for this triangle, and then finding the angles of the equal-sided plane triangle, and computing its sides, it is found that MR $= 19745·99$.

Hence AR $=$ AM $+$ MR $= 67811·63$ metres.

The student is requested to perform the computations in this example that are not given above.

By the method explained in this example, the length of the meridian, traversing France from Dunkirk to the parallel of Montjouy, near Barcelona, was computed. The arc was $= 9° 40' 24·24''$, and its length was $= 1075059$ metres.

In the preceding figure the station A is at Dunkirk, B at Watten, C at Cassel, D at Fiefs, E at Bethune, G at Mesnil, F at Sauti, and H at Bennières.

970. **Problem X.—Given the length of an arc of the meridian, and the latitude of its extremities, or the number of degrees in the arc, to find the length of one degree at the middle latitude.**

RULE.—As the number of degrees (d) in the given arc is to one degree, so is the length of the given arc (z) to the length of an arc of one degree (z').

Or, $$d : 1 = z : z' = \frac{z}{d};\ \text{hence } Lz' = Lz - Ld.$$

EXERCISE

The length of an arc of the meridian between Dunnose and Clifton was found, in the trigonometrical survey of England, to be $= 1036337$ feet, and the amplitude of the arc was $= 2° 50' 23·5''$;

required the length of a degree at the middle latitude, which is $=52° 2' 20''$. . . . . $=364925 \cdot 2$ feet.

In this manner the length of a degree of the meridian at any latitude is found. The length of a degree at the middle latitude between Dunnose and Ardbury Hill was found to be 60864 fathoms, which is about 54 fathoms more than it ought to be, according to the usual elements of the figure of the earth. This and many other anomalous results are commonly referred to local attraction deflecting the plumb-line of the sector employed for measuring the angles.

971. The following Table presents the results of various surveys for determining the length of a degree in different latitudes :—

Country	Middle Latitude			Degrees in Arc measured			Length in Feet of Degree	Observers
Peru . .	1°	31′	0″	3°	7′	3″	362808	Condamine.
India . .	16	8	22	15	57	40	363044	Lambton, Everest.
Cape of Good Hope	33	18	30	1	13	17½	364713	Lacaille.
United States	39	12	0	1	28	45	363786	Mason, Dixon.
Rome . .	42	59	0	2	9	47	364262	Boscovich.
France . .	44	51	2	12	22	13	364535	Delambre, Mechain.
England .	52	35	45	3	57	13	364971	Roy, Kater.
Russia . .	58	17	37	3	35	5	365368	Struve.
Sweden . .	66	20	10	1	37	19	365782	Svanberg.

972. The computations of the angles and distances in a geodetic survey can also be performed by means of the **chordal** triangles—that is, the triangles formed by the chords of the sides of the spherical triangles. The same data are required in this method as in the former, or there must be previously known all the angles of the spherical triangles by observation, and at least one side must be accurately determined. This method was adopted in the survey of England.

Let A, B, C, and a, b, c, be respectively the correct angles and sides of one of the spherical triangles; A_1, B_1, C_1, and a_1, b_1, c_1, the corresponding observed angles and the approximate sides of

the same triangle; and let A′, B′, C′, and a', b', c', be the corresponding angles and sides of the triangle formed by its chords.

If c is the side whose length is accurately known, then is

$$c' = 2 \sin \tfrac{1}{2}c.$$

Find also the sides a, b approximately—that is, a_1 and b_1 by plane trigonometry from the angles A_1, B_1, C_1, and the side c; thus—

$$\text{Sin } a_1 = \frac{\sin A_1}{\sin C_1} \cdot \sin c, \text{ and } \sin b_1 = \frac{\sin B_1}{\sin C_1} \cdot \sin c;$$

and in finding a_1, b_1 five decimal places in the logarithms are sufficient.

973. Let a_2, b_2, c_2 be the **angular** excesses in seconds—that is, the excesses of the correct angles A, B, C above the chordal angles A′, B′, C′ respectively—then

$$16a_2 = R''\left(\frac{b_1 + c_1}{r}\right) \tan \tfrac{1}{2}A_1 - R''\left(\frac{b_1 - c_1}{r}\right) \cot \tfrac{1}{2}A_1,$$

where $R'' = 206264{\cdot}8''$, or $206265''$ = the number of seconds in an arc equal to the radius, and r is the same quantity as r' in Art. 955, or 20888000.

Find in the same manner b_2 and c_2, and then it is evident that the spherical excess is

$$E = a_2 + b_2 + c_2, \text{ for } A' + B' + C' = 180.$$

And if E′ = the observed spherical excess $= (A_1 + B_1 + C_1 - 180)$ (Art. 961), and E_1 = the sum of the errors of the observed angles, or $E_1 = E - E'$, then $\tfrac{1}{3}E_1$ is to be applied with its proper sign to A_1, B_1, C_1 in order to give A, B, and C, supposing that there is an equal chance of error in each; then is

$$A = A_1 + \tfrac{1}{3}E_1,\ B = B_1 + \tfrac{1}{3}E_1, \text{ and } C = C_1 + \tfrac{1}{3}E_1;$$

and $$A' = A - a_2,\ B' = B - b_2, \text{ and } C' = C - c_2.$$

The angles of the chordal triangle are now known, and if correct, then is $$A' + B' + C' = 180, \text{ and } A + B + C - 180 = E.$$

974. The angles and the side c' of the chordal triangle being now known, its other sides can be computed by plane trigonometry; in which computation, however, logarithms with at least seven decimal places must be used. Having found these chords a', b', the accurate values a, b, c of the sides of the spherical triangle can now be found thus:

$$a = a' + \tfrac{1}{24}\frac{a'^3}{r^2}, \text{ and } b = b' + \tfrac{1}{24}\frac{b'^3}{r^2},$$

where r has the same value as in the preceding article.

Any of the exercises formerly given under the problem in Art. 966 can be solved by this method, and will therefore serve as an illustration of it.

975. The distribution of the error E_1, as also of E_1 in Art. 962, must be made according to the relative probabilities of accuracy of the observations, agreeably to the remark in that article. The weight of each observation must be computed, and then the error is to be distributed among the three angles, in the proportion of the reciprocals of the corresponding weights. When an angle is determined by only one observation, its weight must be estimated according to the judgment of the observer. In other cases the observed value of an angle is the quotient arising from dividing the sum of the observed values by their number; and the **observed error** of an observation is its difference from the mean value.

976. **Problem XI.—To find the reciprocal of the weight of the observed value of an angle.**

RULE.—Find the mean of the observations of the angle, and then the errors of the observations; find the sum of the squares of the errors, and divide twice this sum by the square of the number of observations, and the quotient is the reciprocal of the weight.

Let u, v, w be the reciprocals of the weights of the mean observed values A_1, B_1, and C_1. Let n = the number of observations by which the mean value A_1 was determined, and e_1, e_2, e_3,...e_n, the errors of these observations; then

$$u=\frac{2}{n^2}(e_1{}^2+e_2{}^2+e_3{}^2+\ldots e_n{}^2);$$

and in a similar manner the reciprocals v and w are found.

EXAMPLES.—1. Let three observed values of an angle A be $=54°\ 20'\ 32{\cdot}5''$, $54°\ 20'\ 33{\cdot}4''$, and $54°\ 20'\ 33{\cdot}7''$; required the reciprocal of the weight of the mean value.

The mean value, or

$$A_1=\tfrac{1}{3}(163°\ 1'\ 39{\cdot}6'')=54°\ 20'\ 33{\cdot}2''.$$

Hence $\quad e_1=0{\cdot}7''$, $e_2=-0{\cdot}2''$, and $e_3=-0{\cdot}5''$.

And $\quad u=\frac{2}{3^2}({\cdot}49+{\cdot}04+{\cdot}25)=\tfrac{2}{9}\times{\cdot}78=0{\cdot}173.$

2. The mean observed values of three angles are $A_1=62°\ 40'\ 20{\cdot}34''$, $B_1=58°\ 26'\ 32{\cdot}82''$, and $C_1=58°\ 53'\ 13{\cdot}05''$; and the

reciprocals of their weights are found respectively to be = ·25, ·32, and ·18; also the spherical excess is = 2·71″; what are the true angles?

Here E = 2·71″, also E′ = 6·21″; hence $E_1 = -3·5″$. Now, E_1 is to be distributed among A_1, B_1, C_1 respectively as the numbers ·25, ·32, and ·18.

·25	·75 : ·25 = − 3·5″ : − 1·17″
·32	·75 : ·32 = − 3·5 : − 1·49
·18	·75 : ·18 = − 3·5 : − 0·84
·75	

Hence

$$A = A_1 - 1·17'' = 62° \; 40' \; 19·17''$$
$$B = B_1 - 1·49 = 58 \quad 26 \quad 31·33$$
$$C = C_1 - 0·84 = 58 \quad 53 \quad 12·21$$
$$180 \quad 0 \quad 2·71$$

FIGURE OF THE EARTH

977. The figure of the earth is that of an oblate spheroid, whose equatorial diameter exceeds its polar by about $\frac{1}{301}$ part of the latter.

Let $2a$, $2b$, e denote the polar and equatorial diameters of the earth, and its **ellipticity**—that is, the ratio of the difference of these axes to the polar axis; then it has been found that

$$2a = 41704788 \text{ feet} = 7898·63 \text{ miles},$$
$$2b = 41843330 \;\;〃\;\; = 7924·87 \;\;〃$$

and $$e = \frac{b-a}{a} = \frac{69271}{20852394} = ·00332 = \frac{1}{301} \text{ nearly.}$$

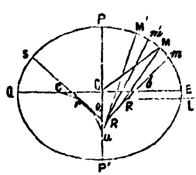
Hence $$2b - 2a = 138542 \text{ feet} = 26·24 \text{ miles}.$$

And the difference of the polar and equatorial radii—that is, the **compression** of each pole = 69271 feet = 13·12 miles.

Since $\frac{b-a}{a} = \frac{1}{301}$ nearly;

therefore $301b = 302a$,

and $a : b = 301 : 302$.



978. The **radius of curvature** of any plane curve on the surface of the earth is the radius of a circle whose curvature is equal to that of the given curve at that point. Thus, the radius of curvature of a meridian at the point M is a line MR, such that the circle described with it from the centre R has a more intimate contact with the elliptic meridian at M than any

other circle. The point R is called the **centre of curvature** for the point M; and the centres of curvature for all the points in the arc PE lie in a curve *aRb*, called the **evolute** of PE, and the radii of curvature, as RM, are tangents to the evolute; so *cra* is the evolute of PQ.

979. The radius of curvature for any point is perpendicular to the tangent at that point; or RM is perpendicular to the tangent at M.

The **normal** to any point is that part of the radius of curvature, produced if necessary, which is intercepted between the point and the axis. Thus, *se* is the normal at the point *s*.

980. The **true** or **astronomical** latitude is the angle contained by the normal and the plane of the equator. Thus, MRL is the true latitude of M.

The **geocentric latitude** of a point on the earth's surface, called also the **reduced latitude**, is the angle contained by the earth's radius at that point and the plane of the equator. Thus, MCE is the geocentric latitude of M.

981. The **true difference of latitude** of two places is the inclination of their normals. The inclination of the normals MR, M′R′ is the difference of the latitudes of M and M′. The normal produced above the earth is called the **vertical**; and the difference between the true and geocentric latitude is called the **angle of the vertical**, or the **reduction** or **correction** of the latitude. This angle is just the inclination of the normal and the earth's radius at the place; thus, angle CMR is the angle of the vertical at M. If l, l' denote the true and geocentric latitudes of a place, the difference of the two is given by the formula,

$$l - l' = 688{\cdot}2242'' \sin 2l - 1{\cdot}1482'' \sin 4l + 0{\cdot}0026'' \sin 6l.$$

When $l = 54° \; 30'$, then is $v = l - l' = 10' \; 59{\cdot}1''$.

982. The length of an arc of any number of degrees is greater the greater the latitude.

If the normals MR, mR contain the same angle as M′R′ and m'R′, then, since the latter normals are longer than the former, and the arcs M′m', Mm are nearly arcs of circles, whose centres are R′ and R, the arc M′m' must exceed Mm.

983. **Problem XII.—To determine the ellipticity of the earth by means of the lengths of two arcs of the meridian measured at different latitudes, and also the number of degrees in the arcs.**

Let l, l' = the latitudes of the middle points of the arcs,
a, a' = " lengths of the arcs in the same denomination,
d, d' = " number of degrees, minutes, or seconds in the arcs.

$$a_1 = \frac{a}{a'}, \text{ and } d_1 = \frac{d'}{d},$$

and e = the ellipticity,

then
$$e = \frac{1 - a_1 d_1}{3(\sin^2 l' - \sin^2 l)}, \text{ or } e = \tfrac{2}{3} \cdot \frac{1 - a_1 d_1}{\cos 2l - \cos 2l'}.$$

Example.—By measurements in Lapland, Svanberg found that an arc of 1°, the middle of which was in latitude = 66° 20′ 10″, was 111488 metres long; and in Peru, Bouguer found that the length of 1°, the middle of the arc being in latitude = 1° 31′, was 110582; required the ellipticity of the earth.

Here $l = 1° \; 31' \; 0''$, $a = 110582$, $d = 1° = 3600''$,
$l' = 66° \; 20' \; 10''$, $a' = 111488$, $d' = 1° = 3600''$.

In this example $d = d'$; hence $\frac{a}{a'} \cdot \frac{d'}{d} = \frac{a}{a'}$;

and $L\frac{a}{a'} = La - La' = 5{\cdot}043685 - 5{\cdot}047229 = \bar{1}{\cdot}996456 = L{\cdot}99187$.

Therefore, $e = \tfrac{2}{3} \cdot \frac{1 - {\cdot}99187}{{\cdot}998599 + {\cdot}677803} = \tfrac{2}{3} \cdot \frac{{\cdot}00813}{1{\cdot}676402} = {\cdot}003233 = \frac{1}{309{\cdot}3}$.

Since here $2l' > 90$, $\cos 2l'$ is negative.

Exercises

1. Given $l = 44° \; 51' \; 2''$, $a = 111108$, $d = 1°$;
also, $l' = 66° \; 20' \; 10''$, $a' = 111488$, $d' = 1°$.

The former data are obtained from the measurements of Delambre, and the latter from Svanberg. The value of e will be $1/301{\cdot}2$.

2. Given $l = 9° \; 34' \; 44''$, $a = 1029100{\cdot}5$ feet, $d = 10210{\cdot}5''$;
also, $l' = 66° \; 20' \; 10''$, $a' = 593277{\cdot}5$ feet, $d' = 5837{\cdot}6''$.

The former data are obtained from Lambton's measurements in India, and the latter from Svanberg's in Lapland. The value of $e = 1/293{\cdot}77$.

984. The ellipticity of the earth can also be determined by observing, by means of a pendulum, the intensity of gravity at

two places, considerably distant, on the earth's surface. The length of the seconds' pendulum is greater the greater the latitude, both because the intensity of gravity is greater the higher the latitude, and because the centrifugal force, which counteracts a part of gravity, is less. By a discussion of the English, Indian, and Russian observations, Clarke (see *Geodesy*) found the value of e to be 1/292·2. The value accepted as the most probable is $1/(300 \pm 3)$.

985. **Problem XIII.—Given the lengths of the seconds' pendulum at two places, whose latitudes are considerably different, to find the ellipticity of the earth.**

Let l, l' = the latitudes of the two places,

p, p' = " lengths of seconds' pendulum at them,

and $p_1 = p \div p'$,

$m = \frac{1}{289}$ = ratio of centrifugal force at the equator to equatorial gravity;

then, if $n = \dfrac{1-p_1}{\sin^2 l' - \sin^2 l} = \dfrac{2(1-p_1)}{(\cos 2l - \cos 2l')} = \dfrac{1-p_1}{\sin (l'+l) \sin (l'-l)}$,

the ellipticity $e = \frac{5}{2}m - n$, where $\frac{5}{2}m = ·0086505$, p and p' may be expressed in any the same denomination.

Example.—The length of the seconds' pendulum at London, in latitude = 51° 31′ 8″, is 39·13929 inches; and at Melville Island, in latitude = 74° 47′ 12″, it is 39·207; required the ellipticity of the earth.

$$\mathrm{L}p - \mathrm{L}p' = 1·5926129 - 1·5933636 = \bar{1}.9992493 = \mathrm{L}·998273,$$

and $$n = \frac{2(1 - ·998273)}{- ·2255934 + ·8622780} = \frac{·003454}{·6366846} = ·005425.$$

Hence $$e = \tfrac{5}{2}m - n = ·0086505 - ·0054250 = ·0032255 = \tfrac{1}{310}.$$

Exercise

At Madras $l = 13° \ 4' \ 9''$, and $p = 39·0234$; and at Melville Island $l' = 74° \ 47' \ 12''$, and $p' = 39·207$; find e. . $e = ·0033299 = \frac{1}{300}$ nearly.

986. The values of the ellipticity do not generally differ much from $\frac{1}{301}$, which has been adopted in this treatise as nearly a mean of the most accurate results. The extreme values, however, derived from accurate operations and observations are $\frac{1}{290}$ and $\frac{1}{334}$.

Such discrepancies in the values of the earth's ellipticity show that it is not of an accurate spheroidal form. That it would be accurately spheroidal is certain were it entirely fluid, for then it would consist of equidense spheroidal strata. Considering, however, the heterogeneous nature of the solid crust of the earth, and

the consequent irregular action of its superficial strata on the spirit-level, the plumb-line, and the pendulum, it is not surprising that the results derived from such a source should not be in perfect accordance.

Those lunar inequalities—that is, the irregularities in the motion of the moon—that are caused by the spheroidal figure of the earth, involve the ellipticity of the earth; and consequently the value of the former elements being known by astronomical observations, the value of the latter can be determined by computation. The result, which, if accurate, must necessarily be nearly the mean value, is found to be $\frac{1}{305\cdot6}$; that is, differs from $\frac{1}{301}$ by only about $\frac{1}{20000}$. This, at the most, would cause a difference of only $\frac{1}{5}$ mile on the earth's radius.

987. **Problem XIV.—Given the length of a small arc on the earth's surface, and its radius of curvature, to find the number of minutes or seconds contained in it.**

RULE.—Divide the arc by the radius of curvature, and multiply the quotient by 3437·75 for the number of minutes; or multiply the quotient by 206265 for the number of seconds.

Let z, r = the length and radius of curvature of the arc,
n°, n', n'' = „ number of degrees, minutes, and seconds respectively in the arc;

then $$n^\circ = 57\cdot29578\frac{z}{r},\quad n' = 3437\cdot75\frac{z}{r},\ \text{and}\ n'' = 206265\frac{z}{r}.$$

Or,
$$\begin{aligned} Ln^\circ &= 1\cdot7581226 + Lz - Lr, \\ Ln' &= 3\cdot5362739 + Lz - Lr, \\ Ln'' &= 5\cdot3144251 + Lz - Lr. \end{aligned}$$

For $2\pi r : z = 360 : n$, for the circumference of circle $= 2\pi r$, where $\pi = 3\cdot1415926$.

Hence $$n^\circ = \frac{360z}{2\pi r},\quad n' = \frac{360 \times 60z}{2\pi r},\quad n'' = \frac{360 \times 60^2 z}{2\pi r},$$

but $$\frac{360 \times 60}{2\pi} = 3437\cdot75,\quad \frac{360 \times 60^2}{2\pi} = 206265,\ \text{and}\ \frac{360}{2\pi} = 57\cdot29578.$$

EXAMPLE.—How many minutes in an arc of 316469 feet on the earth's surface, whose radius of curvature is = 20892000 feet?

Constant log.	=	3·5362739
Lz, 316469	=	5·5003311
		9·0366050
Lr, 20892000	=	7·3199800
Ln', 52·0745	=	1·7166250

Hence $n' = 52\cdot0745 = 52'\ 4\cdot46''$.

When the arc does not amount to 100′, or to about 116 imperial miles, the logarithms need not be carried farther than the fifth decimal place.

EXERCISE

Find the number of seconds in an arc of 423562 feet, its radius of curvature being=20902000 feet. . . . . . $n''=4179{\cdot}79''$.

988. **Problem XV.—Given the degrees contained in an arc on the earth's surface, and its radius of curvature, to find the length of the arc.**

RULE.—Multiply the number of degrees, minutes, or seconds by the radius of curvature, and divide the product by 57·29578, 3437·75, or 206265, according as degrees, minutes, or seconds are given, and the quotient will be the required length of the arc.

Or, $$z=\frac{1}{57{\cdot}29578}rn^{\circ},\ z=\frac{1}{3437{\cdot}75}rn',\ \text{or}\ z=\frac{1}{206265}rn''.$$

Or, $$\begin{aligned} \mathrm{L}z&=\bar{2}{\cdot}2418774+\mathrm{L}n^{\circ}+\mathrm{L}r,\\ \mathrm{L}z&=\bar{4}{\cdot}4637261+\mathrm{L}n'+\mathrm{L}r,\\ \mathrm{L}z&=\bar{6}{\cdot}6855749+\mathrm{L}n''+\mathrm{L}r. \end{aligned}$$

The constant logarithms in these formulæ are the arithmetical complements of those in the preceding problem.

EXAMPLE.—The number of minutes in an arc is=52·0745, and its radius of curvature is=20892000 feet; required the length of the arc.

Constant log.	=	$\bar{4}{\cdot}4637261$
Ln', 52·0745	=	1·7166250
Lr, 20892000	=	7·3199800
Lz, 316469	=	5·5003311

By means of this problem the length of a degree of the meridian at any latitude can be calculated.

EXERCISE

What is the length of an arc of the meridian of 1° 9′ 39·79″ in latitude=48° 50′ 48·59″, its radius of curvature being=20902000, and also the length of an arc of 1° at the same place?

=423562 and 364808·8.

989. **Problem XVI.—Given the number of degrees, minutes, or seconds in an arc, and its length, to find its radius of curvature.**

RULE.—Divide the length of the arc by the number of degrees, minutes, or seconds; and multiply the quotient by 57·29578,

3437·75, or 206265, according as degrees, minutes, or seconds are given, and the product will be the radius of curvature.

Or, $$r=57{\cdot}29578\frac{z}{n^\circ},\ r=3437{\cdot}75\frac{z}{n'},\ r=206265\frac{z}{n''}.$$

Or, $$\mathrm{L}r=1{\cdot}7581226+\mathrm{L}z-\mathrm{L}n^\circ,$$
$$\mathrm{L}r=3{\cdot}5362739+\mathrm{L}z-\mathrm{L}n',$$
$$\mathrm{L}r=5{\cdot}3144251+\mathrm{L}z-\mathrm{L}n''.$$

EXAMPLE.—If the length of an arc of 1° is = 362912·2 feet, what is its radius of curvature?

$$\mathrm{L}r=1{\cdot}7581226+\mathrm{L}z-\mathrm{L}\ 1^\circ=1{\cdot}7581226+5{\cdot}5598015$$
$$=7{\cdot}3179241=\mathrm{L}\ 20793330.$$

EXERCISES

1. What is the radius of curvature of an arc of 52·074′, its length being = 316469 feet? . . . . = 20892200.

2. An arc of 4179·79 seconds is = 423562 feet long; find its radius of curvature . . . . = 20902000.

990. **Problem XVII.—Given the number of degrees in an arc, its length and the latitude of its middle point, and the ellipticity of the earth, to find the polar radius.**

RULE.—Find the radius of curvature of the arc by Art. 989; then find the polar radius by the following formulæ, in which a is the polar radius and l the latitude of the middle of the arc.

1. When the given arc is a part of the meridian,
$$a=r(1+e-3e\sin^2 l),\text{ or } a=\tfrac{1}{2}r(2-e+3e\cos 2l).$$

2. When the arc is perpendicular to the meridian,
$$a=r(1-e-e\sin^2 l),\text{ or } a=\tfrac{1}{2}r(2-3e+e\cos 2l).$$

3. When the latitude $l=o$, or the middle of the arc is on the equator, and the arc is a part of the meridian, find, by Art. 989, the value of its radius of curvature r'; then, if r'' is that of the perpendicular arc, it follows, since $l=o$, that

$$a=r'(1+e),\text{ and } a=r''(1-e);$$

hence $$r':r''=1-e:1+e=1-\tfrac{1}{301}:1+\tfrac{1}{301}=300:302.$$

Or, $$\mathrm{L}r''=\mathrm{L}r'+\mathrm{L}\ 302-\mathrm{L}\ 300.$$

And in this case r'' is just the radius of the equator; hence $r''=b$; and, by the usual formula, $a=b(1-e)=\frac{300}{301}b$, or $\mathrm{L}a=\mathrm{L}b+\mathrm{L}\ 300-\mathrm{L}\ 301$.

EXAMPLE.—In the French survey it was found by Delambre that the difference of latitude between Evaux and Carcassonne was

$=2° 57' 48{\cdot}24''$, and the length of the meridional arc $=168846{\cdot}7$ French toises, the middle latitude being $=44° 41' 48''$; required the axes of the earth, the ellipticity being $\frac{1}{301}$.

Here $l=44° 41' 48''$, $d=2° 57' 48{\cdot}24''=177{\cdot}804'$, and $z=168846{\cdot}7$.

1 French toise : 168846·7 F. t. $=6{\cdot}3946$ English feet : z in feet,

$$Lz = L\ 168846{\cdot}7 + L\ 6{\cdot}3946 = 6{\cdot}0333060.$$

And (Art. 989) $Lr = 3{\cdot}5362739 + Lz - Ld = 7{\cdot}3196383$;

also, $La = L\frac{1}{2}r(2 - e + 3e \cos 2l) = Lr + L{\cdot}998392$

$$= 7{\cdot}3189393 = L\ 20842000.$$

Hence $a = 20842000$, and $b = \frac{301}{300}a = 20911470.$

The values of a given above are also easily derived from the values of R and R′ given in the next problem.

Exercises

1. The length of a degree is $=364535$ feet, and the latitude of its middle point $=44° 51' 2''$; find the polar radius of the earth.
$=20852200$ feet.

2. In the trigonometrical survey it was found that the distance, by General Roy's standard, from Dunnose to Clifton was $=1036337$ feet, and that the middle latitude was $=52° 2' 20''$, and the amplitude $=2° 50' 23{\cdot}5''$; find the semi-axis. $=20849080$ feet.

When this number is reduced to imperial feet by multiplying it by 1·0000691, the value of the axis in imperial feet is $=20850520$; for a foot of General Roy's standard is $=1{\cdot}0000691$ imperial.

991. Problem XVIII.—To find the radius of curvature of a meridian of the earth at any latitude, and of the arc perpendicular to the meridian, its ellipticity and axes being known.

Let R = the radius of curvature of the meridian at any point,
R′ = " " " " arc perpendicular to it;
then, a and e denoting the polar radius and ellipticity, as before (Art. 977), and l the latitude,

$$R = a(1 - e + 3e \sin^2 l),\ R' = a(1 + e + e \sin^2 l);$$

or, $$R = \tfrac{1}{2}a(2 + e - 3e \cos 2l),\ R' = \tfrac{1}{2}a(2 + 3e - e \cos 2l).$$

If r denote the radius of curvature at the given point of the arc of an oblique section, inclined at an angle θ to the meridian, then

$$r = \frac{RR'}{R \sin^2\theta + R' \cos^2\theta},\ \text{or}\ r = \frac{2RR'}{R + R'},\ \text{when}\ \theta = 45°.$$

Instead of r, half the sum of R and R′ may be taken in practice, when $\theta = 45°$.

These expressions are investigated by means of the higher calculus.

In the following example and exercise the mean values obtained for a and e are taken—namely, $a=20853000$, and $e=\frac{1}{301}$, also $\theta=45'$

EXAMPLE.—Find the radii of curvature R, R′, and r for a place in latitude $=48^\circ\ 50'\ 48{\cdot}59''$.

$$R=\tfrac{1}{2}a(2+e-3e\cos 2l)=\tfrac{1}{2}\times 20853000$$
$$(2+\tfrac{1}{301}+\tfrac{3}{301}\times{\cdot}1338757)=10426500\times 2{\cdot}0046566=20901550\,;$$
$$R'=\tfrac{1}{2}a(2+3e-e\cos 2l)=20961550.$$

And $$r=\tfrac{1}{2}(R+R')\text{ nearly}=20931500.$$

In this example $2l>90$; hence $\cos 2l$ is negative, and $-\cos 2l$ becomes $+\cos 2l$.

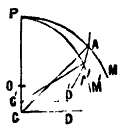
The radii of curvature R, R′ are respectively the radii of the circles of least and greatest curvature at the given place, the radius of any oblique section being always intermediate. The difference between these radii is $=2ae(1-\sin^2 l)=2ae\cos^2 l$, and it is greatest when $l=0$.

EXERCISE

Find the lengths of the radii of curvature R, R′, and r at latitude $=55^\circ\ 40'$. R $=20925450$, R′ $=20969600$, and $r=20947525$.

992. **Problem XIX.—Given the latitude and longitude of a place, and the bearing and distance of another place from the former, to find the difference of latitude and longitude of the two places, and also the bearing of the former in reference to the latter.**

Let A and A′ be the two places, P the pole, and PAM, PA′M′ two meridians; also,



let l, l' = the latitudes of A and A′,

P = APA′, their difference of longitude,

A = MAA′, the azimuth of A′ taken at A,

A′ = M′A′A, " " A " A′,

D = the distance AA′ in feet,

R′ = the radius of curvature of the arc perpendicular to the meridian at A;

then $$l-l'=\frac{D\cos A}{R'\sin 1''}-\frac{D^2\sin^2 A\,.\,\tan l}{2R'^2\sin 1''},$$

which gives the difference of latitude in seconds were the earth spherical. But a correction must be introduced on account of the earth's ellipticity, and the formula then becomes

$$l-l'=\left(\frac{\mathrm{D}\cos\mathrm{A}}{\mathrm{R}'\sin 1''}+\frac{\mathrm{D}^2\sin^2\mathrm{A}\tan l}{2\mathrm{R}'^2\sin 1''}\right)(1+2e\cos^2 l) \quad . \quad [1].$$

Instead of the last factor, which may be denoted by p, its equal $(1+e+e\cos 2l)=p$ may be taken.

Also the difference of longitude in seconds is

$$\mathrm{P}=\frac{\mathrm{D}\sin\mathrm{A}}{\mathrm{R}'\cos l'\sin 1''} \quad . \quad . \quad . \quad . \quad . \quad [2].$$

And the value of A′ in degrees is found by the equation,

$$\mathrm{A}'=180^\circ+\mathrm{A}-\mathrm{P}\frac{\sin\frac{1}{2}(l+l')}{\cos\frac{1}{2}(l-l')} \quad . \quad . \quad . \quad . \quad [3].$$

The last two formulæ are the same for a sphere or spheroid, as no correction is required, on account of ellipticity, for the difference of longitude or the azimuth.

EXAMPLE.—In the trigonometrical survey in France, it was found that the latitude of the Panthéon was $=48^\circ\ 50'\ 48{\cdot}59''$, the azimuth of Dammartin observed at the Panthéon $=133^\circ\ 44'\ 23{\cdot}03''$, and the distance between these two places $=33494{\cdot}32$ metres; required their difference of latitude and longitude, and also the azimuth of the Panthéon in reference to Dammartin.

The distance being given in metres, it must first be reduced to feet.

1 metre : 33494·32 m. = 3·2809 feet : D in feet.

L, 33494·32 . .	$=4{\cdot}5249711$	Also,	$\mathrm{A}=133^\circ\ 44'\ 23{\cdot}03''$
L, 3·2809 . . .	$=0{\cdot}5159930$		$l=\ 48\ \ 50\ \ 48{\cdot}59$
L, D 109891 . .	$=5{\cdot}0409641$		$2l=\ 97\ \ 41\ \ 37{\cdot}18$

And it was found in the example to Art. 991 that $\mathrm{R}'=20969600$.

Also, $p=1+e+e\cos 2l=1+\frac{1}{301}-\frac{1}{301}\times{\cdot}13388=1{\cdot}0028775$.

1. To find the difference of latitude

$$l-l'=\left(\frac{\mathrm{D}\cos\mathrm{A}}{\mathrm{R}'\sin 1''}+\frac{\mathrm{D}^2\sin^2\mathrm{A}\,.\,\tan l}{2\mathrm{R}'^2\sin 1''}\right)(1+e+e\cos 2l).$$

L, D . .	$=5{\cdot}0409641$	L, D^2 . .	$=10{\cdot}0819282$
L, cos A .	$=9{\cdot}8397191-10$	L, $\sin^2\mathrm{A}$.	$=19{\cdot}7176612-20$
L, p . .	$=0{\cdot}0012479$	L, tan l . .	$=10{\cdot}0584930-10$
L, R′ (c) .	$=\bar{8}{\cdot}6784099$	L, p . .	$=\ 0{\cdot}0012479$
L, sin 1″ (c) .	$=5{\cdot}3144251$	L, $\mathrm{R}'^2(c)$.	$=\overline{15}{\cdot}3568198$
L, 749·49 .	$=2{\cdot}8747661$	L, 2 sin 1″(c)	$=\ 5{\cdot}0133951$
1·70		L, 1·696 . .	$=\ 0{\cdot}2295452$
747·79 = 12′ 27·79″			

The arithmetical complements of the logarithms of the quantities

in the denominators of the terms of the formula are indicated above by (*c*), and are of course additive.

Since cos A is negative, the first part of the calculation gives a negative result, and

$$-749{\cdot}49'' + 1{\cdot}70'' = 747{\cdot}79'' = -12'\ 27{\cdot}79'',$$

which is only 0·27″ less than that obtained by a different formula in Francœur's *Géodésie*, p. 212, and with a value of $c = \frac{1}{308{\cdot}65}$ instead of $\frac{1}{301}$, which is adopted in Art. 986, as nearly a mean of the values that are most worthy of confidence.

Latitude of Panthéon	$= l =$	48° 50′ 48·59″
Difference of latitude	$= l - l' =$	+ 12 27·79
Latitude of Dammartin, ...	$l' =$	49 3 16·38

For, since $l - l'$ is negative, l' is greater than l.

2. To find the difference of longitude

$$P = \frac{D \sin A}{R' \cos l' \sin 1''}, \text{ where } l' = 49^\circ\ 3'\ 16{\cdot}38''.$$

L, D . . .	= 5·0409641	L, R′ . . .	= 7·3215901
L, sin A . .	= 9·8588306	L, cos l' . .	= 9·8164692
	14·8997947	L, sin 1″ . .	= 6·6855749
	11·8236342		11·8236342
L, P . . .	= 3·0761605		

Hence P = 1191·69″ = 19′ 51·69″.

3. To find the azimuth of the Panthéon in reference to Dammartin

$$A' = 180^\circ + A - P\frac{\sin \frac{1}{2}(l + l')}{\cos \frac{1}{2}(l - l')}.$$

$$P = 1191{\cdot}69'',\ \tfrac{1}{2}(l + l') = 48^\circ\ 57'\ 2{\cdot}48'',$$

and

$$\tfrac{1}{2}(l - l') = -6'\ 13{\cdot}9''.$$

L, P	= 3·0761605
L, sin $\frac{1}{2}(l + l')$	= 9·8774547
L, cos $\frac{1}{2}(l - l')(c)$	= 10·0000007
L, 898·70″	= 2·9536159

And

$$-898{\cdot}70'' = -0^\circ\ 14'\ 58{\cdot}70''$$

$$180^\circ + A = 313\ \ 44\ \ 23{\cdot}03$$

And

$$A' = 313\ \ 29\ \ 24{\cdot}33$$

This value of A′ is measured from the south in the same direc-

tion as A ; but if it is to be measured in an opposite direction, and also from the south, then will

$$A' = 360° - 313° \, 29' \, 24{\cdot}33'' = 46° \, 30' \, 35{\cdot}67''.$$

When the value of R′ is computed for a given latitude, it may be used without sensible error for all other places whose latitudes differ by less than 3 or 4 degrees from the former.

Exercise

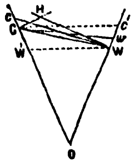
The distance of Black Down from Dunnose is $=314307{\cdot}5$ feet; the latitude of Dunnose $=50° \, 37' \, 7{\cdot}3''$; the azimuth of Black Down, observed at Dunnose, contained between the meridian to the south and the direction of Black Down, is $=95° \, 5' \, 7{\cdot}5''$, the latter place lying westward, and a little north of Dunnose; required the difference of latitude and longitude of these two places, and the azimuth of Dunnose in reference to Black Down.

$l' = 50° \, 41' \, 14{\cdot}07''$, $P = 1° \, 21' \, 2{\cdot}02''$, and $A' = 85° \, 57' \, 31{\cdot}51''$, reckoned from the south towards the east.

THE RELATIVE AND ABSOLUTE HEIGHTS OF THE STATIONS

993. The distances of the stations of a trigonometrical survey are generally so great that refraction sensibly affects the angles of elevation or depression—that is, the vertical angles of position of one station observed at another. (See Art. 587.) The vertical angle of position of one station in reference to another being corrected for refraction, and the distance between them being known, the difference of altitude of the two stations can then be found.

994. **Problem XX.—Given the observed angles of elevation or depression of two stations in reference to each other, and their distance, to find the correct angles and also the refraction, considering it equal for both angles.**



Let C and W be the two stations, and O the earth's centre, or rather the centre of curvature (Art. 978), OC, OC′ two radii of curvature through C and W, and CH, WH two perpendiculars to them—that is, two horizontal lines through C and W. W appears from refraction to be elevated to w; and similarly C appears at c when seen from W; also, let C′ be taken, so that OC′ = OC, and let OW′ = OW; also, let angle

HCw $=e$, the observed angle of depression of W,
HWc $=e'$, " " " C,
WCw $=r=$ CWc, the refraction supposed equal for both,
COW $=o$, the angle at centre of curvature,
C′CW $=v$, " real angular depression of W below C,
and CWW′ $=v'$ " " elevation of C above W;
then is $2r \;=o-(e+e')$ and $v=\frac{1}{2}o-(e+r)$.
When r is not required, then $v=\frac{1}{2}(e-e')$. When e or e' is an angle of elevation, its sign must be changed; also, if z and h are the distances between C and W, and the height C′W of C above W, and if v is expressed in seconds, then is

$$h=vz\sin 1'', \text{ or } Lh=\bar{6}\cdot6855749+Lv+Lz.$$

EXAMPLE.—In the British trigonometrical survey, it was found that two stations of a large triangle connecting the borders of England and Scotland were $=235018\cdot6$ feet or 44·511 miles distant; from Cross Fell, one of these stations, Wisp Hill, the other station, was seen depressed $=30'\ 48''$, and from Wisp Hill, Cross Fell was depressed $=2'\ 31''$; required the refraction and the height of Cross Fell above Wisp Hill.

Here $\qquad e=30'\ 48'',\ e'=2'\ 31'',\ z=235018\cdot6.$

And if at this latitude (about 55°) an arc of one minute (found as in Art. 988) be about 6094·5 feet, then is z equal to $38'\ 33\cdot7''=o$; therefore

$$2r=o-(e+e')=38'\ 33\cdot7''-33'\ 19'', \text{ and } r=2'\ 37\cdot35''.$$

Hence $\qquad v=\frac{1}{2}o-(e+r)=19'\ 16\cdot85''-33'\ 25\cdot35''.$

$$=-14'\ 8\cdot5''=-848\cdot5'';$$

also, $\qquad Lh=L\sin 1''+Lv+Lz=\bar{6}\cdot6855749+2\cdot9286518$

$$+5\cdot3711022=2\cdot9853289;$$

and $\qquad h=-966\cdot8$ feet.

So that the height of Wisp Hill above Cross Fell is $-966\cdot8$ feet—that is, the latter station is higher than the former by 966·8 feet. In this example the refraction amounts to about $\frac{1}{15}$ of the intercepted arc.

The formulæ are easily proved. Since angle HCC′ + C′CO = 90°, and in the isosceles triangle OCC′, $\frac{1}{2}o+C=90$; hence HCC′ $=\frac{1}{2}o$; and from this it is evident that $v=\frac{1}{2}o\sim(e+r)=(e+r)-\frac{1}{2}o$ in this case. Similarly, it can be proved that $v'=\frac{1}{2}o-(e'+r)$. But $v=v'$; hence $e+r-\frac{1}{2}o=\frac{1}{2}o-e'-r$, and $2r=o-(e+e')$.

Again, in the triangle CC′W, angle C′ may be considered to be a right angle without sensible error in computing h, which is small in comparison with z, and CC′ $=z$; hence $z:h=1:\sin v$, and $h=z$

sin v. But if v is expressed in seconds, $\sin v : \sin 1'' = v : 1''$, as v is small; hence $\sin v = v \sin 1''$, and $h = vz \sin 1''$.

995. If the height of the eye when observing the vertical angle of position of a station, taken at another station, is higher than the latter station, which is generally the case, the height of the eye being that of the instrument, the observed angle will thus be too great or too small by a few seconds, according as it is an angle of depression or elevation. Thus, if z is the distance of the stations in feet, h' the height of the eye in feet above one station, and v'' the angle in seconds, subtended by the height h' at the distance z —that is, at the other station; then $z : h' = 1 : \sin v''$, and

$$\sin v'' = v'' \sin 1'',$$

and $$\sin v'' = \frac{h'}{z}\,; \text{ hence } v'' = \frac{h'}{z \sin 1''}.$$

Or, $$\mathrm{L}v'' = 5{\cdot}3144251 + \mathrm{L}h' - \mathrm{L}z.$$

Sometimes, instead of observing the elevation or depression of the top of the second station, it is that of the top of the signal that is observed, and of course a correction must also be made on this account exactly similar to that in the last paragraph.

996. It is evident that the distance between any two stations is represented by the corresponding side of one of the spherical triangles described on the imaginary sphere mentioned in Art. 946. By computing the radius of curvature at the mean latitude of these stations, this distance can be reduced to the level of the sea at that latitude; and if their distance, measured by an arc concentric with the sphere, and passing through one of the stations, be required, it can be found in a similar way by using a radius equal to the sum of the height of this station, and the sphere's radius previously mentioned for the middle latitude.

Exercise

At the station of Black Comb in Cumberland, Scilly Bank appeared depressed $= 49' \, 14''$; and at Scilly Bank, Black Comb was observed to be elevated $= 31' \, 31''$, the distance between the stations being $= 121028$ feet; required the refraction and the height of Black Comb above Scilly Bank, the height of the instrument at both stations being $= 5\frac{1}{2}$ feet.

The correction of the angles of depression and elevation for the height of the instrument, or $v'' = 9{\cdot}4''$; the refraction $r = 1' \, 14''$, or $\frac{1}{18}$ of the intercepted arc; and the difference of height of the two stations, or $h = 1421{\cdot}43$ feet.

In the survey the heights of the stations are stated to be 500 and 1919; and hence their difference = 1419 feet.

997. When observations are taken at only one of two stations, the amount of refraction at the time of observation cannot be determined; and therefore the mean refraction must be taken, which is about $\frac{1}{12}$ of the angle measured by the distance between the stations at the earth's centre. The effect of horizontal refraction is to increase the height of the station observed by a quantity in feet equal nearly to $\frac{1}{9}$ of the square of the distance in miles. For instance, for a distance of 10 miles, the height would be increased by $\frac{1}{9} \times 100 = 11$ feet.

Let h = the height of an object just visible at the distance d, were there no refraction.

h' = the height of another object at the same distance just visible when there is a mean refraction.

Then if h, h' are expressed in feet and d in miles, $h = \frac{2}{3}d^2$, $h' = \frac{2}{3}(d - \frac{1}{12}d)^2 = \frac{5}{9}d^2$ nearly, and $h - h' = \frac{1}{9}d^2$.

The effect of mean refraction, therefore, would increase the height for a distance of 20 miles by about $\frac{1}{9} \times 20^2$, or 44 feet.

If the refraction at the time of observation were only $\frac{1}{6}$ instead of $\frac{1}{12}$, then would $h' = \frac{2}{3}(d - \frac{1}{6}d)^2 = \frac{2}{3} \cdot (\frac{5}{6})^2 d^2 = \frac{2}{3} \cdot \frac{25}{36}d^2 = \frac{4}{9}d^2$ nearly. And therefore the error on the height, when the refraction is in this extreme state, arising from adopting the mean refraction, would be $= (\frac{5}{9} - \frac{4}{9})d^2 = \frac{1}{9}d^2$; which for a distance of 20 miles would be = 44 feet. But this extreme case rarely occurs. Were the refraction $\frac{1}{10}$, then would $h' = \frac{2}{3}(d - \frac{1}{10}d)^2 = \frac{2}{3} \cdot (\frac{9}{10})^2 d^2 = \frac{27}{50}d^2$. And the error arising from adopting the mean refraction would be $= (\frac{5}{9} - \frac{27}{50})d^2 = \frac{1}{64}d^2$ nearly; which for a distance of 20 miles would give $\frac{1}{64} \times 400 = 6{\cdot}2$ feet.

998. To avoid such errors, it is of importance that the zenith distances should be taken simultaneously at every two stations of each triangle, in order that the refraction for each pair of reciprocal observations may be the same as nearly as possible. The relative heights of the stations being known by the foregoing process, the absolute heights can easily be found by determining that of one station. A station is to be chosen for this purpose near to the sea; but instead of computing its height by observing the depression of the horizon, which is very uncertain on account of the unknown horizontal refraction, rendered still more irregular by the vapours exhaled at the surface of the sea, the more correct method of levelling or of Mensuration of Heights and Distances (Art. 589 to 603) is to be adopted in preference.

CURVE-TRACING

999. The mutual dependence of two quantities which may each take a succession of values may in general be represented by an algebraical formula containing these as variables (see above, p. 299). When the one quantity is given, the other may be calculated either exactly or to any desired degree of approximation. Thus, when a ball is thrown with a given initial velocity, the path it describes is perfectly definite, so that the height it has reached at any instant has a definite relation to the time which has elapsed from the beginning of the motion.

Thus, with the notation of Art. 699, if y is the height reached at time t, the relation between y and t is given by the formula,

$$y = vt \sin e - \tfrac{1}{2}gt^2.$$

Also, if we represent by x the horizontal projection of the distance travelled in the time t, we have for the relation between x and t the formula,

$$x = vt \cos e.$$

We may represent each of these relations by a geometrical construction, which not only shows to the eye almost at a glance the nature of the relation, but also enables us to arrive rapidly and unerringly at important quantitative conclusions. This is the method of curve-tracing, the curve or graph being obtained by a series of points which are laid down on paper in such a way that the distances of any one of these points from two chosen lines represent to scale the corresponding values of the two quantities involved in the formula.

Thus, to fix our ideas, let $v \sin e$ and $v \cos e$ be respectively 30 and 40 feet per second, so that the equations above become,

$$y = 30t - 16t^2,$$
$$x = 40t.$$

Draw the two lines, or 'axes,' OX and OY at right angles to each other (perpendicularity is not essential, but it greatly simplifies the interpretation), and let time be measured along the horizontal

axis OX, and y (or x) along the vertical axis. Arrange the corresponding numbers as follows in rows, so that the top number in

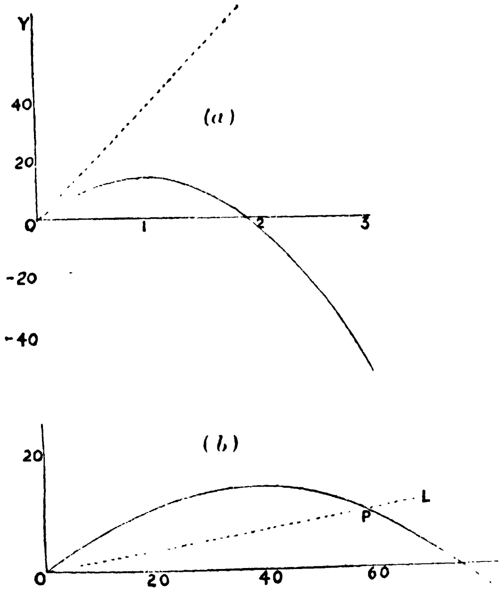


Fig. 1.

any column is the time, and the second and third numbers in the same column the corresponding values of y and x.

$t=$	0	0·5	1	1·5	2	2·5	3
$y=$	0	11	14	9	−4	−25	−54
$x=$	0	20	40	60	80	100	120

Mark off convenient scales along the two axes; lay off a point on OX 0·5 to the right of O, and measure a vertical height 11 above this point; similarly, measure height 14 above the point 1 to the right of the starting-point or origin, O; then height 9 above the

point 1·5 to the right, and so on as indicated in the figure. These are only a few of the infinite number of points which may be similarly calculated; but a little consideration will show that all points obtained by similar calculation will lie in a continuous sequence, and will trace out a curve passing through the few already calculated. Hence, if we draw with free hand a smooth, continuous curve passing through these marked in we shall get a curve which may be regarded as a representation of the formula; and the more numerous the points filled in by calculation the better will the representation be. Note that somewhere between 1·5 seconds and 2 seconds y changes sign, and must thereafter be measured *down* instead of *up*. At some intermediate instant y will have value zero. This instant is easily calculated, for if y vanishes so must the quantity

$$t\,(30-16t);$$

but this vanishes when either $t=0$, or $t=30/16=1·875$. If the graph had been drawn with great care and accuracy this value could have been read off. Thus the method may, under certain conditions, lead to the solution of an equation.

Now, in exactly the same manner, the simpler graph of

$$x=40t$$

may be drawn. It is shown in dotted line in the diagram, and is a straight line passing through the origin, because x and t vanish together.

But we may get a third curve from the numbers given in the Table—that curve, namely, which shows the relation between x and y. In this case let us take new axes and measure x horizontally and the corresponding y vertically. The result is shown in fig. 1, *b*. Broadly speaking, this graph x, y has the same characteristics as the graph t, y. The value of y passes through a maximum value, then becomes zero, and finally becomes negative. By eliminating t between the original equations, we find that x and y are connected by the formula,

$$y=\tfrac{3}{4}x-\tfrac{1}{100}x^2=0·75x-0·01x^2.$$

Now, since y is the height reached at any assigned time, and x is the horizontal projection of the distance travelled in that time, it is clear that the point marked out by any pair of corresponding values of x and y is a point passed through by a projected ball. In fact, the graph x, y is the trajectory. From it we see at a glance what the range is on a horizontal plane; and we can also find by an obvious construction what the range is along any

inclined plane such as that indicated in the figure by the line OL cutting across the trajectory at the point P, whose measured height is 10 and distance from O 58.

1000. The equations which are represented by graphs are called the equations of the graphs or curves. The t, y and x, y graphs just given belong to the class of curves known as parabolas. One term involves the square of one of the variables. The general type of the parabolic equation is

$$y=a+bx+cx^2.$$

In finding a complete graphical representation of this equation we must bear in mind the possibility of there being negative as well as positive values of x and y. If a, b, c are all positive numbers—take, for example, the numbers 12, 8, 1—then whatever positive value of x be chosen, the corresponding value of y will also be positive. Also, as x is taken larger and larger, y will grow simultaneously larger, but at a greater rate. This is seen in the Table below. When x is zero, $y=12$ $(=c)$. When x is taken negative we must measure to the left of the origin, and y will continue diminishing in value to a certain point. But evidently not indefinitely; because x^2 is necessarily positive for all negative values of x, and will become more and more dominant the larger x is taken. In fact, when x is very large and negative y will be positive, whatever the positive values of a, b, c may be. The following Table shows how the values of x and y correspond for the case in which a, b, c have values 12, 8, 1.

x	− 100	− 10	− 7	− 6	− 5	− 4	− 3	− 2	− 1	0	+ 1	+ 2	+ 5	+ 10	+ 100
y	+ 9212	+ 32	+ 5	0	− 3	− 4	− 3	0	+ 5	+ 12	+ 21	+ 32	+ 77	+ 192	+ 10812

The corresponding graph is shown in fig. 2. The points where the curve cuts the axis of x are the roots of the quadratic equation $x^2+8x+12=0$. The quantity y has a minimum value -4, which occurs when $x=-4$. With any other positive values given to a, b, c, a curve broadly similar to the one sketched will be obtained.

How will the properties of the graph be altered if a and b are made negative? Draw the graph of $y=x^2-8x-12$. A sufficient series of values are as follows:—

x	− 10	− 3	− 2	− 1	0	+ 2	+ 4	+ 8	+ 9	+ 10	+ 20 &c.
y	+ 168	+ 21	+ 8	− 3	− 12	− 24	− 28	− 12	− 2	+ 8	+ 228 &c.

This graph is also shown in fig. 2. It intersects the x axis in the points for which $x=4+\sqrt{28}$, and $x=4-\sqrt{28}$—that is, about 9·29 and $-1·29$. These values may be picked out from the graph or worked out directly from the quadratic equation when $y=0$. The two graphs in fig. 2 intersect at one point, the co-ordinates or distances from the axes of which have the values $x=-1·5$, $y=+2·25$. In this simple case these values are easily calculated as simultaneously satisfying the two equations.

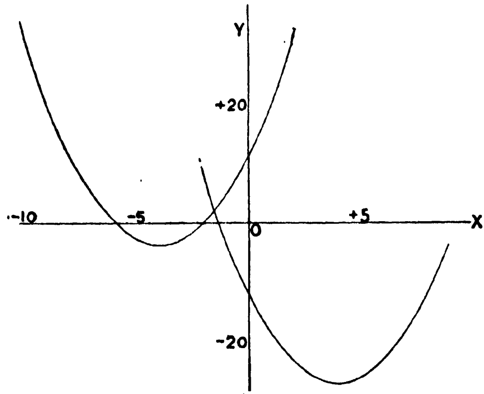


Fig. 2.

1001. The method hinted at here might be of great service in solving a somewhat complex equation which could not be solved directly by ordinary methods. For example, the position of equilibrium for two rods jointed together and resting on the convex surface of a parabola, vertex up, is determined by the equations,

$$Pa \sin^3\theta + 4(P+Q)m \cot \phi = 0,$$
$$Qb \sin^3\phi + 4(P+Q)m \cot \theta = 0;$$

in which P, Q are the weights; a, b the distances from the joint of the centres of gravity; θ, ϕ the inclinations of the rods; and m the distance between the vertex and focus of the parabola. Suppose, to take a special case, that

$$\frac{4(P+Q)m}{Pa} = \frac{1}{\sqrt{2}}, \quad \frac{4(P+Q)m}{Qb} = 1,$$

conditions which may be satisfied in many ways; then the equations become,

$$\sqrt{2}\sin^3\theta = \cot\phi,$$
$$\sin^3\phi = \cot\theta.$$

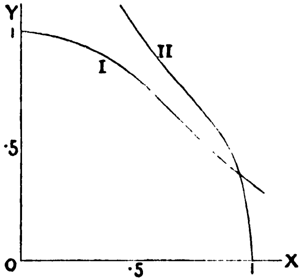


Fig. 3.

Squaring and writing x for $\sin^2\theta$ and y for $\sin^2\phi$, we get the simpler forms,

$$y(1+2x^3)=1, \tag{I}$$
$$x(1+\ y^3)=1. \tag{II}$$

To eliminate y and find the equation in x only would lead to a troublesome form; but we can very quickly get an approximation to the required solution by drawing the two graphs and finding their point or points of intersection. In this case, since x and y are sines of angles, we need consider no values greater than unity. We may quickly, with the help of a Table of cube-roots, get the following points on the two curves in the important region:

Equation I						Equation II				
x	0	0·5	0·6	0·79	1	1	0·9	0·8	0·6	0·5
y	1	0·8	0·7	0·5	0·33	0	0·48	0·63	0·87	1

The graphs are shown in fig. 3, intersecting at a point for which x is very nearly equal to 0·95. This value substituted in Equation II. gives $y=3·76$; but if we use this value for y in Equation I. we find $x=0·94$. Hence, to get the accurate point of intersection

we must move a little to the right. It will be found that $x=0{\cdot}953$ and $y=0{\cdot}366$ satisfy both curves to the third significant figure. Hence the angles of inclination of the two rods in the case proposed are the angles whose sines are the square roots of these numbers—namely, 77° 24′ and 37° 15′.

1002. Curve-tracing is of great value in the comparison of experimental results. Numerical values of two related quantities are obtained from the experiment; and each pair of corresponding numbers is used to determine a point in the diagram exactly as in the cases already discussed. Through the points so determined a continuous curve is drawn, and from a study of the form and peculiarities of this curve important conclusions may often be made as to the law connecting the quantities. In certain cases it may be possible to assign an equation between the x and y coordinates; but even if this be not possible, much information may be gained by a mere glance at the curve.

A very good and yet simple example of an experimental curve is the one obtained from the experiment first devised and executed by Boyle for comparing the volume and pressure of a given mass or quantity of air or other gas. A few of Boyle's original measurements are given in the following Table, the first row being numbers measuring the volumes of the air, and the second row containing the corresponding heights of mercury column by means of which the pressure was increased :

Volume of air . . x	12	11	10	9	8	7	6	5	4	3
Increments of pressure y	0	$2\frac{13}{16}$	$6\frac{3}{16}$	$10\frac{2}{16}$	$15\frac{1}{16}$	$21\frac{3}{16}$	$29\frac{11}{16}$	$41\frac{9}{16}$	$58\frac{2}{16}$	$88\frac{7}{16}$

To get the true pressures we should add to the quantities y the height of the barometer column which measures the atmospheric pressure ($29\frac{1}{8}$ according to Boyle's statement). But instead of doing this let us use these numbers for determining the points on the representative curve. The result is shown in fig. 4, and is of a form which at once suggests that the one quantity is an inverse function of the other. What we have called x is the volume, but the quantity y is not the whole pressure. Is it possible to represent this curve by an equation of the form

$$x(y+a)=b,$$

where a and b are constant numbers, the best values of which are to be determined by taking into account all the observations? If

we substitute $29\frac{1}{8}$ for a, as Boyle did, and put $x=v$ and $y+a=p$, we shall find that the relation

$$pv = \text{constant}$$

is almost exactly satisfied throughout. This is Boyle's Law for gases at constant temperature, and is equivalent to the statement that the pressure is inversely as the volume.

By shifting the x axis downwards through a distance equal to $29\frac{1}{8}$, and then measuring the pressure p from this as a new axis, we have the graph representing the equation pv=constant, as shown in fig. 5. This particular curve is a special form of hyperbola, the axes being the asymptotes to the curve—that is, the

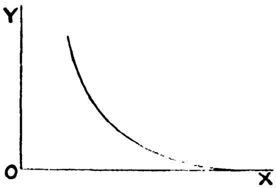


Fig. 4.

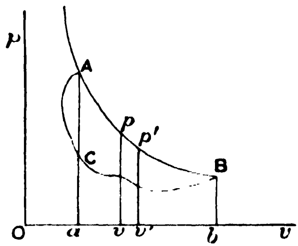


Fig. 5.

tangents at infinity which the infinite branches continually approach but never reach.

1003. Another important application of graphs may be illustrated by means of this curve. Let us imagine the air to be contained in a cylindrical tube, and to be slowly compressed by the motion inwards of a piston fitting the tube perfectly air-tight. The work done is measured by the average force multiplied by the distance through which it acts—that is, in this case by the product,

$$\text{pressure} \times \text{area} \times \text{distance} = p \times \text{change of volume}.$$

Let the change of volume be from v to v', as shown on the curve. The average pressure will be intermediate to the values vp and $v'p'$. The work done will be less than the rectangle pv', and greater than the rectangle $p'v$. Hence if we consider a succession of very small changes taking place continuously, we shall find that ultimately, as the air changes from the state pv to the state $p'v'$ by the representative point moving along the curve, the work done will be accurately measured by the area $pvv'p'p$, bounded by the

vertical ordinates and the intercepts on the curve and on the x axis. Thus, as the gas is compressed from the state B to the state A, the work done by the compressing force is measured by the area AabBA. Similarly, if the air expands from volume Oa to volume Ob at constant temperature, it will pass through states represented by successive points on the graph pv=constant, and work will be done by the gas to an amount measured by the same area.

But we may suppose the air to pass from state A to state B under conditions in which the temperature does not remain constant. In this case the successive states will be represented by points tracing out a curve of a different form from that given by

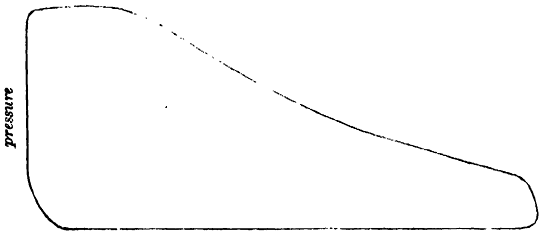


Fig. 6.

Boyle's Law. We may suppose ACB to be such a curve; and we may suppose the air to pass through the complete cycle from A to B along the hyperbola and back to A by the lower path. In this cycle the work done by the air in passing along AB by the equal-temperature path is measured by the area AabBA; but the work done on the air is measured by the area ACBbaA. Hence, since we have made ACB the lower curve, there is on the whole work done by the gas to an amount measured by the area ACBA.

Here, then, we have an illustration of a graph which not only gives information regarding the successive states of a physical process, but also, by means of the area enclosed by a closed path, gives the measure of another important physical quantity.

This particular example has a direct application in estimating by means of the indicator diagram the work done by the expanding steam in the piston-chamber of a steam-engine. By a very simple device the engine is made to trace its own curve, giving the relation of pressure and volume. A pencil is fixed to a piston in connection with the large piston-chamber, and rises and falls as the pressure

of the steam increases and decreases. At the same time by the reciprocating motion of the large piston the paper on which the pencil presses is moved to and fro. At each complete stroke the pencil marks out on the paper a closed curve, the area of which measures the mechanical work done by the expanding and contracting steam. An example of the kind of graph obtained directly from a steam-engine is shown in fig. 6. Beginning at the upper left-hand corner and passing to the right, we see that the pressure of the steam in the piston-chamber reaches a maximum, and then falls away as the steam expands until the end of the stroke. The lower part of the graph is then traced as the piston is pushed back through the space in which the steam is now condensed, exerting therefore a very small pressure. On completion of the second part of the stroke steam again enters the chamber under consideration, and goes through the same cycle of pressure and volume as before. The area of the graph represents the mechanical work done by the steam during one stroke.

1004. Another characteristic feature of graphs from which valuable information may be gained is the slope at each point. Looking back at the various forms depicted, we see that the straight line is the only graph which has the same slope throughout. In every other case the slope changes as we pass along the curve. In the case of the hyperbola (figs. 4 and 5) the slope is continuously decreasing as we pass along the curve to the right, and the slope measures the rate of increase of volume per unit decrease of pressure, or the rate of decrease of volume per unit increase of pressure. But this latter quantity divided by the volume measures the compressibility of the substance. This interpretation can be applied to all forms of *pv* curves, whether of gases, liquids, solids, or mixtures of these.

In the case of the various parabolic graphs depicted, the slope changes sign as we pass from one end to the other—that is, from being a descending slope it becomes an ascending slope. The transition from one to the other occurs at a point for which one of the quantities has a minimum or maximum value. The precise determination of maximum and minimum points in any given graph is a question of great importance in many cases.

1005. If we inspect closely the two curves in fig. 3 we shall see that the slope first increases, then diminishes, in I, and first diminishes and then increases in II, as we pass from the highest points downward to the right. At some intermediate point the

slope must for the moment be stationary. It has, in fact, a minimum value. Such a point is called a point of inflection. It is mathematically defined as a point at which the rate of change of slope vanishes; and the determination of the positions of such points of inflection also forms in many cases an important inquiry.

TABLES

THE METRIC SYSTEM

In this system of measures the values of the denominations proceed by tens—that is, ten of one denomination are equivalent to one of the next higher; and hence it is sometimes called the **decimal** system of measures. The names of the different denominations are formed by placing before the name of the **unit** the prefixes **deca** for ten, **hecto** for a hundred, **kilo** for a thousand, **deci** for a tenth, **centi** for a hundredth, and **milli** for a thousandth.

1. Measures of Length

The unit is the **metre**, and the denominations are in order, beginning at the highest—the kilometre, hectometre, decametre, metre, decimetre, centimetre, millimetre.

The decametre is = 10 metres, the hectometre = 10 decametres = 100 metres; and so on. So the decimetre = $\frac{1}{10}$ of a metre, the centimetre = $\frac{1}{10}$ of a decimetre = $\frac{1}{100}$ of a metre; and so on.

The metre . . . = 3·2808992 imperial feet.
An imperial foot . . = 3·047945 decimetres.

2. Superficial or Square Measure

The unit is the **are**, and the denominations in order are the hectare, decare, are, deciare, centiare.

1 hectare . . . . . = 2·47105 acres.
1 acre . . . . . . = 0·40468 hectare.

The are is a square decametre.

The only multiple of the **are** which is in use is the hectare; the only sub-multiple is the centiare.

3. Solid Measure

The **stere** is the unit, and the denominations are the decastere, stere, decistere.

1 stere . . . . . = 35·31716 cubic feet.
1 cubic foot . . . . = 2·831485 centisteres.

The stere is a cubic metre.

4. Measures of Capacity

The **litre** is the unit, and the denominations are the hectolitre, decalitre, litre, decilitre, centilitre.

1 litre . . . . = 1·760773 pints.
1 gallon . . . . = 4·543458 litres.

The litre is a cubic decimetre.

5. Measures of Weight

The **gramme** is the unit, and the denominations are the kilogramme, hectogramme, decagramme, gramme, decigramme, centigramme, milligramme.

1 gramme . . . . . = 15·432349 grains.
1 oz. avoirdupois . . . . = 28·34954 grammes.

A gramme is the weight of a cubic centimetre of distilled water in vacuo at its maximum density, or at 39° F.

TABLE

LENGTHS OF CIRCULAR ARCS TO RADIUS = 1

D	Arc	D	Arc	D	Arc	M	Arc	S	Arc	T	Arc
1	·0174533	61	1·0646508	121	2·1118484	1	2909	1	48	1	1
2	·0349066	62	1·0821041	122	2·1293017	2	5818	2	97	2	2
3	·0523599	63	1·0995574	123	2·1467550	3	8727	3	145	3	2
4	·0698132	64	1·1170107	124	2·1642083	4	11636	4	194	4	3
5	·0872665	65	1·1344640	125	2·1816616	5	14544	5	242	5	4
6	·1047198	66	1·1519173	126	2·1991149	6	17453	6	291	6	5
7	·1221730	67	1·1693706	127	2·2165682	7	20362	7	339	7	6
8	·1396263	68	1·1868239	128	2·2340214	8	23271	8	388	8	6
9	·1570796	69	1·2042772	129	2·2514747	9	26180	9	436	9	7
10	·1745329	70	1·2217305	130	2·2689280	10	29089	10	485	10	8
20	·3490659	80	1·3962634	140	2·4434610	20	58178	20	970	20	16
30	·5235988	90	1·5707963	150	2·6179939	30	87266	30	1454	30	24
40	·6981317	100	1·7453293	160	2·7925268	40	116355	40	1939	40	32
50	·8726646	110	1·9198622	170	2·9670597	50	145444	50	2424	50	40
60	1·0471976	120	2·0943951	180	3·1415927	60	174533	60	2909	60	48

A TABLE OF THE AREAS OF THE SEGMENTS OF A CIRCLE,

THE DIAMETER OF WHICH IS UNITY, AND SUPPOSED TO BE DIVIDED INTO 1000 EQUAL PARTS

Height	Area	Height	Area	Height	Area	Height	Area	Height	Area
·001	·000042	·048	·013818	·095	·037909	·142	·068225	·189	·103116
·002	·000119	·049	·014248	·096	·038497	·143	·068924	·190	·103900
·003	·000219	·050	·014681	·097	·039087	·144	·069626	·191	·104686
·004	·000337	·051	·015119	·098	·039681	·145	·070329	·192	·105472
·005	·000471	·052	·015561	·099	·040277	·146	·071034	·193	·106261
·006	·000619	·053	·016008	·100	·040875	·147	·071741	·194	·107051
·007	·000779	·054	·016458	·101	·041477	·148	·072450	·195	·107843
·008	·000952	·055	·016912	·102	·042081	·149	·073162	·196	·108636
·009	·001135	·056	·017369	·103	·042687	·150	·073875	·197	·109431
·010	·001329	·057	·017831	·104	·043296	·151	·074590	·198	·110227
·011	·001533	·058	·018297	·105	·043908	·152	·075307	·199	·111025
·012	·001746	·059	·018766	·106	·044523	·153	·076026	·200	·111824
·013	·001969	·060	·019239	·107	·045140	·154	·076747	·201	·112625
·014	·002199	·061	·019716	·108	·045759	·155	·077470	·202	·113427
·015	·002438	·062	·020197	·109	·046381	·156	·078194	·203	·114231
·016	·002685	·063	·020681	·110	·047006	·157	·078921	·204	·115036
·017	·002940	·064	·021168	·111	·047633	·158	·079650	·205	·115842
·018	·003202	·065	·021660	·112	·048262	·159	·080380	·206	·116651
·019	·003472	·066	·022155	·113	·048894	·160	·081112	·207	·117460
·020	·003749	·067	·022653	·114	·049529	·161	·081847	·208	·118271
·021	·004032	·068	·023155	·115	·050165	·162	·082582	·209	·119083
·022	·004322	·069	·023660	·116	·050805	·163	·083320	·210	·119898
·023	·004619	·070	·024168	·117	·051446	·164	·084060	·211	·120713
·024	·004922	·071	·024680	·118	·052090	·165	·084801	·212	·121530
·025	·005231	·072	·025196	·119	·052737	·166	·085545	·213	·122348
·026	·005546	073	·025714	·120	·053385	·167	·086290	·214	·123167
·027	·005867	·074	·026236	·121	·054037	·168	·087037	·215	·123988
·028	·006194	·075	·026761	·122	·054690	·169	·087785	·216	·124811
·029	·006527	·076	·027290	·123	·055346	·170	·088536	·217	·125634
·030	·006866	·077	·027821	·124	·056004	·171	·089288	·218	·126459
·031	·007209	·078	·028356	·125	·056664	·172	·090042	·219	·127286
·032	·007559	·079	·028894	·126	·057327	·173	·090797	·220	·128114
·033	·007913	·080	·029435	·127	·057991	·174	·091555	·221	·128943
·034	·008273	·081	·029979	·128	·058658	·175	·092314	·222	·129773
·035	·008638	·082	·030526	·129	·059328	·176	·093074	·223	·130605
·036	·009008	·083	·031077	·130	·059999	·177	·093837	·224	·131438
·037	·009383	·084	·031630	·131	·060673	·178	·094601	·225	·132273
·038	·009763	·085	·032186	·132	·061349	·179	·095367	·226	·133109
·039	·010148	·086	·032746	·133	·062027	·180	·096135	·227	·133946
·040	·010538	·087	·033308	·134	·062707	·181	·096904	·228	·134784
·041	·010932	·088	·033873	·135	·063389	·182	·097675	·229	·135624
·042	·011331	·089	·034441	·136	·064074	·183	·098447	·230	·136465
·043	·011734	·090	·035012	·137	·064761	·184	·099221	·231	·137307
·044	·012142	·091	·035586	·138	·065449	·185	·099997	·232	·138151
·045	·012555	·092	·036162	·139	·066140	·186	·100774	·233	·138996
·046	·012971	·093	·036742	·140	·066833	·187	·101553	·234	·139842
·047	·013393	·094	·037324	·141	·067528	·188	·102334	·235	·140689

Height	Area	Height	Area	Height	Area	Height	Area	Height	Area
·236	·141538	·289	·188141	·342	·237369	·395	·288476	·448	·340793
·237	·142388	·290	·189048	·343	·238319	·396	·289454	·449	·341788
·238	·143239	·291	·189956	·344	·239268	·397	·290432	·450	·342783
·239	·144091	·292	·190865	·345	·240219	·398	·291411	·451	·343778
·240	·144945	·293	·191774	·346	·241170	·399	·292390	·452	·344773
·241	·145800	·294	·192685	·347	·242122	·400	·293370	·453	·345768
·242	·146655	·295	·193597	·348	·243074	·401	·294350	·454	·346764
·243	·147513	·296	·194509	·349	·244027	·402	·295330	·455	·347760
·244	·148371	·297	·195423	·350	·244980	·403	·296311	·456	·348756
·245	·149231	·298	·196337	·351	·245935	·404	·297292	·457	·349752
·246	·150091	·299	·197252	·352	·246890	·405	·298274	·458	·350749
·247	·150953	·300	·198168	·353	·247845	·406	·299256	·459	·351745
·248	·151816	·301	·199085	·354	·248801	·407	·300238	·460	·352742
·249	·152681	·302	·200003	·355	·249758	·408	·301221	·461	·353739
·250	·153546	·303	·200922	·356	·250715	·409	·302204	·462	·354736
·251	·154413	·304	·201841	·357	·251673	·410	·303187	·563	·355733
·252	·155281	·305	·202762	·358	·252632	·411	·304171	·464	·356730
·253	·156149	·306	·203683	·359	·253591	·412	·305156	·465	·357728
·254	·157019	·307	·204605	·360	·254551	·413	·306140	·466	·358725
·255	·157891	·308	·205528	·361	·255511	·414	·307125	·467	·359723
·256	·158763	·309	·206452	·362	·256472	·415	·308110	·468	·360721
·257	·159636	·310	·207376	·363	·257433	·416	·309096	·469	·361719
·258	·160511	·311	·208302	·364	·258395	·417	·310082	·470	·362717
·259	·161386	·312	·209228	·365	·259358	·418	·311068	·471	·363715
·260	·162263	·313	·210155	·366	·260321	·419	·312055	·472	·364714
·261	·163141	·314	·211083	·367	·261285	·420	·313042	·473	·365712
·262	·164020	·315	·212011	·368	·262249	·421	·314029	·474	·366711
·263	·164900	·316	·212941	·369	·263214	·422	·315017	·475	·367710
·264	·165781	·317	·213871	·370	·264179	·423	·316005	·476	·368708
·265	·166663	·318	·214802	·371	·265145	·424	·316993	·477	·369707
·266	·167546	·319	·215734	·372	·266111	·425	·317981	·478	·370706
·267	·168431	·320	·216666	·373	·267078	·426	·318970	·479	·371705
·268	·169316	·321	·217600	·374	·268046	·427	·319959	·480	·372704
·269	·170202	·322	·218534	·375	·269014	·428	·320949	·481	·373704
·270	·171090	·323	·219469	·376	·269982	·429	·321938	·482	·374703
·271	·171978	·324	·220404	·377	·270951	·430	·322928	·483	·375702
·272	·172868	·325	·221341	·378	·271921	·431	·323919	·484	·376702
·273	·173758	·326	·222278	·379	·272891	·432	·324909	·485	·377701
·274	·174650	·327	·223216	·380	·273861	·433	·325900	·486	·378701
·275	·175542	·328	·224154	·381	·274832	·434	·326891	·487	·379701
·276	·176436	·329	·225094	·382	·275804	·435	·327883	·488	·380700
·277	·177330	·330	·226034	·383	·276776	·436	·328874	·489	·381700
278	·178226	·331	·226974	·384	·277748	·437	·329866	·490	·382700
·279	·179122	·332	·227916	·385	·278721	·438	·330858	·491	·383700
·280	·180020	·333	·228858	·386	·279695	·439	·331851	·492	·384699
·281	·180918	·334	·229801	·387	·280669	·440	·332843	·493	·385699
·282	·181818	·335	·230745	·388	·281643	·441	·333836	·494	·386699
283	·182718	·336	·231689	·389	·282618	·442	·334829	·495	·387699
·284	·183610	·337	·232634	·390	·283593	·443	·335823	·496	·388699
·285	·184522	·338	·233580	·391	·284569	·444	·336816	·497	·389699
·286	·185425	·339	·234526	·392	·285545	·445	·337810	·498	·390699
·287	·186329	·340	·235473	·393	·286521	·446	·338804	·499	·391699
·288	·187235	·341	·236421	·394	·287499	·447	·339799	·500	·392699

NUMBERS OF FREQUENT USE IN CALCULATION

Numbers			Logarithms	Arithmetical Complements *
1 mètre expressed in feet . . .	=	3·2808992......	0·5159929	$\bar{1}$.4840071
1 foot expressed in decimetres . .	=	3·0479449......	0·4830071	$\bar{1}$·5169929
1 gallon expressed in litres . .	=	4·543458.......	0·6573865	$\bar{1}$·3426135
1 litre expressed in pints . . .	=	1·760773.......	0·2457033	$\bar{1}$·7542967
1 oz. avoirdupois expressed in grammes	=	28·34954........	1·4525460	$\bar{2}$·5474540
1 gramme expressed in grains . .	=	15·432349.......	1·1884320	$\bar{2}$·8115680
Circumf. of circle to diameter 1, or π	=	3·141592654....	0·4971499	$\bar{1}$·5028501
Square of this number, or π^2 . .	=	9·869604400....	0·9942998	$\bar{1}$·0057002
Area of circle to diameter 1, or $\frac{1}{4}\pi$.	=	·78539816.....	$\bar{1}$·8950899	0·1049101
Volume of sphere to diameter 1, or $\frac{1}{6}\pi$	=	·52359878.....	$\bar{1}$·7189986	0·2810014
Area of circle to circumference 1 .	=	·0795775......	$\bar{2}$·9007904	1·0092096
Degrees in arc that is=radius . .	=	57·29577951.....	1·7581226	$\bar{2}$·2418774
Minutes in same	=	3437·74677........	3·5362739	$\bar{4}$·4637261
Length of arc of 1° to radius 1 . .	=	·01745329.....	$\bar{2}$·2418773	1·7581227
" " 1′ " 1 . .	=	·0002908882...	$\bar{4}$·4637260	3·5362740
" " 1″ " 1 . .	=	·0000048481368	$\bar{6}$·6855747	5·3144253
Degrees in circumference of circle .	=	360.............	2·5563025	$\bar{3}$·4436975
Minutes " " " .	=	21600.............	4·3344538	$\bar{5}$·6655462
Seconds in 1 hour	=	3600.............	3·5563025	$\bar{4}$·4436975
" " 24 hours	=	86400.............	4·9365137	$\bar{5}$·0634863
Length of tropical year in days .	=	365·24224........	2·5625810	$\bar{3}$·4374190
Length of same in seconds . .	=	31556930.............	7·4990948	$\bar{8}$·5009052
Time of earth's rotation in seconds .	=	86164·0997.........	4·9353264	$\bar{5}$·0646736
Length of seconds' pendulum at Edinburgh	=	39·1555 in......	1·5927928	$\bar{2}$·4071072
Length of seconds' pendulum at London	=	39·1393 in......	1·5926130	$\bar{2}$·4073870
Force of gravity in feet at Edinburgh	=	32·20415........	1·5079109	$\bar{2}$·4920891
" " " London .	=	32·19078........	1·5077220	$\bar{2}$·4922780
British mile in feet	=	5280.............	3·7226339	$\bar{4}$·2773661
Geographical or nautical mile in feet	=	6075·6............	3·7835892	$\bar{4}$·2164108
Earth's polar axis in feet . . .	=	41711200.............	7·6202527	$\bar{8}$·3797473
Earth's equatorial diameter in feet .	=	41850600.............	7·6217017	$\bar{8}$·3782983
Mean diameter of earth in miles .	=	7918·4............	3·8983433	$\bar{4}$·1016567
Modulus of common logarithms .	=	·43429448.....	$\bar{1}$·6377843	0·3622157
Its reciprocal or natural logarithm of 10	=	2·30258509.....	0·3622149	$\bar{1}$·6377851
Base of natural system	=	2·718281828459.	0·4342945	$\bar{1}$·5657055
Its reciprocal	=	·36787944.....	$\bar{1}$·5657055	0·4342945

* The arithmetical complement of a logarithm given in this column is such that when added to the logarithm, the sum is 0.

FOUR PLACE LOGARITHMS OF NUMBERS AND CIRCULAR FUNCTIONS.

For rough and rapid calculation it is sometimes convenient to use four place logarithms. It has consequently been thought advisable to give three Tables of such logarithms of numbers, of sines and cosines, and of tangents and co-tangents. Calculations made by use of these Tables will be more accurate than can be obtained with the slide-rule in its ordinary form, and will be very nearly as rapid.

The Tables are to be used in the ordinary way (for instructions in the use of logarithms, see the introduction to *Chambers's Mathematical Tables*). It will be noticed that the Table of the logarithms of numbers begins in a somewhat expanded form, the logarithms for four-figured numbers between 1000 and 1110 being given complete. Thereafter it is sufficient to tabulate them as usual for three significant figures in the number, the logarithm for any number involving four figures being got by simple interpolation. The single numbers at the tops of the columns give the next digit of the number whose earlier digits are in the left-hand column.

The logarithms of the circular functions are tabulated for every 6 minutes of arc—that is, to tenths of degrees. The Table is therefore in a form convenient for working with degrees and decimals of a degree; and interpolation can readily be effected either to minutes or to hundredths of degrees.

LOGARITHMS OF NUMBERS

Num.	0	1	2	3	4	5	6	7	8	9	Diff.
100	0000	004	009	013	017	022	026	030	035	039	4
101	043	048	052	056	060	065	069	073	078	082	4
102	086	090	095	099	103	107	111	116	120	124	4
103	128	133	137	141	145	149	154	158	162	166	4
104	170	175	179	183	187	191	195	199	204	208	4
105	212	216	220	224	228	233	237	241	245	249	4
106	253	257	261	265	269	274	278	282	286	290	4
107	294	298	302	306	310	314	318	322	326	330	4
108	334	338	342	346	350	354	358	362	366	370	4
109	374	378	382	386	390	394	398	402	406	410	4
110	414	418	422	426	430	434	438	441	445	449	4
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	38
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	35
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	33
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	30
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	28
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	26
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	25
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	24
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	22
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	21
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	20
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	19
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	18
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	18
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	17
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	16
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	16
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	15
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	15
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	14
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	14
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	13
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	13
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	13
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	12
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	12
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	12
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	11
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	11
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	11
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	10
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	10
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	10
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	10
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	10
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	9
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	9
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	9
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	9

LOGARITHMS OF NUMBERS

Num.	0	1	2	3	4	5	6	7	8	9	Diff.
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	8
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	8
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	8
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	8
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	8
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	7
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	7
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	7
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	7
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	7
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	7
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	6
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	6
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	6
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	6
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	6
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	6
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	6
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	5
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	5
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	5
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	5
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	5
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	5
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	5
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	5
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	5
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	4

LOGARITHMS OF SINES AND COSINES

Logarithmic Sines (0° – 45°)

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	60′	
0°	Inf. Neg.	7·2419	5429	7190	8439	9408	0̄200	0̄870	1̄450	1̄961	8·2419	89°
1	8·2419	2832	3210	3558	3880	4179	4459	4723	4971	5206	8·5428	88
2	8·5428	5640	5842	6035	6220	6397	6567	6731	6889	7041	8·7188	87
3	8·7188	7330	7468	7602	7731	7857	7979	8098	8213	8326	8·8436	86
4	8·8436	8543	8647	8749	8849	8946	9042	9135	9226	9315	8·9403	85
5	8·9403	9489	9573	9655	9736	9816	9894	9970	0̄046	0̄120	9·0192	84
6	9·0192	0264	0334	0403	0472	0539	0605	0670	0734	0797	9·0859	83
7	9·0859	0920	0981	1040	1099	1157	1214	1271	1326	1381	9·1436	82
8	9·1436	1489	1542	1594	1646	1697	1747	1797	1847	1895	9·1943	81
9	9·1943	1991	2038	2085	2131	2176	2221	2266	2310	2353	9·2397	80
10	9·2397	2439	2482	2524	2565	2606	2647	2687	2727	2767	9·2806	79
11	9·2806	2845	2883	2921	2959	2997	3034	3070	3107	3143	9·3179	78
12	9·3179	3214	3250	3284	3319	3353	3387	3421	3455	3488	9·3521	77
13	9·3521	3554	3586	3618	3650	3682	3713	3745	3775	3806	9·3837	76
14	9·3837	3867	3897	3927	3957	3986	4015	4044	4073	4102	9·4130	75
15	9·4130	4158	4186	4214	4242	4269	4296	4323	4350	4377	9·4403	74
16	9·4403	4430	4456	4482	4508	4533	4559	4584	4609	4634	9·4659	73
17	9·4659	4684	4709	4733	4757	4781	4805	4829	4853	4876	9·4900	72
18	9·4900	4923	4946	4969	4992	5015	5037	5060	5082	5104	9·5126	71
19	9·5126	5148	5170	5192	5213	5235	5256	5278	5299	5320	9·5341	70
20	9·5341	5361	5382	5402	5423	5443	5463	5484	5504	5523	9·5543	69
21	9·5543	5563	5583	5602	5621	5641	5660	5679	5698	5717	9·5736	68
22	9·5736	5754	5773	5792	5810	5828	5847	5865	5883	5901	9·5919	67
23	9·5919	5937	5954	5972	5990	6007	6024	6042	6059	6076	9·6093	66
24	9·6093	6110	6127	6144	6161	6177	6194	6210	6227	6243	9·6259	65
25	9·6259	6276	6292	6308	6324	6340	6356	6371	6387	6403	9·6418	64
26	9·6418	6434	6449	6465	6480	6495	6510	6526	6541	6556	9·6570	63
27	9·6570	6585	6600	6615	6629	6644	6659	6673	6687	6702	9·6716	62
28	9·6716	6730	6744	6759	6773	6787	6801	6814	6828	6842	9·6856	61
29	9·6856	6869	6883	6896	6910	6923	6937	6950	6963	6977	9·6990	60
30	9·6990	7003	7016	7029	7042	7055	7068	7080	7093	7106	9·7118	59
31	9·7118	7131	7144	7156	7168	7181	7193	7205	7218	7230	9·7242	58
32	9·7242	7254	7266	7278	7290	7302	7314	7326	7338	7349	9·7361	57
33	9·7361	7373	7384	7396	7407	7419	7430	7442	7453	7464	9·7476	56
34	9·7476	7487	7498	7509	7520	7531	7542	7553	7564	7575	9·7586	55
35	9·7586	7597	7607	7618	7629	7640	7650	7661	7671	7682	9·7692	54
36	9·7692	7703	7713	7723	7734	7744	7754	7764	7774	7785	9·7795	53
37	9·7795	7805	7815	7825	7835	7844	7854	7864	7874	7884	9·7893	52
38	9·7893	7903	7913	7922	7932	7941	7951	7960	7970	7979	9·7989	51
39	9·7989	7998	8007	8017	8026	8035	8044	8053	8063	8072	9·8081	50
40	9·8081	8090	8099	8108	8117	8125	8134	8143	8152	8161	9·8169	49
41	9·8169	8178	8187	8195	8204	8213	8221	8230	8238	8247	9·8255	48
42	9·8255	8264	8272	8280	8289	8297	8305	8313	8322	8330	9·8338	47
43	9·8338	8346	8354	8362	8370	8378	8386	8394	8402	8410	9·8418	46
44	9·8418	8426	8433	8441	8449	8457	8464	8472	8480	8487	9·8495	45
	60′	54′	48′	42′	36′	30′	24′	18′	12′	6′	0′	

Logarithmic Cosines (45° – 90°)

LOGARITHMS OF SINES AND COSINES

Logarithmic Sines (45° – 90°)

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	60′	
45°	9·8495	8502	8510	8517	8525	8532	8540	8547	8555	8562	9·8569	44°
46	9·8569	8577	8584	8591	8598	8606	8613	8620	8627	8634	9·8641	43
47	9·8641	8648	8655	8662	8669	8676	8683	8690	8697	8704	9·8711	42
48	9·8711	8718	8724	8731	8738	8745	8751	8758	8765	8771	9·8778	41
49	9·8778	8784	8791	8797	8804	8810	8817	8823	8830	8836	9·8843	40
50	9·8843	8849	8855	8862	8868	8874	8880	8887	8893	8899	9·8905	39
51	9·8905	8911	8917	8923	8929	8935	8941	8947	8953	8959	9·8965	38
52	9·8965	8971	8977	8983	8989	8995	9000	9006	9012	9018	9·9023	37
53	9·9023	9029	9035	9041	9046	9052	9057	9063	9069	9074	9·9080	36
54	9·9080	9085	9091	9096	9101	9107	9112	9118	9123	9128	9·9134	35
55	9·9134	9139	9144	9149	9155	9160	9165	9170	9175	9181	9·9186	34
56	9·9186	9191	9196	9201	9206	9211	9216	9221	9226	9231	9·9236	33
57	9·9236	9241	9246	9251	9255	9260	9265	9270	9275	9279	9·9284	32
58	9·9284	9289	9294	9298	9303	9308	9312	9317	9322	9326	9·9381	31
59	9·9331	9335	9340	9344	9349	9353	9358	9362	9367	9371	9·9375	30
60	9·9375	9380	9384	9388	9393	9397	9401	9406	9410	9414	9·9418	29
61	9·9418	9422	9427	9431	9435	9439	9443	9447	9451	9455	9·9459	28
62	9·9459	9463	9467	9471	9475	9479	9483	9487	9491	9495	9·9499	27
63	9·9499	9503	9507	9510	9514	9518	9522	9525	9529	9533	9·9537	26
64	9·9537	9540	9544	9548	9551	9555	9558	9562	9566	9569	9·9573	25
65	9·9573	9576	9580	9583	9587	9590	9594	9597	9601	9604	9·9607	24
66	9·9607	9611	9614	9617	9621	9624	9627	9631	9634	9637	9·9640	23
67	9·9640	9643	9647	9650	9653	9656	9659	9662	9666	9669	9·9672	22
68	9·9672	9675	9678	9681	9684	9687	9690	9693	9696	9699	9·9702	21
69	9·9702	9704	9707	9710	9713	9716	9719	9722	9724	9727	9·9780	20
70	9·9730	9733	9735	9738	9741	9743	9746	9749	9751	9754	9·9757	19
71	9·9757	9759	9762	9764	9767	9770	9772	9775	9777	9780	9·9782	18
72	9·9782	9785	9787	9789	9792	9794	9797	9799	9801	9804	9·9806	17
73	9·9806	9808	9811	9813	9815	9817	9820	9822	9824	9826	9·9828	16
74	9·9828	9831	9833	9835	9837	9839	9841	9843	9845	9847	9·9849	15
75	9·9849	9851	9853	9855	9857	9859	9861	9863	9865	9867	9·9869	14
76	9·9869	9871	9873	9875	9876	9878	9880	9882	9884	9885	9·9887	13
77	9·9887	9889	9891	9892	9894	9896	6897	9899	9901	9902	9·9904	12
78	9·9904	9906	9907	9909	9910	9912	9913	9915	9916	9918	9·9919	11
79	9·9919	9921	9922	9924	9925	9927	9928	9929	9931	9932	9·9934	10
80	9·9934	9935	9936	9937	9939	9940	9941	9943	9944	9945	9·9946	9
81	9·9946	9947	9949	9950	9951	9952	9953	9954	9955	9956	9·9958	8
82	9·9958	9959	9960	9961	9962	9963	9964	9965	9966	9967	9·9968	7
83	9·9968	9968	9969	9970	9971	9972	9978	9974	9975	9975	9·9976	6
84	9·9976	9977	9978	9978	9979	9980	9981	9981	9982	9983	9·9983	5
85	9·9983	9984	9985	9985	9986	9987	9987	9988	9988	9989	9·9989	4
86	9·9989	9990	9990	9991	9991	9992	9992	9993	9993	9994	9·9994	3
87	9·9994	9994	9995	9995	9996	9996	9996	9996	9997	9997	9·9997	2
88	9·9997	9998	9998	9998	9998	9999	9999	9999	9999	9999	9·9999	1
89	9·9999	9999	$\overline{0}000$	$\overline{0}000$	$\overline{0}000$	$\overline{0}000$	$\overline{0}000$	$\overline{0}000$	$\overline{0}000$	$\overline{0}000$	10·0000	0
	60′	54′	48′	42′	36′	30′	24′	18′	12′	6′	0′	

Logarithmic Cosines (0° – 45°)

LOGARITHMS OF TANGENTS AND COTANGENTS

Logarithmic Tangents (0° – 45°)

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	60′	
0°	Inf. Neg.	7·2419	5429	7190	8439	9409	$\bar{0}200$	$\bar{0}870$	$\bar{1}450$	$\bar{1}962$	8·2419	**89°**
1	8·2419	2833	3211	3559	3881	4181	4461	4725	4973	5208	8·5431	**88**
2	8·5431	5643	5845	6038	6223	6401	6571	6736	6894	7046	8·7194	**87**
3	8·7194	7337	7475	7609	7739	7865	7988	8107	8223	8336	8·8446	**86**
4	8·8446	8554	8659	8762	8862	8960	9056	9150	9241	9331	8·9420	**85**
5	8·9420	9506	9591	9674	9756	9836	9915	9992	$\bar{0}068$	$\bar{0}143$	9·0216	**84**
6	9·0216	0289	0360	0430	0499	0567	0633	0699	0764	0828	9·0891	**83**
7	9·0891	0954	1015	1076	1135	1194	1252	1310	1367	1423	9·1478	**82**
8	9·1478	1533	1587	1640	1693	1745	1797	1848	1898	1948	9·1997	**81**
9	9·1997	2046	2094	2142	2189	2236	2282	2328	2374	2419	9·2463	**80**
10	9·2463	2507	2551	2594	2637	2680	2722	2764	2805	2846	9·2887	**79**
11	9·2887	2927	2967	3006	3046	3085	3123	3162	3200	3237	9·3275	**78**
12	9·3275	3312	3349	3385	3422	3458	3493	3529	3564	3599	9·3634	**77**
13	9·3634	3668	3702	3736	3770	3804	3837	3870	3903	3935	9·3968	**76**
14	9·3968	4000	4032	4064	4095	4127	4158	4189	4220	4250	9·4281	**75**
15	9·4281	4311	4341	4371	4400	4430	4459	4488	4517	4546	9·4575	**74**
16	9·4575	4603	4632	4660	4688	4716	4744	4771	4799	4826	9·4853	**73**
17	9·4853	4880	4907	4934	4961	4987	5014	5040	5066	5092	9·5118	**72**
18	9·5118	5143	5169	5195	5220	5245	5270	5295	5320	5345	9·5370	**71**
19	9·5370	5394	5419	5443	5467	5491	5516	5539	5563	5587	9·5611	**70**
20	9·5611	5634	5658	5681	5704	5727	5750	5773	5796	5819	9·5842	**69**
21	9·5842	5864	5887	5909	5932	5954	5976	5998	6020	6042	9·6064	**68**
22	9·6064	6086	6108	6129	6151	6172	6194	6215	6236	6257	9·6279	**67**
23	9·6279	6300	6321	6341	6362	6383	6404	6424	6445	6465	9·6486	**66**
24	9·6486	6506	6527	6547	6567	6587	6607	6627	6647	6667	9·6687	**65**
25	9·6687	6706	6726	6746	6765	6785	6804	6824	6843	6863	9·6882	**64**
26	9·6882	6901	6920	6939	6958	6977	6996	7015	7034	7053	9·7072	**63**
27	9·7072	7090	7109	7128	7146	7165	7183	7202	7220	7238	9·7257	**62**
28	9·7257	7275	7293	7311	7330	7348	7366	7384	7402	7420	9·7438	**61**
29	9·7438	7455	7473	7491	7509	7526	7544	7562	7579	7597	9·7614	**60**
30	9·7614	7632	7649	7667	7684	7701	7719	7736	7753	7771	9·7788	**59**
31	9·7788	7805	7822	7839	7856	7873	7890	7907	7924	7941	9·7958	**58**
32	9·7958	7975	7992	8008	8025	8042	8059	8075	8092	8109	9·8125	**57**
33	9·8125	8142	8158	8175	8191	8208	8224	8241	8257	8274	9·8290	**56**
34	9·8290	8306	8323	8339	8355	8371	8388	8404	8420	8436	9·8452	**55**
35	9·8452	8468	8484	8501	8517	8533	8549	8565	8581	8597	9·8613	**54**
36	9·8613	8629	8644	8660	8676	8692	8708	8724	8740	8755	9·8771	**53**
37	9·8771	8787	8803	8818	8834	8850	8865	8881	8897	8912	9·8928	**52**
38	9·8928	8944	8959	8975	8990	9006	9022	9037	9053	9068	9·9084	**51**
39	9·9084	9099	9115	9130	9146	9161	9176	9192	9207	9223	9·9238	**50**
40	9·9238	9254	9269	9284	9300	9315	9330	9346	9361	9376	9·9392	**49**
41	9·9392	9407	9422	9438	9453	9468	9483	9499	9514	9529	9·9544	**48**
42	9·9544	9560	9575	9590	9605	9621	9636	9651	9666	9681	9·9697	**47**
43	9·9697	9712	9727	9742	9757	9773	9788	9803	9818	9833	9·9848	**46**
44	9·9848	9864	9879	9894	9909	9924	9939	9955	9970	9985	10·0000	**45**
	60′	54′	48′	42′	36′	30′	24′	18′	12′	6′	0′	

Logarithmic Cotangents (45° – 90°)

LOGARITHMS OF TANGENTS AND COTANGENTS

Logarithmic Tangents (45° – 90°)

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	60′	
45°	10·0000	0015	0030	0045	0061	0076	0091	0106	0121	0136	10·0152	44°
46	10·0152	0167	0182	0197	0212	0228	0243	0258	0273	0288	10·0303	43
47	10·0303	0319	0334	0349	0364	0379	0395	0410	0425	0440	10·0456	42
48	10·0456	0471	0486	0501	0517	0532	0547	0562	0578	0593	10·0608	41
49	10·0608	0624	0639	0654	0670	0685	0700	0716	0731	0746	10·0762	40
50	10·0762	0777	0793	0808	0824	0839	0854	0870	0885	0901	10·0916	39
51	10·0916	0932	0947	0963	0978	0994	1010	1025	1041	1056	10·1072	38
52	10·1072	1088	1103	1119	1135	1150	1166	1182	1197	1213	10·1229	37
53	10·1229	1245	1260	1276	1292	1308	1324	1340	1356	1371	10·1387	36
54	10·1387	1403	1419	1435	1451	1467	1483	1499	1516	1532	10·1548	35
55	10·1548	1564	1580	1596	1612	1629	1645	1661	1677	1694	10·1710	34
56	10·1710	1726	1743	1759	1776	1792	1809	1825	1842	1858	10·1875	33
57	10·1875	1891	1908	1925	1941	1958	1975	1992	2008	2025	10·2042	32
58	10·2042	2059	2076	2093	2110	2127	2144	2161	2178	2195	10·2212	31
59	10·2212	2229	2247	2264	2281	2299	2316	2333	2351	2368	10·2386	30
60	10·2386	2403	2421	2438	2456	2474	2491	2509	2527	2545	10·2562	29
61	10·2562	2580	2598	2616	2634	2652	2670	2689	2707	2725	10·2743	28
62	10·2743	2762	2780	2798	2817	2835	2854	2872	2891	2910	10·2928	27
63	10·2928	2947	2966	2985	3004	3023	3042	3061	3080	3099	10·3118	26
64	10·3118	3137	3157	3176	3196	3215	3235	3254	3274	3294	10·3313	25
65	10·3313	3333	3353	3373	3393	3413	3433	3453	3473	3494	10·3514	24
66	10·3514	3535	3555	3576	3596	3617	3638	3659	3679	3700	10·3721	23
67	10·3721	3743	3764	3785	3806	3828	3849	3871	3892	3914	10·3936	22
68	10·3936	3958	3980	4002	4024	4046	4068	4091	4113	4136	10·4158	21
69	10·4158	4181	4204	4227	4250	4273	4296	4319	4342	4366	10·4389	20
70	10·4389	4413	4437	4461	4484	4509	4533	4557	4581	4606	10·4630	19
71	10·4630	4655	4680	4705	4730	4755	4780	4805	4831	4857	10·4882	18
72	10·4882	4908	4934	4960	4986	5013	5039	5066	5093	5120	10·5147	17
73	10·5147	5174	5201	5229	5256	5284	5312	5340	5368	5397	10·5425	16
74	10·5425	5454	5483	5512	5541	5570	5600	5629	5659	5689	10·5719	15
75	10·5719	5750	5780	5811	5842	5873	5905	5936	5968	6000	10·6032	14
76	10·6032	6065	6097	6130	6163	6196	6230	6264	6298	6332	10·6366	13
77	10·6366	6401	6436	6471	6507	6542	6578	6615	6651	6688	10·6725	12
78	10·6725	6763	6800	6838	6877	6915	6954	6994	7033	7073	10·7113	11
79	10·7113	7154	7195	7236	7278	7320	7363	7406	7449	7493	10·7537	10
80	10·7537	7581	7626	7672	7718	7764	7811	7858	7906	7954	10·8003	9
81	10·8003	8052	8102	8152	8203	8255	8307	8360	8413	8467	10·8522	8
82	10·8522	8577	8633	8690	8748	8806	8865	8924	8985	9046	10·9109	7
83	10·9109	9172	9236	9301	9367	9433	9501	9570	9640	9711	10·9784	6
84	10·9784	9857	9932	$\overline{0}008$	$\overline{0}085$	$\overline{0}164$	$\overline{0}244$	$\overline{0}326$	$\overline{0}409$	0494	11·0580	5
85	11·0580	0669	0759	0850	0944	1040	1138	1238	1341	1446	11·1554	4
86	11·1554	1664	1777	1893	2012	2135	2261	2391	2525	2663	11·2806	3
87	11·2806	2954	3106	3264	3429	3599	3777	3962	4155	4357	11·4569	2
88	11·4569	4792	5027	5275	5539	5819	6119	6441	6789	7167	11·7581	1
89	11·7581	8038	8550	9130	9800	$\overline{0}591$	$\overline{1}561$	$\overline{2}810$	$\overline{4}571$	$\overline{7}581$	Inf. Pos.	0°
	60′	54′	48′	42′	36′	30′	24′	18′	12′	6′	0′	

Logarithmic Cotangents (0° – 45°)

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